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Toward a Stronger Constraint for Non-Trivial Inconsistent Theories

Abstract. This article discusses the definition of paraconsistency understood as the property of a consequence relation that does not trivialize inconsistent theories. Some logicians have argued that standard paraconsistency, the requirement of a non-explosive consequence relation, is insufficient for that purpose. In this article, we have a twofold goal. First, we offer an exposition of some attempts to strengthen standard paraconsistency in the literature. After discussing the shortcomings of those attempts, we examine the concepts of triviality in relation to which those concepts of paraconsistency are defined. Then, as our second goal, we propose an alternative definition of paraconsistency that aims to avoid the trivialization of inconsistent theories in a stricter sense.

Keywords: ECQ; explosion; inconsistency; triviality; paraconsistency

1. Introduction

The distinction between inconsistency and triviality and the possibility of constructing non-trivial inconsistent theories have been called "a most telling reason" for paraconsistent logic (see Michael, 2016; Priest et al., 2022). The standard way to obtain logical systems that attain that goal is to reject the rule *ECQ* (*Ex Contradictione Quodlibet*): $A, \neg A \vdash B$. This way, in a paraconsistent logical system, it is not possible to deduce everything from a contradiction.

Some logicians have objected that the standard attempt at paraconsistency, which consists of the rejection of *ECQ*, is too permissive with regard to the paraconsistent goal of constructing non-trivial inconsistent theories. The objection is that a logical system may satisfy standard

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paraconsistency and still nearly trivialize inconsistent theories. This makes standard paraconsistency insufficient for the declared purpose of paraconsistent systems.

In view of the insufficiency of standard paraconsistency, its critics have proposed alternative definitions of paraconsistency that aim to capture all and only the logical systems that are truly suitable for the construction of non-trivial inconsistent theories. Such definitions are Igor Urbas' strict paraconsistency (Urbas, 1990) and Gemma Robles and José Méndez's strong paraconsistency (Robles and Méndez, 2009). Both of these proposals make considerable improvements over standard paraconsistency toward the goal of paraconsistent logic.

This article proposes a discussion of the definition of paraconsistent logic as a description that captures all and only those logical systems adequate for the construction of non-trivial inconsistent theories. A logical system will be considered adequate for the paraconsistent purpose if it does not trivialize theories to any level due to their inconsistency.

In connection with the discussion of the definition of paraconsistency, this article has a twofold goal. In Section 3, we make an exposition of some of the objections and alternative proposals that have been made concerning the standard definition of paraconsistency being insufficient for the goal of supporting non-trivial inconsistent theories. Second, in Section 4, we present our contribution to the debate, which consists of our own criticism of strict paraconsistency and our proposal of a definition of paraconsistency alternative to the standard one, which we deem adequate for the declared purpose of paraconsistent logic. Our proposal involves the realization that paraconsistency is not supposed to avoid all kinds of trivialization of inconsistent theories, but only the trivialization related to inconsistency.

The main discussion is preceded, in Section 2, by a quick introduction to the basic concepts presupposed in our discussion: the concepts of inconsistency, triviality, explosion, and standard paraconsistency. Before our exposition of alternatives to standard paraconsistency, in Section 3.1, we present the definitions of the concepts required for understanding the objections and the alternative definitions, namely, the concepts of quasi-triviality and partial explosion. In Section 4, we also define the concept of triviality that is relevant to our alternative definition: the concept of inconsistency-related triviality. Our discussion is followed, in Section 5, by a summary of its results and the suggestion of some topics for follow-up research.

2. The Route from Explosion to Standard Paraconsistency

Consider the usual definitions of inconsistency and triviality. A theory is inconsistent if a contradiction, i.e., a well-formed formula (wff) and its negation, belongs to it:[1]

DEFINITION 2.1 (Inconsistency). A theory T is inconsistent iff for some wff A of its language $\mathcal{L}$, $A \in T$ and $\neg A \in T$. Accordingly, T is consistent iff for every wff A of $\mathcal{L}$, if $A \in T$, then $\neg A \notin T$.

A theory is trivial if all wffs of the language over which it was constructed belong to it:

DEFINITION 2.2 (Triviality). A theory T is trivial iff for every wff B of $\mathcal{L}$, $B \in T$. Accordingly, T is non-trivial iff for some wff B of $\mathcal{L}$, $B \notin T$.

Classical logic holds the following two theses about the concepts of inconsistency and triviality:

(i) *Inconsistency implies triviality.* This thesis can be expressed in the form of the logical principle *Ex Contradictione (Sequitur) Quodlibet* ('anything follows from a contradiction', hereafter '*ECQ*'), also known as 'the principle of explosion': $A, \neg A \vdash B$, i.e., given a contradiction, any wff can be proved from it. As a result, if a theory constructed over a logic for which this thesis holds is inconsistent (contains a contradiction), then it is also trivial (contains all wffs).

(ii) *Triviality implies inconsistency.* This follows directly from the definition of triviality: if a theory is trivial (contains all wffs), then it contains all contradictions (pairs of a wff and its negation), by which it is inconsistent.

Any logic for which thesis (i) holds is said to have an explosive consequence relation:

DEFINITION 2.3 (Explosion). The consequence relation $\vdash$ of a logic $\mathcal{S}$ is explosive iff the following is a rule of $\mathcal{S}$:

$$ECQ : A, \neg A \vdash B$$

Accordingly, $\vdash$ is not explosive iff *ECQ* is not a rule of $\mathcal{S}$.

[1] There are also studies on weaker ways to define inconsistency. For every concept of inconsistency, there are corresponding concepts of consistency and paraconsistency. We refer the reader to (Robles, 2009; Robles and Méndez, 2009), and related works for a discussion of this topic.

Classical logic, intuitionistic logic, and other well-known logics all have explosive consequence relations. Given that theses (i) and (ii) hold in explosive logics, inconsistency is indistinguishable from triviality in them. In other words, in explosive logics a theory that contains a contradiction is the same as the theory that contains all wffs; a theory being inconsistent is the same as a theory being trivial. As a result, in explosive logics there is no middle ground: a theory is either consistent or trivial.

Paraconsistent logic opposes explosive logics in insisting on the distinction they collapse between inconsistent and trivial theories. There are many motivations for that distinction,[2] but the general idea behind them is that inconsistency does not preclude non-trivial reasoning. Therefore, a logic that makes non-trivial inferences possible in inconsistent contexts is called for; a logic that allows the construction of non-trivial inconsistent theories.

The idea of paraconsistent logic as the logic for the construction of non-trivial inconsistent theories is widespread in the literature.[3] We find it in foundational works such as Newton da Costa's classic paper:

> These new calculi can be used as foundations for non-trivial inconsistent theories, as we shall see. (Da Costa, 1974, p. 498)

We also find it in early surveys on paraconsistent logic. For example, in their *On Paraconsistency*, Graham Priest and Richard Routley wrote:

> The important fact about paraconsistent logics is that they provide the basis for inconsistent but non-trivial theories. In other words, there are sets of statements closed under logical consequence which are inconsistent but non-trivial. This fact is sometimes taken as an alternative definition of 'paraconsistent' and, given that logical consequence is transitive, it is equivalent to the original definition.
> (Priest and Routley, 1983, p. 108)

Graham Priest's entry on paraconsistency to the *Handbook of Philosophical Logic* also reads:

> Paraconsistent logics are those which permit inference from inconsistent information in a non-trivial fashion. (Priest, 2002, p. 287)

[2] A recompilation of various motivations for paraconsistency can be found in (Priest et al., 2022, sect. 2).

[3] Nevertheless, see (Michael, 2016) for a discussion and critique of the idea of constructing non-trivial inconsistent theories as an argument for paraconsistent logics.

