Logic and **L**ogical **P**hilosophy
Online First Articles (2024)
DOI: 10.12775/LLP.2024.001

Heinrich Wansing

Constructive Logic is Connexive and Contradictory

Abstract. It is widely accepted that there is a clear sense in which the first-order paraconsistent constructive logic with strong negation of Almukdad and Nelson, **QN4**, is more constructive than intuitionistic first-order logic, **QInt**. While **QInt** and **QN4** both possess the disjunction property and the existence property as characteristics of constructiveness (or constructivity), **QInt** lacks certain features of constructiveness enjoyed by **QN4**, namely the constructible falsity property and the dual of the existence property.

This paper deals with the constructiveness of the contra-classical, connexive, paraconsistent, and contradictory non-trivial first-order logic **QC**, which is a connexive variant of **QN4**. It is shown that there is a sense in which **QC** is even more constructive than **QN4**. The argument focuses on a problem that is mirror-inverted to Raymond Smullyan's *drinker paradox*, namely the invalidity of what will be called *the drinker truism* and its dual in **QN4** (and **QInt**), and on a version of the Brouwer-Heyting-Kolmogorov interpretation of the logical operations that treats proofs and disproofs on a par. The validity of the drinker truism and its dual together with the greater constructiveness of **QC** in comparison to **QN4** may serve as further motivation for the study of connexive logics and suggests that constructive logic is connexive and contradictory (the latter understood as being negation inconsistent).

Keywords: constructive logic; connexive logic; contradictory logics; Drinker principle; Drinker truism; Brouwer-Heyting-Kolmogorov interpretation

1. Introduction

The drinker principle and its dual are bewildering sentences that are valid in classical first-order logic, **QCL**, and that are invalid in both intuitionistic first-order logic, **QInt**, and in the first-order constructive

Special Issue on Paraconsistency. Guest Editors: D. Fazio, T. Jarmużek, J. Malinowski
Received January 14, 2023. *Accepted* September 22, 2023. *Published online* January 2, 2024

paraconsistent logic with strong negation from Almukdad and Nelson [1], **QN4**.[1]

The drinker principle and its dual are invalid in **QC**, the connexive version of **QN4**, as well. Unlike the drinker principle and its dual, two other sentences, the drinker truism and its dual, may be seen as plausible and appealing principles. Here the problem is that **QCL**, **QInt**, and **QN4** fail to validate these theses. However, the drinker truism and its dual are not only valid in **QC** but can also be justified from the point of view of an improved version of the Brouwer-Heyting-Kolmogorov (BHK) interpretation of the logical operations. This interpretation may be called *the connexive López-Escobar interpretation*. I interpret these observations as showing that there is a sense in which the connexive logic **QC** is more constructive than **QInt** and **QN4**. Since it is the connexivity (actually, the "hyperconnexivity") of the constructive logic **QC**[2] that enables the validation of the drinker truism and its dual, the present study suggests that constructive logic is connexive and hence contra-classical. The step from **QInt** to **QN4** secures the constructible falsity property and the dual of the existence property as indicators of constructiveness, and the step from **QN4** to **QC** delivers the drinker truism and its dual as valid. In addition to providing further motivation for the study of connexive logics, the greater constructivity of the non-trivial negation inconsistent logic **QC** in comparison to the constructiveness of **QInt** and **QN4** suggests that constructive logic is not only connexive but also contradictory.

2. The first-order logics QInt, QInt$^+$, QCL, QN4, and QC

Since it suffices for present purposes, we use a simple first-order language $\mathcal{L}$ without function symbols and identity predicate. The set of terms contains denumerably many individual variables and for every individual **c** from a given, non-empty set of individuals, it contains an individual constant c. We use the letters x and y for arbitrary individual variables, the letter a for an arbitrary constant, and the letter t for an arbitrary term. Moreover, the vocabulary of $\mathcal{L}$ comprises the connectives $\sim$ (unary), $\wedge$ (binary), $\vee$ (binary), $\rightarrow$ (binary), the universal and particular quantifiers

[1] A natural deduction proof system for **QN4** can be found, in [20, p. 96] and a sequent calculus for **QN4** is given in [1] and, independently, already in [24]. As far as the present author recalls, the name '**N4**' was introduced in [26].

[2] For surveys on connexive logic see [14, 29].

