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Phrasal Coordination Relatedness Logic

Abstract. I presented a sub-classical *relating logic* based on a relating via an NL-inspired relating relation $R^{\mathrm{c}}_{\mathrm{ss}}$. The relation $R^{\mathrm{c}}_{\mathrm{ss}}$ is motivated by the NL-phenomenon of phrasal (subsentential) coordination, exhibiting an important aspect of contents relating among the arguments of binary connectives. The resulting logic $\mathcal{L}^{\mathrm{c}}_{\mathrm{ss}}$ can be viewed as a relevance logic exhibiting a contents related relevance, stronger than the variable-sharing property of other relevance logics like **R**.

Note that relating here is not "tailored" to justify some predetermined logic; rather, the relating relation is *independently justified*, and induces a logic not previously investigated.

Keywords: relating logic; sub-sentential coordination; relevance logic

1. Introduction

A hallmark of logical object-languages is that they are *freely generated* from some set of atomic formulas. Thus, when considering the connectives of classical logic, for *any* formulas φ and ψ, their conjunction $\varphi \wedge \psi$, disjunction $\varphi \vee \psi$, and implication $\varphi \rightarrow \psi$ are well-formed formulas. Furthermore, for propositional logics the atomic generators are usually viewed as *propositional variables*, amenable for arbitrary uniform substitutions.

I consider this free generation as an *over-abstraction* of the syntax of natural language, of which formal logical object-languages originate as abstractions.

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One facet of this over-generation was discussed in [3]. Here, I want to consider another facet of this over-generation, namely the *disregarding of contents relating* when constructing compound sentences.[1]

It is certainly possible to use contents based relatedness in the semantics, mainly as a *filter on truth-conditions*, as in *relatedness semantics* [6], following Epstein [2]. In this semantics, an *arbitrary* binary relation R is imposed on formulas of the object-language, a relation used, usually as a filter on truth-conditions, in defining a logic model-theoretically. A detailed history of relating logic can be found in [7].

The idea of employing relatedness in terms of content already is not completely new. For example, Krajewski [8] considers relatedness by stipulating an *arbitrary* relatedness relation among propositional variables and among predicate names, extended to formulas in a certain way.

In this paper I also investigate content considerations in the syntactic formation rules. I define a logic $\mathcal{L}_{\rm ss}$ by imposing a rather *non-arbitrary*, fixed relation $R_{\rm ss}$, motivated by an intriguing phenomenon in natural language (NL). The phenomenon is the ability of NL to express *sub-sentential (phrasal) coordination*:[2] as explained in Section 2.

In order to impose $R_{\rm ss}$, the object-language $L_{\rm ss}$ of the logic $\mathcal{L}_{\rm ss}$ cannot be propositional, as the definition of $R_{\rm ss}$ depends on sub-atomic components of atomic sentences in a (quantification free) first-order language, as specified in Section 3.1.

2. Meaning connection among arguments of binary connectives in natural languages

While sentential combination via sentential connectives is also present in (some) natural languages, the latter have a richer structure allowing also for *sub-sentential (phrasal) coordination*, either as *constituent*[3] *coordination* or as *non-constituent coordination*, as in[4]

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[1] I use 'sentences' and 'formulas' as synonyms.

[2] In an abuse of nomenclature, for convenience, I include implication also as a coordination. It is not in accordance with standard linguistics nomenclature but should cause no confusion.

[3] Constituency is a syntactic property the exact details of which, theory dependent, are immaterial here.

[4] Natural language sentences and phrases considered are in italics and are always mentioned, never used.

Mary sings and/or dances
Mary loves Bill and/or John
Bill and/or John love Mary
Mary is pretty and loves John (1)
Mary loves and Sue hates John
Fred bought a shirt for Bill and a sweater for George

This kind of coordination is more frequently used than its sentential counterpart. For an overview [see 5].

In this paper, coordination with *plural predication* like

John and Mary are siblings

is excluded. In addition, natural languages allow *anaphora*, as in[5]

If Mary is happy, she smiles (2)

Consider also the following recursive coordinations.