> The major motivation behind paraconsistent logic has always been the thought that in certain circumstances we may be in a situation where our information or theory is inconsistent, and yet where we are required to draw inferences in a sensible fashion. (Priest, 2002, p. 288)

The same idea also appears in recent introductions to the topic, such as the entry on paraconsistent logic to the *Stanford Encyclopedia of Philosophy* co-authored by Graham Priest, Koji Tanaka, and Zach Weber:

> A most telling reason for paraconsistent logic is, prima facie, the fact that there are theories that are inconsistent but non-trivial. If we admit the existence of such theories, their underlying logics must be paraconsistent. (Priest et al., 2022, sect. 2.1)

The standard way to obtain logical systems that attain the goal of allowing the construction of non-trivial inconsistent theories is by rejecting the logical rule *ECQ*, the principle of explosion. The consequence relation of the resulting logical system is not explosive and, therefore, the resulting logical system is called ‘paraconsistent’. The standard definition of paraconsistent logic goes as follows:[4]

Definition 2.4 (Standard Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in the standard sense iff its consequence relation $\vdash$ is not explosive, i.e., the following is not a rule of S:

$$ECQ : A, \neg A \vdash B$$

Standard paraconsistency is certainly a milestone for the distinction between inconsistency and triviality. However, as we will see in the next section, according to some logicians standard paraconsistency is still unsatisfactory and does not allow the construction of truly non-trivial inconsistent theories.

3. Strong and Strict Paraconsistency

As argued above, standard paraconsistency distinguishes inconsistent theories from the trivial theory, thus describing logical systems in which non-trivial inconsistent theories can be constructed. Nevertheless, some logicians have argued against the permissiveness of standard paraconsistency. Priest and Routley wrote the following:

[4] For a presentation of the standard definition of paraconsistent logic, see (Priest et al., 2022, sect. 1).

> Furthermore, other logics that are technically paraconsistent, such as minimal logic, are not *interestingly paraconsistent* because, although they avoid the disaster of entirely trivialising inconsistent theories, they have the same effect for a whole syntactically determined class of statements. (Priest and Routley, 1983, p. 113)

The remark was taken up in later works by Priest:

> [. . .] there are logics that are paraconsistent but not really appropriate for the use. (Priest, 2002, p. 288)

> Paraconsistency, in the sense just defined, is not a sufficient condition for a consequence relation to be a sensible one with which to handle inconsistent information. (Priest, 2007, p. 130)

In the same line of criticism, Urbas wrote:

> For clearly there are logics which satisfy the letter of D1 [standard paraconsistency] while brazenly flouting its spirit. (Urbas, 1990, p. 345)

Robles and Méndez also wrote:

> Is there really much difference between a theory that allows us to affirm everything and another that lets us deny everything, no matter if it is valid, not valid, contingent? (Robles, 2009, p. 189)[5]

In the works quoted above, the logicians mentioned realized that it is possible to satisfy the definition of standard paraconsistency without properly allowing inconsistent theories to be non-trivial. As ECQ must fail in a standard paraconsistent logic, not all wffs of the language will be derivable from an inconsistency in a given theory. However, nothing prevents inconsistent theories constructed on a standard paraconsistent logic from being nearly trivial, e.g., having all wffs whose main connective is a negation or a conditional belong to them. The examples that the authors above mention are systems like Johansson's minimal logic, which is standard paraconsistent but allows $ECQ_{\neg} : A, \neg A \vdash \neg B$, or Arruda and da Costa's J_2 to J_4 systems, which are also standard paraconsistent but allow $ECQ_{\rightarrow} : A, \neg A \vdash B \rightarrow C$.[6]

[5] See also (Robles and Méndez, 2009, pp. 364–365)

[6] The example of minimal logic is discussed by both Urbas (1990) and Robles and Méndez (2009). Urbas (1990) additionally discusses the example of Arruda and da Costa's systems.

This calls for a more restrictive definition of paraconsistency, one that does not allow logics that nearly trivialize inconsistent theories to bear the name. However, there are those logicians who believe that such a pursuit is a hopeless one. On the topic of the search for more restrictive definitions of paraconsistency, Priest writes that such a definition seems unattainable:

> It is possible to try to tighten up the definition of 'paraconsistent' in various ways. But it seems unlikely that there is any purely formal necessary and sufficient condition for the spirit of paraconsistency: inconsistent information may make a nonsense of a consequence relation in so many, and quite different, ways. (Priest, 2007, pp. 130–131)

For Priest (2002, p. 288), the standard definition of paraconsistency aims simply to set a minimal condition for a logic to be paraconsistent. The permissiveness of the definition is thus justified by its goal of fitting the great variety of logics that are considered paraconsistent. According to the logician, the permissiveness mentioned can be corrected by more precise constraints established in the definition of particular paraconsistent systems (Priest, 2007, p. 131).

Pulcini and Varzi (2018) agree that the attempts made so far to place further constraints on the standard definition of paraconsistency turn out to be too restrictive, since they end up excluding well-establish paraconsistent systems.[7] This way, the authors believe that the pursuit of stronger constraints for paraconsistency in the way conducted so far does not seem to be "on the right track" (Pulcini and Varzi, 2018, p. 5491).

Despite the criticism above, some logicians have proposed more restrictive definitions of paraconsistency. Their goal is to obtain a definition of paraconsistent logic that applies only to logics that do not trivialize their inconsistent theories not only in the technical sense (not allowing all wffs to be derivable from an inconsistency), but in a broad sense (e.g., not allowing all negations nor all conditionals to be derivable from an inconsistency). In this section, we will discuss those attempts. Still, first, we need to provide a precise account of what is meant by 'triviality in a broad sense', from which paraconsistent logics are supposed to protect their inconsistent theories.

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[7] For examples of paraconsistent systems that the authors claim to be inadequately excluded by more restrictive attempts of defining paraconsistency, see (Pulcini and Varzi, 2018, p. 5491).

3.1. Quasi-Triviality and Partial Explosion

Given the fact that an inconsistent theory may be non-trivial, i.e., not contain all wffs, and still contain all negations or all conditionals, for example, we notice that the possibility of constructing inconsistent theories that are non-trivial in the sense of the Definition 2.2 is insufficient to characterize paraconsistent logics in the intended sense. This way, an adequate characterization of paraconsistent logic as a logic that does not trivialize inconsistent theories in any sense demands a broader definition of triviality, one according to which a theory that contains all negations or all conditionals is trivial.

Of course, the argument used for the insufficiency of Definition 2.2 as a description of the kind of triviality that paraconsistency is supposed to avoid on the ground that a theory may not contain all wffs and still contain all negations or all conditionals can be iterated for a definition that classifies as trivial a theory containing all wffs that have a given main connective. We can now say that the definition of triviality in the broader sense classifies as non-trivial a theory that does not contain all negations but still contains all negations of conjunctions or all negations of conditionals, and so on. This argument can be iterated infinitely over the complexity of wffs.

According to Urbas (1990, pp. 345–347), the lesson that we learn from the reasoning above is that the definition of triviality we are looking for is sufficiently broad to classify as trivial any theory containing all wffs of the language that have a given syntactic structure. We can then define a matching concept of explosion, according to which a consequence relation is explosive if it allows passage from inconsistency to any form of triviality. A paraconsistent logic in the intended sense will be one that does not allow any form of explosion, i.e., one that does not allow any form of trivialization of inconsistent theories.

Let us call triviality in the broad sense 'quasi-triviality'[8] and let us define it as follows:

Definition 3.1 (Quasi-Triviality). A theory T is quasi-trivial iff there is a wff B in $\mathcal{L}$ such that for every uniform substitution $s(B)$ of B, $s(B) \in T$.

[8] The expression is borrowed from (Robles, 2009, p. 189, def. 8). The definition is based on Urbas' *-triviality (Urbas, 1990, p. 346, definition D2).

According to the definition above, a theory is quasi-trivial if all uniform substitutions of some wff belong to it. Triviality is a subcase of quasi-triviality, namely, the one in which all uniform substitutions of an atomic wff (i.e., all wffs) belong to a theory.[9]

We can now define a matching broad sense of explosion which we will call 'partial explosion'[10] and define as follows:

DEFINITION 3.2 (Partial Explosion). A consequence relation $\vdash$ of a logic $\mathcal{S}$ is partially explosive iff for some wff A of $\mathcal{L}$ and for every uniform substitution $s(B)$ of some wff B of $\mathcal{L}$, $A, \neg A \vdash s(B)$.