$\forall$ and $\exists$, for every $n > 0$ denumerably many n-place predicate symbols, and brackets, (and). The set of atomic $\mathcal{L}$-formulas and the set of all $\mathcal{L}$-formulas are recursively defined as usual, and the binary connective $\leftrightarrow$ is defined by $(A \leftrightarrow B) := ((A \to B) \wedge (B \to A))$. A literal is an atomic formula or a negated atomic formula. We use P and Q for arbitrary predicate symbols (of an appropriate arity) and the letters A, B, C, D, A_1, A_2 for arbitrary formulas. The notions of a free variable, a bound variable, and a term t being free (for replacement with) a variable in a formula are defined as usual. We will often omit outermost brackets of $\mathcal{L}$-formulas and we write $A(t/x)$ for the result of replacing all free occurrences of x in A by t. The language $\mathcal{L}^+$ is obtained from $\mathcal{L}$ by omitting the negation symbol, $\sim$.

We first present the predicate logics **QInt**, **QInt**$^+$, **QCL**, **QN4**, and **QC** as axiomatic proof systems **HQInt**, **HQCL**, **HQN4**, and **HQC**.

DEFINITION 1. The schematic axioms and rules of **HQInt** are:

a1	$A \to (B \to A)$
a2	$(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
a3	$A \to (B \to (A \wedge B))$
a4	$(A \wedge B) \to A$
a5	$(A \wedge B) \to B$
a6	$A \to (A \vee B)$
a7	$B \to (A \vee B)$
a8	$(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
a9	$(A \to B) \to ((A \to \sim B) \to \sim A)$
a10	$\sim A \to (A \to B)$
a11	$A(t) \to \exists x A(x)$ (t is free for x in A)
a12	$\forall x A(x) \to A(t)$ (t is free for x in A)
r1	$\dfrac{A \quad A \to B}{B}$
r2	$\dfrac{A \to B(x)}{A \to \forall x B(x)}$ (x not free in A)
r3	$\dfrac{A(x) \to B}{\exists x A(x) \to B}$ (x not free in B)

If axioms a9 and a10 are omitted, one obtains an axiomatization **HQInt**$^+$ of first-order positive intuitionistic logic, **QInt**$^+$, in the language $\mathcal{L}^+$.

DEFINITION 2. The system **HQCL** is obtained from **HQInt** by adding $\sim\sim A \to A$ as further axiom.

DEFINITION 3. The schematic axioms and rules of the paraconsistent calculus **HQN4** are a1–a8, i.e., the axioms of first-order positive intuitionistic logic, r1–r3, and

a13	$\sim\sim A \leftrightarrow A$
a14	$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
a15	$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
a16	$\sim(A \to B) \leftrightarrow (A \wedge \sim B)$
a17	$\sim\exists x A \leftrightarrow \forall x \sim A$
a18	$\sim\forall x A \leftrightarrow \exists x \sim A.$

DEFINITION 4. The schematic axioms and rules of **HQC** are those of **HQN4** except that a16 is replaced by

a19 $\sim(A \to B) \leftrightarrow (A \to \sim B).$

DEFINITION 5. Let $\Delta \cup A$ be a set of formulas and $\lambda \in \{\mathbf{HQInt}, \mathbf{HQInt}^+, \mathbf{HQCL}, \mathbf{HQN4}, \mathbf{HQC}\}$. We write $\Delta \vdash_\lambda A$ and say that A is derivable from Δ in λ iff there is a sequence of formulas $\langle A_1, \cdots, A_n, A\rangle$ with $n \geqslant 0$ and such that every formula in the sequence either belongs to Δ, is an axiom of λ, or is obtained from formulas preceding it in the sequence by means of one of the rules $r1$–$r3$.

If it is clear from the context which $\lambda \in \{\mathbf{HQInt}, \mathbf{HQInt}^+, \mathbf{HQCL}, \mathbf{HQN4}, \mathbf{HQC}\}$ is meant, we simply write $\Delta \vdash A$ instead of $\Delta \vdash_\lambda A$. For a later purpose, we define the weight of an $\mathcal{L}$-formula.