Mary can either [sing and dance] or [sing and play the guitar]

As mentioned above, such sentences are more dominant in ordinary discourse compared to *sentential* coordination as in

Mary is happy and/or grass is green
If Mary is happy, grass is green (3)

It is important to realise that because I treat only propositional connectives, I am not concerned here with sentences with quantified subject and/or object, like

everyone/every girl/someone/some girl loves Bill and/or John

involving issues of conjunction reduction [see, e.g., 9 and further references therein], the latter not always preserving semantic equivalence ('$\equiv$'-identity of meaning). While

Everyone sings and dances $\equiv$ *Everyone sings and everyone dances*

we have

Everyone sings or dances $\not\equiv$ *Everyone sings or everyone dances*

[5] The form '*If Mary is happy then she smiles*' is less colloquial.

For the simple, non-quantified subjects and objects used here, the semantic equivalence preservation by a translation to sentential coordination is justified.

Thus, the sentences in (1) and (2) are semantically equivalent to their respective expansions to sentences with sentential coordinations.

Mary sings and/or Mary dances
Mary loves Bill and/or Mary loves John
Bill loves Mary and/or John loves Mary
Mary is pretty and Mary loves John (4)
Mary loves John and Sue hates John
Fred bought a shirt for Bill and Fred bought a coat for George
If Mary is happy, Mary smiles

And for the recursive coordinations:

[Mary can sing and Mary can dance] or
[Mary can sing and Mary can play the guitar] (5)

I will take coordinated sentences resulting from translation of sentences with sub–sentential coordination or with anaphoric references as indicating the *semantic connection* between the combined subsentence; the connection arising from *sharing* a sub-sentential phrase.

3. The logic $\mathcal{L}_{\text{ss}}$

In this section I introduce the logic $\mathcal{L}_{\text{ss}}$, taking its name from *sub-sentential* coordination as discussed above.

3.1. The object-language L_{ss}

The object-language L_{ss} is a fragment of L_{qf}, the quantifier-free standard first-order language over the connectives $C = \{\neg, \wedge, \vee. \rightarrow\}$ (negation, conjunction, disjunction and implication, respectively). By '$*$' is meant any binary connective in C. The L_{ss}-fragment is obtained by restricting L_{qf} to formulas called R_{ss}-proper, where R_{ss} is a relating relation, as specified in the next section. Individual constants are ranged over by a, b, and closed formulas by φ, ψ.

3.2. The relating relation R_{ss}

The relating relation R_{ss} mimics the NL subsentential phrase sharing discussed in Section 2. For the language L_{ss}, this amounts to sharing an individual constant or a predicate name (of any arity).

Before defining he relating relation R_{ss}, consider, as motivating examples, regimenting the NL-sentences in (4) and (5) in L_{qf}.

Example 3.1. The L_{qf}-regimentation of the sentences in (4) and (5) under the obvious choice of constants and predicate names look as follows.

$S(m) \wedge / \vee D(m)$	shared m
$L(m, b) \wedge / \vee L(m, j)$	shared $m,\ L$
$L(b, m) \wedge / \vee L(j, m)$	shared $m,\ L$
$P(m) \wedge L(m, j)$	shared m
$L(m, j) \wedge H(s, j)$	shared j
$B(f, s, b) \wedge B(f, c, b)$	shared B, f, b
$H(m) \rightarrow S(m)$	shared m
$(S(m) \wedge D(m)) \vee (S(m) \wedge P(m, g))$	shared $S(m))$ ⊣

Example 3.2. Similarly, the regimentation of the sentences in (3) look as follows.

$$H(m) \wedge G(g)$$
$$H(m) \rightarrow G(g)$$

Clearly, no sharing is present among the coordinated subformulas. ⊣

Definition 3.1 (shared subsentential phrase relatedness). L_{qf}-formulas φ and ψ are *subsentential phrase sharing related*, denoted $\varphi R_{ss}\psi$, iff one of the following conditions is satisfied:

1. φ and ψ share an individual constant.
2. φ and ψ share a predicate name.

The non-R_{ss}-relating of φ and ψ is indicated by $\varphi \not{R}_{ss}\psi$. Let $\Gamma R_{ss}\varphi$ iff $\psi R_{ss}\varphi$ for every $\psi \in \Gamma$.