According to the definition, a consequence relation is partially explosive if it makes an inconsistent theory quasi-trivial, i.e., if a contradiction belonging to a theory makes every wff of some given syntactic structure also belong to it. Explosion is a sub-case of partial explosion, namely, the one in which B is an atomic wff, so the set of all of its uniform substitutions is the set of all wffs.

The principle of explosion corresponding to Definition 2.3 is ECQ : $A, \neg A \vdash B$. ECQ is related to the set of all wffs $\mathcal{F}$, for it allows a contradiction to entail any element of $\mathcal{F}$. Now, for Definition 3.2 of partial explosion above, the corresponding principle of explosion will be a restricted version of ECQ, which we will call 'ECQ_*', for a sequence of connectives $*$ that defines a subset of wffs $\mathcal{F}_*$ of $\mathcal{F}$ of all wffs that have the syntactic structure described by $*$.

For example, the restricted version of ECQ for the set of all negations $\mathcal{F}_\neg$ is $ECQ_\neg$: $A, \neg A \vdash \neg B$ and, for the set of all conditionals $\mathcal{F}_\rightarrow$, $ECQ_\rightarrow$: $A, \neg A \vdash B \rightarrow C$, and so on. Requiring $ECQ_\neg$ to fail in a logic $\mathcal{S}$ implies that, for $\mathcal{F}_\neg$ and every inconsistent theory T of $\mathcal{S}$, there is some wff $\neg B \in \mathcal{F}_\neg$ such that $\neg B \notin T$, and similarly for other connectives. By rejecting a restricted version of ECQ as a rule, a paraconsistent logic avoids the partial explosion and, consequently, the quasi-trivialization of inconsistent theories regarding the set of wffs to which that restricted version of ECQ is related (e.g., $\mathcal{F}_\neg$ for $ECQ_\neg$, $\mathcal{F}_\rightarrow$ for $ECQ_\rightarrow$, and so on).

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[9] Quasi-triviality is closely related to Batens' (1980) notion of 'A-destructive'. A theory T constructed from a set of wffs Γ is B-destructive (in our usage of the variables) if for all $s(B)$ of B, $s(B) \in T$, but B is not a theorem of the underlying logic and not for all $s(B)$ of B, $s(B) \in \Gamma$. For the pertinent definitions in Batens' own formulation, see (Batens, 1980, pp. 201–202).

[10] The expression is borrowed from (Carnielli et al., 2007, p. 14, def. 9). The definition is based on Urbas' *-explosion (Urbas, 1990, p. 346, def. D3).

With the concepts of quasi-triviality and partial explosion at hand, we can now take a tour of the proposals of some logicians of concepts of paraconsistency stronger than standard paraconsistency meant to avoid the quasi-trivialization of inconsistent theories.

3.2. Strong Paraconsistency

The first step from standard paraconsistency toward an adequate definition of paraconsistency is to prevent not only all wffs from belonging to inconsistent theories but also all wffs with the same main primitive connective, e.g., all negations or all conditionals. Such a definition of paraconsistency would prevent both the trivialization and the quasi-trivialization (for primitive connectives) of inconsistent theories. That can be accomplished by requiring not only *ECQ* to fail in a paraconsistent logic $\mathcal{S}$ but also restricted versions of *ECQ* for each of the primitive connectives of the language of $\mathcal{S}$.

If we define a paraconsistent logic $\mathcal{S}$ such that the restricted version of *ECQ* for each of the primitive connectives of its language is not a rule of $\mathcal{S}$, we obtain a strong paraconsistent logic in the sense proposed by Gemma Robles and José Méndez.[11] For a propositional language with the usual five connectives $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$, the definition of strong paraconsistency goes as follows:

Definition 3.3 (Strong Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in a strong sense iff for some wff A and for every wff B and C the following does not hold:

(1) $ECQ_{\neg} : A, \neg A \vdash \neg B$
(2) $ECQ_{\wedge} : A, \neg A \vdash B \wedge C$
(3) $ECQ_{\vee} : A, \neg A \vdash B \vee C$
(4) $ECQ_{\rightarrow} : A, \neg A \vdash B \rightarrow C$
(5) $ECQ_{\leftrightarrow} : A, \neg A \vdash B \leftrightarrow C$

A logic $\mathcal{S}$ is strong paraconsistent iff for every set of wffs $\mathcal{F}_*$ such that $*$ is a primitive connective of the language of $\mathcal{S}$, there is a restricted version of *ECQ*, ECQ_*, which is not a rule of $\mathcal{S}$. Another way to formulate the definition of strong paraconsistency is to say that, for every primitive connective $*$, not all uniform substitutions of a wff $*B$ or $B * C$ belong

[11] For the definition of strong paraconsistency, see (Robles, 2009, p. 191, def. 14) and (Robles and Méndez, 2009, p. 367, def. 8.3).

to an inconsistent theory constructed over a strong paraconsistent logic, i.e., these inconsistent theories are not necessarily quasi-trivial.

One could wonder if we should add $ECQ : A, \neg A \vdash B$ as an additional condition to the definition above, in order to prevent not only the quasi-trivialization but also the trivialization of inconsistent theories constructed over strong paraconsistent logics. Another way to put the question is to wonder whether every strong paraconsistent logic is standard paraconsistent. The answer to the latter is affirmative, but it would be redundant to add the failure of ECQ as a requirement in the definition above. Given the failure of any of the restricted versions of ECQ listed in Definition 3.3, the failure of ECQ follows from it. For instance, if $ECQ_{\neg}$ is not a rule of a strong paraconsistent logic $\mathcal{S}$, then for an (non-trivial) inconsistent theory T constructed over $\mathcal{S}$, there must be some wff $\neg B \in \mathcal{F}_{\neg}$ such that $\neg B \notin T$. Consequently, there is also some wff $B \in \mathcal{F}$ such that $B \notin T$, namely, $B = \neg B$. Indeed, for every restricted version of ECQ that strong paraconsistency requires to fail, at least one wff of the syntactic structure specified must not belong to (non-trivial) inconsistent theories constructed over that logic. This way, Definition 3.3 as it is suffices for making every strong paraconsistent logic also standard paraconsistent.

3.3. Absolute Paraconsistency

If we take a closer look at strong paraconsistency, we see that our problem is not yet solved. Strong paraconsistency is an improvement compared to standard paraconsistency in that it prevents the partial explosion and, consequently, the quasi-triviality of its inconsistent theories for each primitive connective. The remaining problem is precisely that it prevents quasi-triviality for those cases only. Nothing prevents an inconsistent theory constructed on a strong paraconsistent logic from being quasi-trivial in relation to a set of all uniform substitutions of a wff with two or more connectives. For example, nothing in the definition of strong paraconsistency prevents that all conjunctions of disjunctions $(B \vee C) \wedge (D \vee F)$, i.e., all elements of the set $\mathcal{F}_{\wedge\vee\vee}$, belong to (non-trivial) inconsistent theories constructed on a strong paraconsistent logic, since $ECQ_{\wedge\vee\vee} : A, \neg A \vdash (B \vee C) \wedge (D \vee F)$ may be valid in the underlying logic.

As noted above, the argument given for the inadequacy of standard paraconsistency can now be used against strong paraconsistency. While standard paraconsistency is inadequate because it does not prevent inconsistent theories from containing all negations or all conditionals, strong paraconsistency is also inadequate because it does not prevent inconsistent theories from containing all negations of conditionals or all conjunctions of disjunctions. This reasoning shows that, if we want to rid inconsistent theories of all kinds of triviality, we should extend strong paraconsistency beyond primitive connectives. We should require restricted versions of *ECQ* for conclusions with more than one connective to fail as well.

However, this time we cannot use the same method that we used for defining strong paraconsistency. Our previous definition required stating a restricted version of *ECQ* that should fail for each set of wffs $\mathcal{F}_*$ such that $*$ is one of the primitive connectives of the formal language in question. Nevertheless, as there are infinitely many sequences $*$ of combinations of primitive connectives, this time we would need a definition of paraconsistency that lists infinitely many restricted versions of *ECQ* that are supposed to fail. Instead of a definition with infinitely many clauses, we should seek a definition that blocks all of these infinitely many cases with a single constraint.