DEFINITION 6. The weight $w(A)$ of an $\mathcal{L}$-formula A is inductively defined as follows:

$$\begin{array}{rcll}
w(l) & = & 0 & \text{for literals } l \\
w(A\sharp B) & = & w(A) + w(B) + 1 & \text{for } \sharp \in \{\wedge, \vee, \to\} \\
w(\sharp x A) & = & w(A(a/x)) + 1 & \text{for } \sharp \in \{\forall, \exists\}, \text{ variables } x, \\
 & & & \text{and constants } a \\
w(\sim\sim A) & = & w(A) + 1 & \\
w(\sim(A \to B)) & = & w(A) + w(\sim B) + 2 & \\
w(\sim(A\sharp B)) & = & w(\sim A) + w(\sim B) + 2 & \text{for } \sharp \in \{\wedge, \vee\} \\
w(\sim\sharp x A) & = & w(\sim A(a/x)) + 1 & \text{for } \sharp \in \{\forall, \exists\}, \text{ variables } x, \\
 & & & \text{and constants } a.
\end{array}$$

The logics **QInt**, **QN4**, and **QC** each have a Kripke semantics [cf. 5, 17, 27, 18], which we need not consider for present purposes, and Gentzen-style proof systems. The Kripke semantics with respect to which the paraconsistent logics **QN4** and **QC** are complete is four-valued, allowing for truth value gaps and gluts. Because of its user-friendliness we here consider the cut-free G3-style sequent calculus **G3C** for **QC** from [18], which for uniformity of notation we call "**G3QC**." We use uppercase Greek letters to stand for finite, possibly empty multisets of formulas, A, Γ to stand for $\{A\} \Cup \Gamma$, and Δ, Γ to stand for $\Delta \Cup \Gamma$, where $\Cup$ is multiset union. Sequents are of the form $\Gamma \Rightarrow A$, and $\varnothing$ stands for the empty multiset.

DEFINITION 7. The rules of the calculus **G3QC** are the following:

Logical axioms:

$P, \Gamma \Rightarrow P \qquad {\sim}P, \Gamma \Rightarrow {\sim}P$, for atomic formulas P

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow C}{(A \wedge B), \Gamma \Rightarrow C}\ L\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \wedge B)}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{(A \vee B), \Gamma \Rightarrow C}\ L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow (A \vee B)}\ R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow (A \vee B)}\ R\vee_2$$

$$\frac{(A \to B), \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{(A \to B), \Gamma \Rightarrow C}\ L{\to} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \to B)}\ R{\to}$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow B}{\forall x A, \Gamma \Rightarrow B}\ L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}\ R\forall \qquad \frac{A(y/x), \Gamma \Rightarrow B}{\exists x A, \Gamma \Rightarrow B}\ L\exists$$

$$\frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}\ R\exists \qquad \frac{A, \Gamma \Rightarrow C}{{\sim}{\sim}A, \Gamma \Rightarrow C}\ L{\sim}{\sim} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow {\sim}{\sim}A}\ R{\sim}{\sim}$$

$$\frac{{\sim}A, {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \vee B), \Gamma \Rightarrow C}\ L{\sim}\vee \qquad \frac{\Gamma \Rightarrow {\sim}A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \vee B)}\ R{\sim}\vee$$

$$\frac{{\sim}A, \Gamma \Rightarrow C \quad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \wedge B), \Gamma \Rightarrow C}\ L{\sim}\wedge$$

$$\frac{\Gamma \Rightarrow {\sim}A}{\Gamma \Rightarrow {\sim}(A \wedge B)}\ R{\sim}\wedge_1 \qquad \frac{\Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \wedge B)}\ R{\sim}\wedge_2$$

$$\frac{{\sim}(A \to B), \Gamma \Rightarrow A \quad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ L{\sim}{\to} \qquad \frac{A, \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}\ R{\sim}{\to}$$

$$\frac{{\sim}A(y/x), \Gamma \Rightarrow B}{{\sim}\forall x A, \Gamma \Rightarrow B}\ L{\sim}\forall \qquad \frac{\Gamma \Rightarrow {\sim}A(t/x)}{\Gamma \Rightarrow {\sim}\forall x A}\ R{\sim}\forall$$

$$\frac{{\sim}A(t/x), {\sim}\exists x A, \Gamma \Rightarrow B}{{\sim}\exists x A, \Gamma \Rightarrow B}\ L{\sim}\exists \qquad \frac{\Gamma \Rightarrow {\sim}A(y/x)}{\Gamma \Rightarrow {\sim}\exists x A}\ R{\sim}\exists$$

where (i) in $R\forall$ and in $R{\sim}\exists$, y must not occur free in $\Gamma, \forall x A$, respectively in $\Gamma, {\sim}\exists x A$ and (ii) in $L\exists$ and in $L{\sim}\forall$, y must not occur free in $\exists x A, \Gamma, B$, respectively in ${\sim}\forall x A, \Gamma, B$.