Corollary 3.1. *R_{ss} is reflexive and symmetric.*

Most importantly, R_{ss} *is not transitive.*

Example 3.3 (non-transitivity of R_{ss}). $P(m)R_{ss}Q(m)$ and $Q(m)R_{ss}Q(n)$ hold but $P(m)\not{R}_{ss}Q(n)$. ⊣

COROLLARY 3.2 (negation). *$\varphi R_{\text{ss}}\psi$ iff $\varphi R_{\text{ss}}\neg\psi$.*

COROLLARY 3.3. *If φ and ψ share some subformula, say χ, then φ and ψ are R_{ss}-related.*

In a sense, under the usual interpretation of first-order logics, R_{ss}-related formulas share some kinds of 'aboutness': either they apply possibly different predications[6] to some individual constant, or they apply the same predication to different individual constants. Clearly, this is an aspect of *shared contents*.

DEFINITION 3.2 (R_{ss}-proper formulas). An L_{qf}-formula φ is R_{ss}*-proper* iff

- φ is atomic, or
- φ is of the form $\chi * \xi$ and $\chi R_{\text{ss}}\xi$ holds, or
- φ is of the form $\neg\psi$ and ψ is R_{ss}-proper.

If φ is not R_{ss}-proper, it is R_{ss}-improper.

COROLLARY 3.4 (decidability of R_{ss}-properness). *For an arbitrary L_{qf}-formula φ, it is decidable whether φ is R_{ss}-proper.*

Let $L_{\text{ss}} := \{\varphi \in L_{\text{qf}} \mid \varphi \text{ is } R_{\text{ss}}\text{-proper}\}$. Clearly, all the formulas in Example 3.1 are R_{ss}-proper, while the formulas in Example 3.2 are not.

The following proposition is an immediate consequence of Definition 3.2.

PROPOSITION 3.1 (R_{ss}-properness propagation).

1. *Atomic sentences are R_{ss}-proper.*
2. *The negation of an R_{ss}-proper φ is R_{ss}-proper.*
3. *The negation of an R_{ss}-improper φ is R_{ss}-improper.*
4. *For $* \in \{\wedge, \vee, \rightarrow\}$:*
   (i) *The $*$-combination of two R_{ss}-proper formulas φ and ψ is:*
   - *R_{ss}-proper if $\varphi R_{\text{ss}}\psi$.*
   - *R_{ss}-improper if $\varphi \not{R}_{\text{ss}}\psi$.*

   (ii) *If at least one of φ and ψ is R_{ss}-improper the $*$-combination is:*
   - *R_{ss} improper if $\varphi \not{R}_{\text{ss}}\psi$.*
   - *R_{ss}-proper if $\varphi R_{\text{ss}}\psi$.*

[6] Here 'predication' should also refer to relating to other individual via some n-ary relation, not just applying a unary predicate name.

Note that the $R_{\rm ss}$-properness *is not compositional.* The following two examples show this.

Example 3.4. Here is an example of two $R_{\rm ss}$-proper sentences the conjunction of which is *not* $R_{\rm ss}$-proper. By definition, atomic sentences are $R_{\rm ss}$-proper, so both $P(a)$ and $Q(b)$ are $R_{\rm ss}$-proper. However, since $P(a)\not R_{\rm ss}Q(b)$, we have that $P(a) \wedge Q(b)$ is $R_{\rm ss}$-improper. ⊣

Example 3.5. Here is an example of a combination of $R_{\rm ss}$-improper formulas that becomes $R_{\rm ss}$-proper. Let $\varphi = P(a) \wedge Q(b)$. Clearly, $P(a)\not R_{\rm ss}Q(b)$, so φ is $R_{\rm ss}$-improper.

Similarly, let $\psi = P(a) \wedge S(c)$, which is $R_{\rm ss}$-improper.

Now consider $\chi = \varphi \vee \psi$. Since φ and ψ share the subformula $P(a)$, we have by Corollary 3.3 $\varphi R_{\rm ss}\psi$. So, χ is $R_{\rm ss}$-proper. ⊣

3.3. Defining $\mathcal{L}_{\rm ss}$

The models interpreting $\mathcal{L}_{\rm ss}$ are the same as those interpreting first-order classical logic, with one extra proviso.