We can achieve the goal above with a strengthening of standard paraconsistency considered although not endorsed by Priest, which we will call ‘absolute paraconsistency’. According to Priest, absolute paraconsistency strengthens standard paraconsistency as follows:

> One might therefore attempt a stronger constraint on the definition of ‘paraconsistent’, such as: for no syntactically definable class of sentences (e.g., negated sentences), Σ, do we have $\alpha, \neg\alpha \vdash \sigma$, for all $\sigma \in \Sigma$. (Priest, 2002, pp. 288–289)

We can formulate that definition in the following way:

Definition 3.4 (Absolute Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in an absolute sense iff for some wff A and B, and for every uniform substitution $s(B)$ of B, the following is not a rule of $\mathcal{S}$:

$$ECQ_* : A, \neg A \vdash s(B).$$

Priest’s syntactically definable classes of sentences Σ are the same as our sets $\mathcal{F}_*$ of all wffs with a given syntactic structure described by their list of connectives $*$. In both cases, we have the sets of all wffs with a given syntactic structure. It is also noteworthy that just as standard paraconsistency is the concept that blocks explosion, absolute paraconsistency is the concept that blocks partial explosion. Strong paraconsistency blocks partial explosion only for some sets $\mathcal{F}_*$, namely, the ones such that $*$ is one of the primitive connectives, while absolute paraconsistency blocks partial explosion for all sets $\mathcal{F}_*$ such that $*$ is any sequence of connectives.

3.4. Strict Paraconsistency

Although absolute paraconsistency fares well in distinguishing inconsistency and triviality, Priest sees a problem with it:

> This seems too strong, however. In many logics, $\alpha, \neg\alpha \vdash \beta$, for every logical truth, β. (Priest, 2002, p. 289)

Urbas identifies the same problem with a definition of paraconsistency that generally prevents all wffs with a given syntactic structure from belonging to inconsistent theories like absolute paraconsistency does:

> The problem, then, is to find a way of extending *-paraconsistency to cover some, but not all, definable connectives, excluding at least theorem-generating ones. (Urbas, 1990, p. 347)

The problem that concerns Urbas and Priest is that the definition of absolute paraconsistency is excessively demanding in that it prevents theorems from belonging to non-trivial inconsistent theories. For example, for the set of all disjunctions of a wff and its negation $B \vee \neg B$ or the set of all conditionals with the same wff as the antecedent and the consequent $B \to B$, there will be a corresponding restricted version of *ECQ* that is not a rule of an absolute paraconsistent logic, by which at least one wff of the form $B \vee \neg B$ and one wff of the form $B \to B$ must fail to belong to non-trivial inconsistent theories constructed on that logic. However, it seems certainly undesirable that the paraconsistency constraint be so demanding that it deductively weakens non-trivial inconsistent theories to the point that some theorem must not be deducible from them.

This way, absolute paraconsistency achieves the distinction that we were looking for between inconsistent and trivial theories at the expense of the deductive power of non-trivial inconsistent theories. The solution is to weaken the definition of absolute paraconsistency in a way that exempts theorems from its deductive restrictions but ideally still prevents inconsistent theories from being quasi-trivial. Urbas proposed the concept of strict paraconsistency to this end:[12]

Definition 3.5 (Strict Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in a strict sense iff for some wff A and B, and for every uniform substitution $s(B)$ of B, the following is not a rule of $\mathcal{S}$:

$$ECQ_* : A, \neg A \vdash s(B),$$

unless for every $s(B)$ of B, $s(B)$ is a theorem of $\mathcal{S}$.

The definition of strict paraconsistency consists precisely of the definition of absolute paraconsistency with an additional clause exempting sets of theorems from restrictions. Strict paraconsistency is stronger than strong paraconsistency because it prevents every wff of (almost) every set of wffs $\mathcal{F}_*$ such that $*$ is any sequence of connectives from being derivable from an inconsistency, in the spirit of absolute paraconsistency. On the other hand, strict paraconsistency is weaker than absolute paraconsistency because it allows every theorem to be derivable from a contradiction. This way, strict paraconsistency allows every wff of a set of wffs $\mathcal{F}_*$ such that $*$ is a sequence of connectives that characterizes the syntactic structure of a theorem to belong to non-trivial inconsistent theories constructed on it.

Urbas' strict paraconsistency seems to solve all the problems we encountered in this section: it prevents the trivialization of inconsistent theories for all (non-theorematic) sets of wffs as well as preserves the deductive power of non-trivial inconsistent theories regarding the deducibility of theorems. Besides, Urbas argues that although strict paraconsistency is much more demanding than standard paraconsistency, most logics that have been defined with the purpose of supporting non-trivial inconsistent theories turn out to be strict paraconsistent.[13]

Moreover, Urbas' proposal of strict paraconsistency as the concept that adequately characterizes paraconsistent logic is acclaimed in the recent literature. For example, Carnielli, Coniglio, and Marcos wrote:

> The requirement that a paraconsistent logic should be boldly paraconsistent was championed by [Urbas, 1990]. The class of boldly paracon-

[12] The definition of strict paraconsistency can be found in (Urbas, 1990, p. 348, def. D5 and D8).

[13] For examples of logics that satisfy strict paraconsistency and logics that do not, see (Urbas, 1990, p. 348-349).

> sistent logics is surely very natural and pervasive.
> (Carnielli et al., 2007, p. 14)

Ripley added:

> [Urbas, 1990] makes a reasonably compelling attempt at the task.
> (Ripley, 2015, p. 773)

Accordingly, some recent contributions on paraconsistency take strict paraconsistency as a desirable property for the paraconsistent logics they propose. For example, Carnielli, Coniglio, and Marcos stated:

> From now on, we will be making an effort, as a matter of fact, to square our paraconsistent logics into this class [. . .].
> (Carnielli et al., 2007, p. 14)

The same can be seen in Castiglioni and Biraben's (2013) 'Strict paraconsistency of truth-degree preserving intuitionistic logic with dual negation', in which the authors proved the strict paraconsistency of the paraconsistent system they discuss. Taking strict paraconsistency to be a desirable property for paraconsistent logics is justified: it is the best concept of paraconsistency available in the literature for the purpose of describing logics that allow the construction of non-(quasi-)trivial inconsistent theories.

4. Beyond Strict Paraconsistency

Standard paraconsistency, the definition of paraconsistent logics as those that reject $ECQ : A, \neg A \vdash B$, has been criticized because it allows other forms of triviality, in particular quasi-triviality, which is intuitively unacceptable from a paraconsistent point of view. Then the quest is open to find a concept of paraconsistency that accomplishes what standard paraconsistency could not. Strict paraconsistency comes very close to the concept of paraconsistency we have been seeking. It avoids both triviality and quasi-triviality, by which it is an improvement over standard and strong paraconsistency. It also allows theories to be regular (i.e., contain all theorems of the underlying logic), by which it can be considered an improvement over absolute paraconsistency as well. However, upon closer examination, we will see that strict paraconsistency is still excessively permissive with respect to notions of triviality not considered so far. In what follows, we will consider two cases of triviality that strict paraconsistency allows.