DEFINITION 8. If the axioms and rules displaying negation are removed from **G3QC**, this results in a G3-style sequent calculus **G3QInt**$^+$ for **QInt**$^+$. If the rules $L{\sim}{\to}$ and $R{\sim}{\to}$ in **G3QC** are replaced by the rules

$$\frac{A, {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\ L{\sim}{\to}' \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}\ R{\sim}{\to}'$$

one obtains a G3-style sequent calculus **G3QN4** for **QN4**.

As other G3-style calculi, **G3QC** and **G3QN4** allow one to prove cut-admissibility.

THEOREM. *The rule*

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow B}\ Cut$$

is (i) *an admissible rule of* **G3QC** *and* (ii) *an admissible rule of* **G3QN4**.

PROOF. For (i) see [18]; (ii) can be shown analogously. ⊣

COROLLARY. 1. *Cut is an admissible rule of* **G3QInt**$^+$.
2. **QC** *is a conservative extension of* **QInt**$^+$.

That is, to the extent to which **G3QInt**$^+$ captures it, constructive derivability is encapsulated in **G3QC**. Both **G3QN4** and **G3QC** are faithfully embeddable into **G3QInt**$^+$ under straightforward and slightly different translations that reflect the difference between the axioms a16 and a19, which can be seen as confirming that **QN4** and **QC** in different ways capture the constructive core of **QInt**.

The logic **QC** is negation inconsistent already in its propositional fragment. Here is a derivation in **G3QC** of one example of a provable contradiction, where we assume that A is a unary predicate letter:

$$\dfrac{\dfrac{\dfrac{A(x), {\sim}A(x) \Rightarrow {\sim}A(x)}{A(x) \Rightarrow {\sim}({\sim}A(x) \to A(x))}\,R{\sim}{\to}}{\varnothing \Rightarrow {\sim}(A(x) \to ({\sim}A(x) \to A(x)))}\,R{\sim}{\to}}{\varnothing \Rightarrow {\sim}\forall x(A(x) \to ({\sim}A(x) \to A(x)))}\,R{\sim}\forall \qquad \dfrac{\dfrac{\dfrac{A(x), {\sim}A(x) \Rightarrow A(x)}{A(x) \Rightarrow {\sim}A(x) \to A(x)}\,R{\to}}{\varnothing \Rightarrow A(x) \to ({\sim}A(x) \to A(x))}\,R{\to}}{\varnothing \Rightarrow \forall x(A(x) \to ({\sim}A(x) \to A(x)))}\,R\forall$$

3. The drinker principle, the drinker truism, and their duals

In [21], Raymond Smullyan presented various "logical curiosities." One is the so-called "drinker principle," a principle that according to Smullyan at first sight seems downright crazy but turns out to be valid in classical logic. Smullyan explained the name "drinker principle" by remarking that in front of students he usually prefaced its study with the following story:

> A man was at a bar. He suddenly slammed down his fist and said, "Gimme a drink, and give everyone elsch a drink, caush when I drink, everybody drinksh!" So drinks were happily passed around the house. Some time later, the man said, "Gimme another drink, and give everyone elsch another drink, caush when I take another drink, everyone takesch another drink!" So, second drinks were happily passed around the house. Soon after, the man slammed some money on the counter and said, "And when I pay, everybody paysh!" [21, p. 209]

The puzzling question is "Does there really exist someone such that if he drinks, everybody drinks?". Upon translation into the language of first-order logic, the question is whether

$$\exists x(P(x) \to \forall y P(y)) \tag{$\mathbb{DP}$}$$

is valid. The fact that classical logic validates $\mathbb{DP}$ is often referred to as "the drinker paradox." Smullyan also mentions a no less at first sight downright crazy dual version of the drinker principle, $\mathbb{DDP}$, namely, "there is someone such that if anybody at all drinks, then he does," i.e.,

$$\exists x(\exists y P(y) \to P(x)). \tag{$\mathbb{DDP}$}$$

The *dual drinker principle* $\mathbb{DDP}$, using the predicate letter A instead of P, is called "Plato's Law" by E. Beth in [2, p. 18], who paraphrases it as "There is an object with the peculiarity that, if there exists one object with the property A, then the first object has to possess the property A." Note that Beth does not give any justification for calling $\mathbb{DDP}$ "Plato's Law."[3] Since $\mathbb{DDP}$ is valid in classical first-order logic, too, a defender of classical logic faces a dual drinker paradox. Independent from Smullyan's discussion of the drinker paradox and its dual and independent from

[3] An anonymous reviewer remarked that she or he would guess that Beth's calling $\mathbb{DDP}$ "Plato's law" is in reference to the theory of the forms, where the existing drinker is the form of all drinkers.