Definition 3.3 (interpretation). A *model* for $\mathcal{L}_{\rm ss}$ is a tuple $\mathcal{M} = \langle D, I\rangle$, where:

- D is a non-empty domain of interpretation.
- I is an interpretation function mapping individual constants to elements of D and predicate symbols to their extensions in D, where the following proviso is imposed, prohibiting some “accidental” $R_{\rm ss}$-relating.

  proviso (normality):
  (a) For constants a and b: if $a \neq b$ then $I[\![a]\!] \neq I[\![b]\!]$.
  (b) For predicate symbols (of any arity) S and T: if $S \neq T$ then $I[\![S]\!] \neq I[\![T]\!]$.

  I is extended to mapping formulas to truth-values as usual.

Definition 3.4 ($\mathcal{L}_{\rm ss}$-logical consequence). φ is a $\mathcal{L}_{\rm ss}$-logical consequence of Γ, denoted $\Gamma \models_{\mathcal{L}_{\rm ss}} \varphi$, iff for every $\mathcal{L}_{\rm ss}$ model and $\mathcal{L}_{\rm ss}$-interpretation I:

$I[\![\varphi]\!] = t$ (i.e., is true) whenever $I[\![\psi]\!] = t$ for every $\psi \in \Gamma$, *and* $\Gamma R_{\rm ss}\varphi$ (i.e., $\psi R_{\rm ss}\varphi$ for every $\psi \in \Gamma$).

I.e., $\mathcal{L}_{\rm ss}$-logical consequence is truth preservation (from assumptions to conclusion) *and* $R_{\rm ss}$-relating of the conclusion to each assumption.

The non-transitivity of $R_{\rm ss}$-relating propagates to non-transitivity of $R_{\rm ss}$-logical consequence.

Proposition 3.2 (non-transitivity of $\models_{\mathcal{L}_{ss}}$). *$\models_{\mathcal{L}_{ss}}$ is non-transitive.*

Example 3.6 (non-transitivity of $\models_{\mathcal{L}_{ss}}$). $P(a) \wedge \neg P(a) \models_{\mathcal{L}_{ss}} Q(a) \wedge \neg Q(a)$. Also, $Q(a) \wedge \neg Q(a) \models_{\mathcal{L}_{ss}} Q(c)$; but $P(a) \wedge \neg P(a) \not\models_{\mathcal{L}_{ss}} Q(c)$. ⊣

4. Properties of $\mathcal{L}_{ss}$

First, the following proposition follows directly from the definition of $\mathcal{L}_{ss}$-logical consequence.

Proposition 4.1 (sub-classicality). *$\mathcal{L}_{ss}$ is sub-classical.*

Next, the following important proposition holds due to the non-transitivity of R_{ss}-logical consequence.

Proposition 4.2 (paraconsistency and paracompleteness of $\mathcal{L}_{ss}$). *$\mathcal{L}_{ss}$ is both paraconsistent and paracomplete.*

Proof. Clearly, an arbitrary ψ need not be R_{ss}-related to a contradiction $\varphi \wedge \neg\varphi$ or to a tautology $\varphi \vee \neg\varphi$. So $\varphi \wedge \neg\varphi \models_{\mathcal{L}_{ss}} \psi$ and $\psi \models_{\mathcal{L}_{ss}} \varphi \vee \neg\varphi$ need not hold. ⊣

Still, some weaker form of explosion and implosion does hold.

Proposition 4.3 (R_{ss}-relating explosion and implosion). *If $\psi R_{ss} \varphi$ (hence, also $\psi R_{ss} \varphi \wedge \neg\varphi$ and $\psi R_{\mathcal{L}_{ss}} \varphi \vee \neg\varphi$), then*

$$\varphi \wedge \neg\varphi \models_{\mathcal{L}_{ss}} \psi \qquad \psi \models_{\mathcal{L}_{ss}} \varphi \vee \neg\varphi$$

Proposition 4.4 (semantic "half" deduction theorem). *If $\Gamma, \varphi \models_{\mathcal{L}_{ss}} \psi$ then $\Gamma \models_{\mathcal{L}_{ss}} \varphi \to \psi$.*

Proof. The argument about truth-propagation is like in classical logic. I therefore present only the argument about R_{ss}-relatedness. If $\Gamma, \varphi \models_{\mathcal{L}_{ss}} \psi$, then $\Gamma R_{ss} \psi$ and $\varphi R_{ss} \psi$. Hence $\Gamma R_{ss} \varphi \to \psi$, by Corollary 3.3. ⊣

The following example shows that the converse of Proposition 4.4 does not hold.