A logic is strict paraconsistent if it rejects $ECQ_* : A, \neg A \vdash s(B)$, for some wff A and B, except for theorematic cases. This definition excludes from strict paraconsistency all logical systems that would allow all formulas with any given syntactic structure to be derivable from a contradiction (except for the theorematic cases). However, strict paraconsistency still allows logical systems in which all formulas obtained by the uniform substitution of a sub-formula of a given formula are derivable from a contradiction. Formally, while strict paraconsistency rejects $ECQ_* : A, \neg A \vdash s(B)$, for some wff A and B, it still allows $ECQ_{*C} : A, \neg A \vdash C * s(B)$, for any sequence of connectives $*$ and some wff A, B and C.[14] Some examples of ECQ_{*C} are: $ECQ_{\wedge C} : A, \neg A \vdash C \wedge s(B)$, $ECQ_{\vee C} : A, \neg A \vdash C \vee s(B)$, $ECQ_{\to C} : A, \neg A \vdash C \to s(B)$, $ECQ_{\wedge C \vee D} : A, \neg A \vdash (C \vee D) \wedge s(B)$, and $ECQ_{\to C \to D} : A, \neg A \vdash (C \to D) \to s(B)$.[15] Intuitively, we may argue that, from a paraconsistent point of view, just as it is unacceptable to prevent that all formulas be derivable from a contradiction but still allow that all conjunctions, all disjunctions, and all conditionals with a conjunctive antecedent do, it is also unacceptable to prevent that all formulas in those subsets of formulas be derivable from a contradiction but still allow that all conjunctions with a fixed conjunct C, all disjunctions with a fixed disjunct C, and all conditionals with a fixed antecedent $C \wedge D$ be derivable from a contradiction.

The sets of formulas $\mathcal{F}_{*C}$ are subsets of the respective sets of formulas $\mathcal{F}_*$ discussed before. For example, the set of all conjunctions with a fixed conjunct C is a subset of the set of all conjunctions, $\mathcal{F}_{\wedge C} \subseteq \mathcal{F}_\wedge$; the set of all conditionals with a fixed antecedent C is a subset of the set of all conditionals, $\mathcal{F}_{\to C} \subseteq \mathcal{F}_\to$; and so on. What we have done here is to consider, for every set of formulas $\mathcal{F}_*$, subsets of it in which all formulas share some non-logical content in a fixed position. Here we can stretch the concept of triviality to its limits. As discussed before, a theory containing all formulas of a given set $\mathcal{F}_*$ is considered quasi-trivial. Now, consider a theory that contains all formulas of a subset of $\mathcal{F}_*$ such that every element of that subset contains the same non-logical content in a given fixed position. For example, if we take the set of

[14] As before, we can think of $*$ as the sequence of connectives that determines the syntactical structure of the formula $C * s(B)$. Clearly, in this case the first connective of the sequence must be a binary connective that links C and $s(B)$.

[15] More complexity could be added to this representation of restricted versions of ECQ to make it unequivocal for all cases. As for our present purpose a few examples are sufficient, we will refrain from adding such complexity.

all conditionals $\mathcal{F}_{\rightarrow}$, consider a theory containing all conditionals with a fixed antecedent C. Should we not consider this theory nearly trivial according to some stricter sense of triviality? We can call the concept of triviality corresponding to the latter case 'sub-triviality', as it concerns the set of all formulas of a given syntactical structure obtained from uniform substitution over a sub-formula of a given wff:

DEFINITION 4.1 (Sub-Triviality). A theory T is sub-trivial iff there is a wff $C * B$ in $\mathcal{L}$ such that for some wff B and C, a sequence of connectives $*$ of $\mathcal{L}$, and every uniform substitution $s(B)$ of B, $C * s(B) \in T$.

A new definition of triviality comes with a corresponding definition of explosion:

DEFINITION 4.2 (Sub-Explosion). A consequence relation $\vdash$ of a logic $\mathcal{S}$ is sub-explosive iff for some wff A and every wff $C * s(B)$ of $\mathcal{L}$ such that $s(B)$ is obtained from a wff B of $\mathcal{L}$ by uniform substitution, $A, \neg A \vdash C * s(B)$.

According to the definition above, a consequence relation is sub-explosive if it makes an inconsistent theory sub-trivial, i.e., if a contradiction belonging to a theory makes every wff obtained by uniform substitution of a sub-formula of a given formula belong to it. Partial explosion is a sub-case of sub-explosion, namely, the case for which C is an empty string of symbols of the language in question.

Now, one might think that the concept of triviality has been stretched too far, that sub-triviality is too strict, or that this form of triviality might be acceptable. However, it is essentially the same argument that allows the passage from triviality to quasi-triviality that suggests the shift from quasi-triviality to sub-triviality. Basically, any class of wffs that can be syntactically characterized can be taken to constitute some form of triviality, for we have all wffs of a given syntactical structure belonging to a theory.

A second argument against strict paraconsistency may be obtained from arguments presented by Béziau (2000) and Pulcini and Varzi (2018, p. 5491). The logicians noticed that it is possible to extend Urbas' (1990) reasoning according to which an adequate paraconsistent logic must reject all versions of ECQ for restricted conclusions, such as $ECQ_{\neg}$, $ECQ_{\rightarrow}$, $ECQ_{\rightarrow\wedge\vee}$, etc., (except when all instances of the conclusion are theorems of the underlying logic), by considering that an adequate paraconsistent logic should reject versions of ECQ for restricted premises too. The example that the authors consider is $EC_{\neg}Q : \neg A, \neg\neg A \vdash B$.

A logic may be standard paraconsistent, so that ECQ is not valid in it, and still validate $EC_{\neg}Q$,[16] which someone who endorses Urbas' position should deem unacceptable.

Indeed, from the point of view of the paraconsistent ideal of supporting non-trivial inconsistent theories, $EC_{\neg}Q$ is unacceptable just like $ECQ_{\neg}$, i.e., deriving all wffs from a negation and its negation is unreasonable just like deriving all negations from a wff and its negation. Versions of ECQ obtained by restricting its premises should be rejected just like the ones obtained by restricting its conclusion. In the same line of thought, restricted versions of ECQ obtained from imposing restrictions on both premises and conclusion, like $EC_{\neg}Q_{\neg} : \neg A, \neg\neg A \vdash \neg B$ and $EC_{\neg}Q_{\rightarrow} : \neg A, \neg\neg A \vdash B \rightarrow C$, should also be avoided by an adequate concept of paraconsistency. While strict paraconsistency prevents all versions of ECQ obtainable by imposing restrictions on the conclusion of ECQ (except for theorematic cases), it does not avoid the versions of ECQ obtainable by imposing restrictions on the premises of ECQ or both on premises and conclusion.

How can we overcome the permissiveness of strict paraconsistency? Perhaps we could add supplementary clauses to the definition of strict paraconsistency. Besides demanding the rejection of $ECQ_{*} : A, \neg A \vdash s(B)$ (with an exception for theorems), we could require that an adequate paraconsistent logic also reject $ECQ_{*C} : A, \neg A \vdash C * s(B)$, i.e., that its inconsistent theories may be non-sub-trivial, and that it reject all rules similar to $EC_{\neg}Q : \neg A, \neg\neg A \vdash B$, obtainable by imposing further restrictions on the premises of $EC_{\neg}Q$ or some restriction on the conclusion, with an exception for theorems in every case. Let us call the resulting concept of paraconsistency 'extended strict paraconsistency':

Definition 4.3 (Extended Strict Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in an extended strict sense iff for some wff A and B, for every uniform substitution $s(B)$ of B, given a possibly empty well-formed string of symbols C, a sequence of connectives $*$, and two sequences $\neg_n$ and $\neg_m$ of negation symbols such that $n \geq 0$ and $m = n+1$, all instances of the following are not rules of $\mathcal{S}$:

$$EC_{\neg_n}Q_{*C} : \neg_n A, \neg_m A \vdash C * s(B),$$

unless every $C * s(B)$ is a theorem of $\mathcal{S}$.

[16] We refer the reader to the examples of systems that reject ECQ and yet validate $EC_{\neg}Q$ given by Béziau (2000) and Pulcini and Varzi (2018, p. 5491).

In the definition above we have combined all three cases of restrictions of *ECQ* we needed to avoid, namely, ECQ_*, ECQ_{*C}, and all cases similar to $EC_\neg Q$ obtainable by imposing further restrictions on the premises or the conclusion, into a single general rule. In $EC_{\neg_n}Q_{*C} : \neg_n A, \neg_m A \vdash C * s(B)$, if C is empty and n=0, then $EC_{\neg_n}Q_{*C} = ECQ_*$, and $*$ simply determines the syntactic structure of B. If $n > 0$, then we have some version of *ECQ* of the style pointed out by Béziau (2000) and Pulcini and Varzi (2018, p. 5491), i.e., obtained by imposing restrictions on the premises of *ECQ*, and the conclusion may be all uniform substitutions of any wff. If C is not empty, then we have some of the cases we have pointed out above for which the conclusion contains fixed non-logical content, and $*$ also determines the binary connective that links the fixed part of the formula and the non-fixed part.