Beth's discussion of "Plato's Law," both principles have been studied in first-order superintuitionistic logic [see 4, 12, 15, 19]. The drinker paradox is called "Wel_2" and the dual drinker paradox "Wel_1" in [19]. The drinker paradox is called "F" in [12] and [15], and the dual drinker paradox is called "G" in [15].[4] Smullyan is mentioned in [32], but neither [4] nor [12], [15], or [19] are referred to there.

Whilst the validity of $\mathbb{DP}$ and $\mathbb{DDP}$ in classical logic is bewildering (paradoxical), there are some principles the invalidity of which in **QN4** (as well as **QInt** and **QCL**) can be seen as problematic. I will call these principles "the drinker truism" and "the dual drinker truism":

$$\sim\exists x(P(x) \to \sim\exists y P(y)) \tag{$\mathbb{DT}$}$$

("It is false that there is someone such that if she drinks, then it is false that someone drinks.")

$$\sim\exists x(\forall y P(y) \to \sim P(x)). \tag{$\mathbb{DDT}$}$$

("It is false that there is someone such that if everybody drinks, it is false that she drinks.")

The two formulas $\mathbb{DT}$ and $\mathbb{DDT}$ are valid in **QC** and have simple derivations in **G3QC**:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(z) \Rightarrow P(z)}{P(z) \Rightarrow \exists y P(y)}\,R\exists}{P(z) \Rightarrow {\sim}{\sim}\exists y P(y)}\,R{\sim}{\sim}}{\varnothing \Rightarrow {\sim}(P(z) \to {\sim}\exists y P(y))}\,R{\sim}{\to}}{\varnothing \Rightarrow {\sim}\exists x(P(x) \to {\sim}\exists y P(y))}\,R{\sim}\exists \qquad \dfrac{\dfrac{\dfrac{\dfrac{P(z), \forall y P(y) \Rightarrow P(z)}{\forall y P(y) \Rightarrow P(z)}\,L\forall}{\forall y P(y) \Rightarrow {\sim}{\sim}P(z)}\,R{\sim}{\sim}}{\varnothing \Rightarrow {\sim}(\forall y P(y) \to {\sim}P(z))}\,R{\sim}{\to}}{\varnothing \Rightarrow {\sim}\exists x(\forall y P(y) \to {\sim}P(x))}\,R{\sim}\exists$$

The provability of $\mathbb{DT}$ and $\mathbb{DDT}$ in **HQC** is guaranteed by axiom a19. With it, $\mathbb{DT}$ is provably interderivable in **HQC** with $\forall x(P(x) \to \exists y P(y))$, and $\mathbb{DDT}$ is provably interderivable in **HQC** with $\forall x(\forall y P(y) \to P(x))$.

[4] It was shown by Casari [4] that Wel_1, the dual drinker paradox, is valid on any first-order Kripke frame with constant domain if the frame's partial order is a well-order (i.e., has no infinite descending chains) and that Wel_2, the drinker paradox, is valid on any first-order Kripke frame with constant domain if the frame's partial order is a dual well-order (i.e., has no infinite ascending chains). Komori [12] showed that the logic LF defined as the addition of F, i.e., the drinker paradox, to first-order intuitionistic logic has no characteristic Kripke model but enjoys the disjunction property. Nakamura [15] proved that LF has a stronger disjunction property, called "HD" after Harrop, and that the logic LG defined as the addition of G, i.e., the dual drinker paradox, to first-order intuitionistic logic has no characteristic Kripke model but enjoys the HD property and hence the disjunction property. Moreover, he showed that both LF and LG lack the existence property.