Example 4.1. By inspection, we have

$$P(b), P(b) \to (P(a) \to Q(a)) \models_{\mathcal{L}_{ss}} (P(a) \to Q(a))$$

However,

$$P(b), P(b) \to (P(a) \to Q(a)), P(a) \not\models_{\mathcal{L}_{ss}} Q(a)$$

because $P(b) \not{R}_{ss} Q(a)$. ⊣

5. A natural-deduction system for $\mathcal{L}_{\text{ss}}$

In this section, I attend to presenting a proof system, a natural-deduction (ND) system $\mathcal{N}_{\text{ss}}$, for $\mathcal{L}_{\text{ss}}$. Recall that the the objects language L_{ss} is a sublanguage of the classical first-order quantifier-free language L_{qf}.

The point of departure is Gentzen's ND-system *NK* for classical logic [4]. However, the latter is not concerned with R_{ss}-relating, so it has to be modified to account for the latter too.

The basic idea in the modification is to add to each *NK*-rule R_{ss}-*relating of the conclusion to each premise* as a side condition. However, because of the non-transitivity of the logical consequence relation, this modification does not suffice, as exemplified by the following example.

Example 5.1 (non-transitivity of derivation). Consider the following "derivation" (where $\vdash_{\mathcal{N}_{\text{ss}}}$ is defined below) for the invalid

$$P(b), P(b) \to (P(a) \to Q(a)), P(a) \vdash_{\mathcal{N}_{\text{ss}}} Q(a)$$

The invalidity of

$$P(b), P(b) \to (P(a) \to Q(a)), P(a) \models_{\mathcal{L}_{\text{ss}}} Q(a)$$

stems from $P(b) \not{R}_{\text{ss}} Q(a)$.

The derivation is applying twice in consecution the modified modus-ponens rule (implication elimination $(\to E_{\text{ss}})$).

$$\dfrac{\dfrac{P(b) \quad P(b) \to (P(a) \to Q(a))}{P(a) \to Q(a)}\,(\to E_{\text{ss}}) \qquad P(a)}{Q(a)}\,(\to E_{\text{ss}})$$

In this derivation:

- Each formula is R_{ss}-proper.
- In each application of $(\to E_{\text{ss}})$, the conclusion is R_{ss}-related to each premise.
- Still, the conclusion of the derivation is *not* R_{ss}-related to each initial assumption. ⊣

Consequently, the side condition on each *NK*-rule has to be stronger:

1. *The conclusion in each rule has to be* R_{ss}*-related to each premise* and *to each open assumption on which that premise depends.*

2. *For assumption discharging rules, the R_{ss}-relatedness is also imposed on* at least one *sub-derivation from the discharged assumption* (see remark below).

Denote the combination of those two conditions, serving together as a side-condition on rules, by $R^{\mathrm{c}}_{\mathrm{ss}}$.

The way to implement the side condition $R^{\mathrm{c}}_{\mathrm{ss}}$ is by adopting the technique employed in the ND-system for the relevant logic **R** [1]. Each assumption is uniquely labelled by an index, and the rules propagate the dependence on assumption, recording the set α of indices of assumptions on which a formula φ in a derivation depends in the form of φ_α.

While for **R** the index sets α are employed for *tracking use* of an assumption in a derivation (in order to avoid vacuous discharge), in $\mathcal{N}_{\mathrm{ss}}$ those indices are employed to impose R_{ss}-relating of a conclusion to the open assumption on which it depends.

Note that an assumption discharged by an application of a rule is no longer in the index of the conclusion, and is exempt from being R_{ss}-related to it.

We thus obtain the following definition of the side condition $R^{\mathrm{c}}_{\mathrm{ss}}$ imposed on each *NK*-rule ρ, to get the corresponding $\mathcal{N}_{\mathrm{ss}}$-rule ρ_{ss}.