Now, have we finally found the adequate concept of paraconsistency? It would seem not. Although extended strict paraconsistency applies great improvement on strict paraconsistency, there seems to be no guarantee that one will not come up with some other notion of triviality and some other way to restrict *ECQ* to which extended strict paraconsistency is vulnerable. But beyond this matter of future objections, we also have some concrete worries. Blocking $ECQ_{*C} : A, \neg A \vdash C * s(B)$ seems to make extended strict paraconsistency too restrictive. Indeed, when $C \neq A$, avoiding ECQ_{*C} seems desirable, but it turns out to be questionable when $C = A$. Consider, for instance, the following two examples: $ECQ_{\vee A} : A, \neg A \vdash A \vee s(B)$ and $ECQ_{\rightarrow A} : A, \neg A \vdash s(B) \rightarrow A$. As extended strict paraconsistency requires the rejection of all instances of ECQ_{*C} that are non-theorematic, $ECQ_{\vee A}$ and $ECQ_{\rightarrow A}$ would be rejected too. For that, our paraconsistent systems would have to give up either monotonicity $A \vdash B \Rightarrow A, C \vdash B$ or disjunction introduction $A \vdash A \vee B$ and *Verum Ex Quodlibet* (*VEQ*) $A \vdash B \rightarrow A$. However, each of these rules separately seems to pose no problem to paraconsistency. Besides, $ECQ_{\vee A}$ and $ECQ_{\rightarrow A}$ do not seem to be harmful restrictions of *ECQ*. After all, if we accept $A \vdash A \vee s(B)$ for consistent contexts, why would we then reject $A, \neg A \vdash A \vee s(B)$ when the context is made inconsistent? On the other hand, the solution is not as simple as saying that we must reject ECQ_{*C} for $C \neq A$ and accept it for $C = A$. Not all instances of ECQ_{*A} are desirable. For example, there seems to be no reason to accept $ECQ_{\wedge A} : A, \neg A \vdash A \wedge s(B)$. It seems we are once again in a predicament, and inconsistency has made a fool of us once more.

If we take extended strict paraconsistency to be adequate, despite the argument above that some instances of ECQ_{*C} are acceptable, we ought to determine, for the examples considered above and similar cases, which rule must be rejected in a paraconsistent system in order for it to satisfy the definition of extended strict paraconsistency. In the cases considered above, in order not to have the rules $ECQ_{\vee A} : A, \neg A \vdash A \vee s(B)$ and $ECQ_{\to A} : A, \neg A \vdash s(B) \to A$, our paraconsistent systems must reject either monotonicity or disjunction introduction and *VEQ*, as argued above. In general, any rule that introduces sub-triviality in a consistent context can be transformed into a version of *ECQ* by applying monotonicity.

Urbas (1990) has a clear position regarding the problem presented above: paraconsistent logic should abandon both *VEQ* and monotonicity. Urbas argues that *VEQ* is related to a series of problems for paraconsistency: if a logic contains *VEQ*, contraposition, and *modus ponens*, any inconsistent theory constructed on it contains all negations, which is no other than $ECQ_{\neg} : A, \neg A \vdash \neg B$, and, in particular, one contradiction belonging to an inconsistent theory implies all contradictions belonging to it, for each of the wffs of the theory.[17] Urbas' point is that by letting go of these rules we get rid of arbitrary conclusions not only from inconsistent premises but from premises in general (Urbas, 1990, p. 350), while we can maintain other rules that make connectives well-behaved and symmetric, such as contraposition for negation (Urbas, 1990, pp. 351–353).

Nevertheless, as Urbas (1990, p. 350) himself recognizes, at this point we are no longer discussing paraconsistency requirements, for our discussion has no longer to do with the goal of paraconsistent logic of distinguishing inconsistency and triviality. In Urbas' case, although his arguments against *VEQ* and monotonicity make way for interesting paraconsistent systems, they cannot be taken to define what paraconsistency is and simply exclude systems with *VEQ* and monotonicity from the realm of paraconsistency. Urbas' objections to *VEQ* and monotonicity, as he makes explicit, are based not on a notion of what paraconsistency should be, but rather have utter relevantist motivations (Urbas, 1990, pp. 349–353).[18]

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[17] These arguments are presented in detail in (Urbas, 1990, pp. 352–353).

[18] Similarly, Batens' (1980) concept of 'logically conservative', which also aims to block arbitrary conclusions from any set of premises (except if the conclusion is a theorem or the set of premises is already trivial in the sense in question), would be

How can we distinguish arguments for or against the inclusion of certain rules in paraconsistent systems that have genuine paraconsistent grounds from arguments that, as Urbas', have different motivations? The criterion seems to be the relationship of the argument to the very goal of paraconsistent logic as an underlying system for the construction of non-trivial inconsistent theories. Urbas' argument against *VEQ* and monotonicity does not concern inconsistent contexts in particular, but rather aims to eliminate arbitrary conclusions from any set of premises. Thus, as Urbas' argument does not concern the goal of paraconsistent logic, imposing such restrictions on the definition of paraconsistency is excessive. Another clear example of excessive restrictions can be seen in the case of absolute paraconsistency. The question of whether we should require inconsistent theories to be regular, allow both regular and non-regular inconsistent theories, or require inconsistent theories to be non-regular (as absolute paraconsistency does) is not a matter of paraconsistency. Whatever our motivation is for or against the regularity of theories, such motivations would hold regarding not only the regularity of inconsistent theories but also the regularity of consistent theories. As a general guideline in defining paraconsistency, wherever a trait of the logic is not related to the paraconsistent goal, we will simply follow classical logic.[19]

Perhaps Béziau (2000) was correct, and this approach of defining paraconsistency as a list of negative properties, i.e., a list of all rules a paraconsistent system should not have, is insufficient. At least in the present study, we have found problems even with our ultimate attempt at such a task, the concept of extended strict paraconsistency. According to that concept of paraconsistency, as it is clear that $ECQ_{\vee A} : A, \neg A \vdash A \vee s(B)$ and $ECQ_{\to A} : A, \neg A \vdash s(B) \to A$ introduce some form of triviality, then we should find a way of avoiding those rules given the purpose of paraconsistent logic of ridding inconsistent theories from triviality. However, there seems to be reasonable ground to insist that the kind of triviality discussed here is not harmful from a paraconsistent

too strong a paraconsistency constraint as well. The relevant definitions are given in (Batens, 1980, pp. 201–202).

[19] This suggestion of following classical logic wherever possible to the extent that it does not interfere with paraconsistent motivations is outlined in (Da Costa, 1974, p. 498). Of course, one can construct a paraconsistent system that takes on other non-classical aspects. This, however, has no longer to do with paraconsistency, but with different motivations.

point of view, as it can be produced even in consistent contexts. If that is the case, we cannot dictate beforehand that a paraconsistent logic ought to be non-monotonic or ought not to have *VEQ* and disjunction introduction. We cannot exclude systems containing those rules from the realm of paraconsistency, unless we have reasons tied to the goal of paraconsistent logic to reject them. Instead of trying to come up with a paraconsistent reason to give up either disjunction introduction and *VEQ* or monotonicity, for example, we argue that we should take a closer look at the goal of paraconsistent logic.

The most valuable philosophical lesson that the paraconsistent logician teaches us is that explosive logics collapse two concepts: inconsistency and triviality. Then, the paraconsistent logician shows us how it is possible to set up a formal system in which those concepts are not collapsed and how we are now able to construct and work with a number of inconsistent theories that would otherwise all be collapsed into the trivial theory. We believe that a step toward a solution to our present problem is to advocate one further distinction; another valuable philosophical lesson that paraconsistent logic can teach us. Besides distinguishing inconsistency and triviality and formulating the goal of paraconsistent logic as a logic adequate for the construction of non-trivial inconsistent theories, we must now distinguish inconsistency-related triviality from triviality *tout court* and realize that the actual goal of paraconsistent logic is not the construction of inconsistent theories that are non-trivial in any sense of the word, but only in the sense that triviality is related to inconsistency.