If the negated existential $\sim\exists x P(x)$ is read as "No one drinks", it seems that $\mathbb{DT}$ indeed ought to be valid: No one is such that if she drinks, no one drinks. If that is conceded, rendering "No one is such that if she drinks, no one drinks" as $\mathbb{DT}$ causes a problem for **QCL**, **QInt**, and **QN4**. Regarding $\mathbb{DDT}$, the problem is that "No one is such that she doesn't drink if everybody drinks" appears to be valid.

For **QC** (but also for **QN4**) rendering "No one is such that if she drinks, no one drinks" as $\mathbb{DT}$ is confronted with another problem. A state s from a Kripke model for **QC** (or **QN4**) supports the truth of $\sim\exists x P(x)$ iff for every state t above s in the model and every object **a** from the domain of t, the state t supports the falsity of $P(a)$ [see, e.g., 27]. Receiving support of falsity does, however, not mean failure of receiving support of truth, so it may be questionable whether $\sim\exists x P(x)$ expresses "No one drinks" in **QC**.

Can $\mathbb{DT}$ and $\mathbb{DDT}$ be justified by themselves? At least we do not need the full power of axiom a19 to obtain a justification of $\mathbb{DT}$ and $\mathbb{DDT}$. In **QC** (as in **QCL**, **QInt**, and **QN4**), $\sim\exists x A(x)$ is logically equivalent with $\forall x{\sim}A(x)$. Semantically speaking, is it plausible to assume that for any object **a**, $\sim(P(a)\rightarrow\sim\exists y P(y))$ is valid? The Kripke semantics for the logics **QN4** and **QC** suggests to think of logics as theories of information flow [cf. 28]. If a state from a Kripke model for **QN4** or **QC** supports the truth of a conditional $A \rightarrow B$, this can be seen to mean that the state supports the truth of the claim that the information that A is true provides the information that B is true. Moreover, support of truth and support of falsity are taken to be two separate notions in their own right. Neither does support of truth exclude support of falsehood, nor vice versa. It then seems plausible to assume that any state not just fails to support the truth but actually supports the falsity of the claim that the information that $P(a)$ is true provides the information that $\sim P(a)$ is true, i.e., that $P(a)$ is false. Since $\sim\exists y P(y)$ entails $\sim P(a)$ for any object **a**, it seems plausible to assume that for any object **a**, $\sim(P(a)\rightarrow\sim\exists y P(y))$ is valid.

In the Kripke semantics for the logics **QN4** and **QC**, for a given elementary property $P(x)$ it is not precluded that there is a state, say s, which for some (or even any) entity **a** of a given domain of individuals provides contradictory information speaking in favour of $P(a)$ being true and $P(a)$ being false, i.e., the state s supports both the truth and the falsity of $P(a)$. Although such a situation is not precluded, $\mathbb{DT}$ and $\mathbb{DDT}$ retain their plausibility. The meaning of the constructive conditional and the strong negation is such that $\mathbb{DT}$ and $\mathbb{DDT}$ are valid, so that even the considered state s supports the truth of $\mathbb{DT}$ and $\mathbb{DDT}$ (if validity is defined as support of truth at any state). Similarly, at the propositional level, the validity of $(P(a) \vee Q(a)) \to \sim(\sim P(a) \wedge \sim Q(a))$, for example, is unproblematic. Any state s supports the truth of the latter formula even if s is a state that supports the truth of $P(a)$, $Q(a)$, $\sim P(a)$, and $\sim Q(a)$ for every object $\mathbf{a}$ from the domain under consideration.

Further justification for $\mathbb{DT}$ and $\mathbb{DDT}$ emerges from the point of view of a certain proof/disproof interpretation of the logical operations that improves on the BHK-interpretation, see Section 5.

4. Constructiveness

4.1. Characteristics of constructiveness

An uncontroversial characteristic of constructiveness of a logic based on $\mathcal{L}$ is the invalidity and unprovability of the Law of Excluded Middle (LEM), $A \vee \sim A$. In **QInt** the LEM is interderivable with Double Negation Elimination (DNE), $\sim\sim A \to A$ which may be seen as a problematic feature of **QInt**, if the DNE could be given a justification that is acceptable from a constructive point of view for a convincing notion of negation different from intuitionistic negation. Moreover, there are two other properties of **QInt** that have been put forward as indicating that **QInt** formulated in $\mathcal{L}$ is a system of constructive logic, namely the disjunction property and the existence property, that both fail for **QCL** [see 5, Section 5.4]:

if $\vdash A \vee B$ then ($\vdash A$ or $\vdash B$) (Disjunction property)

if $\vdash \exists x A(x)$ then $\vdash A(c/x)$ for some constant c (Existence property)

where $\{A \vee B, \exists x A(x)\}$ is a set of closed formulas.[5]