Definition 5.1 ($R^{\mathrm{c}}_{\mathrm{ss}}$). Consider any *NK*-rule ρ with n indexed premises $\pi_{i\alpha_i}$, $1 \leqslant i \leqslant n$, and conclusion ψ_β. Let $\alpha = \cup_{1\leqslant i\leqslant n}\alpha_i$. Then,

1. $\pi_i R_{\mathrm{ss}} \psi$ for $1 \leqslant i \leqslant n$.
2. $\varphi_j R_{\mathrm{ss}} \psi$ for every $j \in \alpha$, where φ_j is the open assumption indexed j.

The rules of $\mathcal{N}_{\mathrm{ss}}$ are displayed below:

$$\frac{}{\varphi_i : \varphi_{\{i\}}}\ (Ax_{\mathrm{ss}})$$

$$\frac{\varphi_\alpha \quad \psi_\beta}{(\varphi\wedge\psi)_{\alpha\cup\beta}}\ (\wedge I_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}} \qquad \frac{(\varphi\wedge\psi)_\alpha}{\varphi_\alpha}\ (\wedge_1 E_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}} \qquad \frac{(\varphi\wedge\psi)_\alpha}{\psi_\alpha}\ (\wedge_2 E_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}}$$

$$\frac{\begin{array}{c}[\varphi]_i\\ \vdots\\ \psi_\alpha\end{array}}{(\varphi\to\psi)_{\alpha-\{i\}}}\ (\to I^i_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}} \qquad \frac{\varphi_\alpha \quad (\varphi\to\psi)_\beta}{\psi_{\alpha\cup\beta}}\ (\to E_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}}$$

$$\frac{\varphi_\alpha}{(\varphi\vee\psi)_\alpha}\ (\vee_1 I_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}} \qquad \frac{\psi_\alpha}{(\varphi\vee\psi)_\alpha}\ (\vee_2 I_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}}$$

$$\dfrac{(\varphi \vee \psi)_\alpha \quad \begin{matrix}[\varphi]_i \\ \vdots \\ \chi_\beta\end{matrix} \quad \begin{matrix}[\psi]_j \\ \vdots \\ \chi_\gamma\end{matrix}}{\chi_{\alpha \cup \beta \cup \gamma - \{i,j\}}} \; (\vee E^{i,j}_{ss}), R^c_{ss}$$

$$\dfrac{\begin{matrix}[\varphi]_i \\ \vdots \\ \psi_\beta\end{matrix} \quad \begin{matrix}[\varphi]_j \\ \vdots \\ (\neg\psi)_\gamma\end{matrix}}{(\neg\varphi)_{\beta \cup \gamma - \{i,j\}}} \; (\neg I^{i,j}_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}} \qquad \dfrac{(\neg\neg\varphi)_\alpha}{\varphi_\alpha} \; (dne_{\mathrm{ss}}), R^{\mathrm{c}}_{\mathrm{ss}}$$

Remark 5.1. 1. Note that the bad derivation in Example 5.1 is blocked, as the second application of $(\rightarrow E_{\mathrm{ss}})$ violates the side condition $R^{\mathrm{c}}_{\mathrm{ss}}$ in $Q(a)$ not being R_{ss}-related to the premise $P(b)$.

2. Note that while the conditional introduction rule $(\rightarrow I)$ has one sub-derivation, requiring ψ to be R_{ss}-related to the discharged assumption φ, the disjunction elimination rule $(\vee E)$ has two sub-derivations, requiring the arbitrary conclusion χ to be R_{ss}-related to at least one disjunct (a discharged assumption); this ensure already the R_{ss}-relatedness of χ to the major premise, the disjunction $\varphi \vee \psi$.

This is exemplified[7] in Example 5.2. ⊣

Clearly, the structural rule of Weakening (adding an arbitrary assumption) cannot be admissible, as the assumption does not need to be R_{ss}-related to the conclusion. A weaker form of Weakening, namely, adding an assumption R_{ss}-related to the conclusion, is admissible. The axiom could, therefore, be taken as

$$\dfrac{}{\mathit{\Gamma}, \varphi_i : \varphi_i} \; (Ax_{\mathrm{ss}}, \ \mathit{\Gamma} \ R_{\mathrm{ss}} \ \varphi)$$

Example 5.2. I show below that

First, note that

$$P(a) \rightarrow P(b), P(b) \rightarrow Q(b) \ \models_{\mathcal{L}_{\mathrm{ss}}} (P(a) \vee P(b)) \rightarrow Q(b)$$

Classical validity is obvious, and the conclusion is clearly R_{ss}-related to both assumptions. The derivation is in Figure 1.