Triviality, quasi-triviality, and sub-triviality are said to be 'inconsistency-related' if a theory possesses all wffs of a given set of formulas because it is inconsistent, i.e., the derivation of all wffs of a given set of formulas requires a contradiction. Otherwise, triviality, quasi-triviality, and sub-triviality will not be said to be inconsistency-related. The definition of inconsistency-related triviality can be rendered as follows:

Definition 4.4 (Inconsistency-related triviality). An inconsistent theory T constructed over the language $\mathcal{L}$ is (quasi-/sub-)trivial in an inconsistency-related sense iff for every wff $C * s(B)$ of $\mathcal{L}$ such that C is a possibly empty well-formed string of symbols of $\mathcal{L}$, $*$ is a sequence of connectives of $\mathcal{L}$ that determines the syntactical structure of $C * s(B)$, and $s(B)$ is obtained by uniform substitution over the wff B of $\mathcal{L}$, the following conditions hold:

(i) $C * s(B) \in T$, and
(ii) for every consistent set of wffs $\Gamma \subseteq T$, there is some $C * s(B)$ such that $C * s(B) \notin \Gamma$.

The idea of the definition above is that the (quasi-/sub-)triviality of an inconsistent theory is due to its inconsistency iff all the wffs that trivialize the theory belong to it but do not belong to any consistent subset of it; otherwise, inconsistency is not the cause of the triviality in question.

The definition above includes the three concepts of triviality we have been working with in this article: triviality as given in Definition 2.2, the case in which C is empty and B is an atom; quasi-triviality as given in Definition 3.1, the case in which C is empty and B may not be an atom; and sub-triviality as given in Definition 4.1, the case in which C is not empty and B may be an atom or not. This way, we can use the definition above to determine whether triviality is inconsistency-related in any of those three senses. For the case of triviality as given in Definition 2.2, our definition of inconsistency-related triviality is straightforward: if T is the trivial theory (i.e., the theory that contains all wffs of $\mathcal{L}$), of course no consistent subset of it will be trivial in the relevant sense, for no proper subset of T will be trivial in this sense. Therefore, triviality *tout court* will always be inconsistency-related. For quasi-triviality and sub-triviality, on the other hand, it is possible to have a trivial theory T in any of these two senses and yet have consistent subsets of T that are trivial as well, especially depending on whether inconsistency is defined as a wff and its negation belonging to a theory or in a weaker sense. When it is possible to find a consistent subset of T that is trivial, triviality will not be inconsistency-related; for other cases, it will. Thus, our definition draws a distinction for these stricter notions of triviality between inconsistecy-related and unrelated cases.

We propose that the actual aim of paraconsistent logic is to rid inconsistent theories of (quasi-/sub-)triviality in the inconsistency-related sense only, not of all kinds of triviality. The triviality introduced by disjunction introduction, i.e., the fact that if for some wff A, $A \in T$, then $A \vee s(B) \in T$ for every uniform substitution $s(B)$ of some wff B of the language of T, for example, holds in consistent and inconsistency theories alike. Therefore, it is not the job of a paraconsistency constraint to block that kind of triviality. It is only reasonable that triviality already present in a consistent theory be imported into an inconsistent extension of that theory.[20] A paraconsistency constraint is supposed to avoid that new forms of triviality come up in an inconsistent theory that were not present in any of its consistent subsets. The task that follows is to find a definition of paraconsistency that adequately expresses its aim.

We can define a concept of paraconsistency satisfying the description above relying on the definition of inconsistency-related triviality, defining paraconsistent logics as those that do not make inconsistent theories trivial in the inconsistency-related sense:

Definition 4.5 (Consistency-Based Paraconsistency). A logic $\mathcal{S}$ is paraconsistent in a consistency-based sense iff for every inconsistent theory T constructed over $\mathcal{S}$ and for every wff $C * s(B)$ of $\mathcal{L}$ such that C is a possibly empty well-formed string of symbols of $\mathcal{L}$, $*$ is a sequence of connectives of $\mathcal{L}$ that determines the syntactical structure of $C * s(B)$, and $s(B)$ is obtained by uniform substitution over the wff B of $\mathcal{L}$, the following condition holds:

- If $C * s(B) \in T$, then for some consistent set $\Gamma \subseteq T$, $C * s(B) \in \Gamma$.

This definition of paraconsistency seems to achieve all that was intended. It maintains the nice features of strict paraconsistency, namely, blocking quasi-triviality more efficiently than standard and strong paraconsistency, and allowing theorems to be derivable from inconsistent premises. It is also an improvement over strict paraconsistency and extended strict paraconsistency, as it blocks the undesired sub-triviality that strict paraconsistency allows while permitting the inconsistency-unrelated cases that extended strict paraconsistency blocks, e.g., the sub-triviality introduced by disjunction introduction and *VEQ*. Besides, it also works as a solution for the difficulty pointed out by Béziau (2000) and Pulcini and Varzi (2018, p. 5491), as consistency-based paraconsistency also blocks undesirable versions of *ECQ* with restricted premises. Finally, this definition of paraconsistency attends Béziau's (2000) requirement that an adequate definition of paraconsistency should present positive criteria besides the usual negative ones.

[20] This idea is already present in Batens' concept of A-destructive (Batens, 1980, pp. 201–202). For Batens, a logic is B-destructive (in our usage of the variables) only if, given a theory T constructed from a set of wffs Γ, for all uniform substitutions $s(B)$ of a wff B, $s(B) \in T$, but, for some $s(B)$, $s(B) \notin \Gamma$. The idea behind this clause is that, if for all $s(B)$ of B, $s(B) \in \Gamma$, then of course for all $s(B)$, $s(B) \in T$, at least if our logic is reflexive, and, in this context, there is nothing *destructive* about a logic in which this happens.

Consistency-based paraconsistency uses consistent theories as a parameter to determine which cases of triviality are inconsistency-related and which are not. If all wffs $C * s(B)$ as specified in Definition 4.5 belong to some consistent set of wffs, then the triviality resulting from all wffs $C * s(B)$ belonging to a theory is not related to inconsistency and should not be barred by paraconsistent logic. On the other hand, if all wffs $C * s(B)$ belong to inconsistent theories only, then the triviality in question is clearly inconsistency-related and is precisely the kind of triviality that paraconsistent logic is supposed to avoid.

It is noteworthy that every consistency-based paraconsistent logic is standard, strong, and strict paraconsistent, for consistency-based paraconsistency avoids triviality and quasi-triviality, i.e., neither *ECQ* nor any form of ECQ_* is allowed, except for the case of theorems. On the other hand, no consistency-based paraconsistent logic is extended strict paraconsistent nor absolute paraconsistent, for it allows some forms of sub-triviality and makes an exception for the case of theorems, e.g., consistent-based paraconsistency allows $ECQ_{\vee A}$ and $ECQ_{\rightarrow A}$ discussed above, and also the restricted versions of ECQ_* such that all uniform substitutions of the conclusion are theorems. We claim that allowing restricted versions of *ECQ* like the ones above as rules of a paraconsistent logic is not a shortcoming but a virtue of consistency-based paraconsistency. The sub-triviality allowed by consistency-based paraconsistency in the cases of disjunction introduction and *VEQ* discussed above, for example, can be seen simply as the result of applying monotonicity to extend rules that are considered reasonable in consistent contexts to inconsistent contexts.

This way, consistency-based paraconsistency solves the difficulty we had discussed above, according to which strict paraconsistency is excessively permissive because it allows inconsistent theories to be sub-trivialized while extended strict paraconsistency is excessively restrictive because it forces paraconsistent logics to give up rules that seem harmless to the paraconsistent goal, like disjunction introduction and *VEQ*. However, there is a second kind of difficulty pointed out by Urbas (1990, pp. 352–353) that remains. As already mentioned above, certain combinations of rules, like *VEQ*, contraposition, and *modus ponens*, make inconsistent theories trivial in a sense inadmissible for paraconsistent logic. Although an adequate paraconsistent system must lack one of the rules of the problematic combinations, we argue that it cannot be determined beforehand, in the very definition of paraconsistency, which one it must lack, as each one of those rules separately is acceptable from a paraconsistent point of view. Thus, our consistency-based paraconsistency only solves the problem for combinations of triviality-introducing rules and monotonicity. For these cases, we realize that an adequate concept of paraconsistency makes it so none of those rules has to be abandoned for the sake of paraconsistency. However, regarding the difficulty pointed out by Urbas (1990, pp. 352–353) for other combinations of rules, we adopt a pluralist approach in the lines described by Béziau (2000), according to which an adequate paraconsistency constraint may allow various incompatible alternatives as genuine paraconsistent systems.[21]

5. Conclusion

In this article, we have discussed the definition of paraconsistent logic. Our parameter for assessing the definitions proposed in the literature was the declared goal of paraconsistent logic as the logic adequate for the construction of non-trivial inconsistent theories. Classical logic and other explosive logics are obviously inadequate for this purpose, which was the motivation for proposing paraconsistent logic in the first place. Despite its great advancement toward allowing the construction of non-trivial inconsistent theories, the shortcomings of standard paraconsistency with relation to this goal have been criticized by a number of scholars working on paraconsistent logic. The list of critics of standard paraconsistency includes Priest (2002, 2007), who nonetheless believes that a stronger definition of paraconsistency cannot be proposed; Robles (2009) and Robles and Méndez (2009), who propose strong paraconsistency instead; and Urbas (1990), who proposes strict paraconsistency instead. We agree with their objections to standard paraconsistency and we agree on important points with the solutions proposed by the latter authors.

Urbas' proposal is the closest to an adequate definition of paraconsistent logic. Strict paraconsistency blocks restricted versions of *ECQ* that would nearly trivialize inconsistent theories more efficiently than

[21] This is also in line with Priest's argument that the permissiveness of a paraconsistency constraint can be justified by its goal of fitting the great variety of logics that are considered paraconsistent (Priest, 2002, p. 288). However, the effort made here and in the work we have discussed is aimed at eliminating logics that do not align with the paraconsistent goal from that great variety of logical systems.

standard or strong paraconsistency. Urbas' article also makes a compelling argument for going beyond paraconsistent logic into relevance logic. However, taken as a paraconsistency constraint, the concept of strict paraconsistency proposed by Urbas, despite its great advantages over other proposals, still presents a number of flaws, some of which have been pointed out by other authors and some of which we hope to have made a convincing case against in the discussion above.

Our contribution to the topic started with the identification of unsatisfactory aspects of strict paraconsistency and the proposal of solutions to its flaws. We have identified some restricted versions of *ECQ* that strict paraconsistency allows and would therefore be a threat to the inconsistent theories constructed over a strict paraconsistent logic. Such restricted versions of *ECQ* can be divided into two groups: the ones related to the sets of all wffs in which some non-logical content is fixed and uniform substitution is allowed over one of its sub-formulas, and the ones pointed by Béziau (2000) and Pulcini and Varzi (2018, p. 5491) related to imposing restrictions over the premises of *ECQ*, which can also be extended to include mixed cases in which some kind of restriction is imposed on both the premises and the conclusion of *ECQ*.

We have also shown that an attempt in the same line of thought as Urbas' at extending strict paraconsistency to block the restricted versions of *ECQ* pointed out above as problematic, which we called 'extented strict paraconsistency', does not work. This time, we obtain a paraconsistency constraint that is too demanding, as it requires the rejection of triviality that can come about even in consistent contexts, also requiring the rejection of the rules that bring about that trivialization, such as disjunction introduction, *VEQ*, and monotonicity. We argued that rules like disjunction introduction, differently from *ECQ*, bring about a kind of trivialization to inconsistent theories that is harmless from a paraconsistent point of view because it applies to consistent theories as well. Therefore, as we take the goal of paraconsistent logic to be the elimination of asymmetries in the trivialization of consistent and inconsistent theories, we argue that eliminating the kind of trivialization that affects consistent and inconsistent theories alike is beyond the scope of a paraconsistency constraint.

From the outset, the critics of standard paraconsistency argued that defining a paraconsistent logic as a non-explosive logic is insufficient. In view of the problems discussed throughout this paper, we argue that defining a paraconsistent logic as a non-partially-explosive logic (theorems exempted) as proposed by strict paraconsistency is also insufficient. Going beyond strict paraconsistency and defining a paraconsistent logic as a non-partially-explosive and non-sub-explosive logic turns out to be too restrictive, for we end up depriving paraconsistent systems of deductive inferences that can be carried out even in consistent contexts. Our attempt at extending strict paraconsistency blocks sub-triviality due to inconsistency and sub-triviality unrelated to inconsistency alike, while the goal of paraconsistent logic should be to avoid the former only. This way, we propose that paraconsistent logic be defined not simply as a non-(partially-/sub-)explosive logic or a logic for non-(quasi-/sub-)trivial inconsistent theories (theorems exempted in all cases) but as a logic that blocks inconsistency-related (quasi-/sub-)triviality, i.e., a logic that does not allow inconsistent theories to be trivial nor quasi-trivial nor sub-trivial in reason of their inconsistency.

Accordingly, we propose that paraconsistent logic be defined in the consistency-based sense: only those inference rules that either do not introduce (quasi-/sub-)triviality or introduce (quasi-/sub-)triviality in consistent and inconsistent theories alike can be rules of a paraconsistent system, excluding those rules that introduce (quasi-/sub-)triviality in inconsistent theories only, such as *ECQ* and most of its restricted versions. Therefore, from standard paraconsistency to consistency-based paraconsistency, the goal of paraconsistent logic should be reformulated from ‘a logic for the construction of non-trivial inconsistent theories’ to ‘a logic for the construction of inconsistent theories that are non-trivial in the inconsistency-related sense’.

Our discussion of strengthening standard paraconsistency focused on how restrictive a paraconsistency constraint is supposed to be at the inferential level. Recent studies on paraconsistent logic explore the extension of standard paraconsistency to the metainferential level.[22] Barrio et al. (2018, p. 95) notice that metainferential standard paraconsistency is vulnerable to partial explosion and suggest that it would be desirable to eliminate partial-explosion following Urbas’ (1990) considerations. This way, it would be interesting to extend the discussion presented in this paper about strengthening standard paraconsistency at the inferential level to the metainferential level. Another interesting follow-up on this study would be to verify which well-established paraconsistent systems

[22] On the topic of metainferential explosion and metainferential paraconsistency, see (Barrio et al., 2018), (Da Ré et al., 2022), and (Pailos and Da Ré, 2023, ch. 10.3).

are consistency-based paraconsistent, considering that this investigation was focused on the adequacy of definitions of paraconsistency only, without examining particular systems. Finally, strict paraconsistency and consistency-based paraconsistency could be defined using weaker concepts of inconsistency, such as those considered by Robles and Méndez (2009), given that in this occasion we worked only with inconsistency as contradiction.

As discussed throughout this article, the goal of paraconsistent logic as described by Da Costa (1974) and Priest et al. (2022) is not necessarily met by a logic that satisfies the definition of standard paraconsistency. Urbas' (1990) and Robles and Méndez's (2009) works furnish us with the required tools (the concepts of partial explosion and quasi-triviality) and establish some of the conditions that must be met for a logic to prevent the trivialization (in a broad sense) of inconsistent theories. As we have seen above, Urbas' strict paraconsistency comes quite close to the goal. If the reader finds our criticism and proposal adequate, namely, our criticism of strict paraconsistency, the reformulation of the goal of paraconsistent logic, and the definition of consistency-based paraconsistency, not as the final word on the topic of defining paraconsistency, but as a discussion of the guidelines for an adequate definition, then we might have given a step further toward a definition of paraconsistent logic in the intended sense: a logic truly adequate for the construction of non-trivial inconsistent theories.

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