Showing $R^{\mathrm{c}}_{\mathrm{ss}}$: I show for exemplification the justification (holding of the side condition) of two of the derivation steps. The other steps are justified too, as can be seen by inspection.

[7] I thank a referee of this journal for raising the issue of disjunction.

$$
\dfrac{\dfrac{[P(a) \vee P(b)]_3 \quad \dfrac{\dfrac{[P(a)]_4 \quad P(a) \rightarrow P(b)_1}{P(b)_{1,4}}\ (\rightarrow E_{\text{ss}}) \quad P(b) \rightarrow Q(b)_2}{Q(b)_{1,2,4}}\ (\rightarrow E_{\text{ss}}) \quad \dfrac{[P(b)]_5 \quad P(b) \rightarrow Q(b)_2}{Q(b)_{2,5}}\ (\rightarrow E_{\text{ss}})}{Q(b)_{1,2,3}}\ (\vee I^{4,5}_{\text{ss}})}{(P(a) \vee P(b)) \rightarrow Q(b)_{1,2}}\ (\rightarrow I^{3}_{\text{ss}})
$$

Figure 1.

1. Consider the last application of $(\rightarrow I^3_{\rm ss})$:
   - First, the conclusion $(P(a) \vee P(b)) \rightarrow Q(b)$ is $R_{\rm ss}$-related to its (direct) premise, sharing $Q(b)$, which depends on $\{1, 2, 3\}$.
   - The conclusion is $R_{\rm ss}$-related to assumption 1, sharing both $P(a)$ and $P(b)$.
   - Similarly, the conclusion is $R_{\rm ss}$-related to assumption 2, sharing both $P(a)$ and $Q(b)$.
   - Assumption 3 is discharged by the application of the rule, so need not be checked for being $R_{\rm ss}$-related to the conclusion. It happens, though, to be, sharing the sub-formula $(P(a) \vee P(b))$.
2. Consider the application of $(\vee E^{4,5}_{\rm ss})$:
   - First, its conclusion $Q(b)_{1,2,3}$ is indeed $R_{\rm ss}$-related to its direct premise, the disjunction $P(a) \vee P(b)$ by sharing b.
   - While in the left sub-derivation the conclusion $Q(b)_{1,2,4}$ is not $R_{\rm ss}$-related to the discharged disjunct $P(a)$, the conclusion $Q(b)_{2,5}$ of the right sub-derivation *is* $R_{\rm ss}$-related to the second disjunct, the discharged $P(b)$, sharing b. $\dashv$

6. Conclusion

I presented a sub-classical *relating logic* based on a relatedness via an NL-inspired relating relation $R^{\rm c}_{\rm ss}$. The relation $R^{\rm c}_{\rm ss}$ is motivated by the NL-phenomenon of phrasal (subsentential) coordination, exhibiting an important aspect of contents relatedness among the arguments of binary connectives.

The resulting logic $\mathcal{L}_{\rm ss}$ can be viewed as a relevance logic exhibiting a contents related relevance, stronger than the variable-sharing property of other relevance logics like **R**.

Note that relatedness here is not "tailored" to justify some predetermined logic; rather, the relating relation is *independently justified*, and induces a logic not previously investigated.

Future work may include:

- A more thorough examination of the logic $\mathcal{L}^{\rm c}_{\rm ss}$, in particular providing a completeness proof for $\mathcal{N}_{\rm ss}$.
- Strengthening the side condition $R^{\rm c}_{\rm ss}$ so as to validate the *full* (not just "half") semantic deduction theorem.
- Incorporate into $R_{\rm ss}$ some *lexically derived* relating. For example

  If the sky is cloudy it will rain

where '*cloudy*' and '*rain*' can be considered related by the underlying lexical semantics.

- Recovering sharing relating applicable to quantification.

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