David Nelson seems to have been the first to suggest that the disjunction property and the existence property ought to be complemented by their duals, the constructible falsity property and the dual existence property. For **QN4** and for **QC** these properties hold:

if $\vdash \sim(A \wedge B)$ then ($\vdash \sim A$ or $\vdash \sim B$) (Constructible falsity property)

[5] Van Dalen [5, p. 174] points out that the proofs of the disjunction property and the existence property that he presents are carried out in a non-constructive classical meta-theory. Whether the proofs of the salient results about **QC** can be given in **QC** as meta-theory is a topic for another study.

if $\vdash \sim\forall x A(x)$ then $\vdash \sim A(c/x)$ for some constant c
(Dual existence property)

where $\{\sim(A \wedge B), \sim\forall x A(x)\}$ is a set of closed formulas.

Whereas the provability of the LEM blocks the disjunction property for **QCL**, the provability of the Law of Non-Contradiction (LNC), $\sim(A \wedge \sim A)$, blocks the constructible falsity property for **QInt**.

4.2. The Brouwer-Heyting-Kolmogorov interpretation

Often, intuitionistic logic is identified with constructive logic. Troelstra and van Dalen [22, p. 9], for example, in "discussing pure logic ... treat "constructive" and "intuitionistic" as synonymous." The constructive understanding of the logical operators occurring in $\mathcal{L}$-formulas is then usually explained in term of the so-called *Brouwer-Heyting-Kolmogorov interpretation* (BHK-interpretation) of the logical operations. In the literature, one can find various for present purposes insignificantly different versions of the BHK-interpretation. In any case the BHK-interpretation comes as a recursive definition of the notion "π is a proof of $\mathcal{L}$-formula A." We identify a proof of an open formula $A(x)$ with a proof of $\forall x A(x)$, and for ease of later extension by refutation clauses suggested by Edgar López-Escobar, we will work with the BHK-interpretation as presented by Jean-Yves Girard [9, p. 5f.], who believes that "Heyting's *semantics of proofs*" is "[o]ne of the greatest ideas in logic" [9, p. 71]. With an adjustment of notation and a slight modification of the clauses for quantified formulas [cf. 5, p. 154], Girard's version of the BHK-interpretation is as follows:

1. For atomic sentences, we assume that we know intrinsically what a proof is; for example, pencil and paper calculation serves as a proof of "$27 \times 37 = 999$".
2. A proof of $A \wedge B$ is a pair $\langle \pi_1, \pi_2 \rangle$ consisting of a proof π_1 of A and a proof π_2 of B.
3. A proof of $A \vee B$ is a pair $\langle i, \pi \rangle$ with:
   - $i = 0$, and π is a proof of A, or
   - $i = 1$, and π is a proof of B.
4. A proof of $A \rightarrow B$ is a function f, which maps each proof π of A to a proof $f(\pi)$ of B.
5. In general, the negation $\sim A$ is treated as $A \rightarrow \bot$ where $\bot$ is a sentence with no possible proof.

6. A proof of $\forall x A$ is a function f, which maps each point $\mathbf{a}$ of the domain of definition to a proof $f(\mathbf{a})$ of $A(a/x)$.
7. A proof of $\exists x A$ is a pair $\langle \mathbf{a}, \pi \rangle$ where $\mathbf{a}$ is a point of the domain of definition and π is a proof of $A(a/x)$.

Some comments on the first clause of the above version of the BHK-interpretation are in place. The example of an elementary statement that is given is a statement from number theory and contains a function symbol and the identity predicate that are not present in the vocabulary of our first-order language $\mathcal{L}$. Elementary sentences provable in some mathematical or empirical theory are usually considered to be necessarily true against the background of that theory. In general, a proof of an elementary closed $\mathcal{L}$-formula can be understood in a much weaker sense, namely as providing evidence speaking in favour of the formula's truth. The proof would "tell the formula true." It is a construction that supports the truth of the formula and does not necessarily show that the formula is true or even necessarily true. We will assume that proofs of atomic sentences are atomic, non-compound proofs. Van Dalen [5, p. 154] gives another example and explains that "a proof of '2 + 3 = 5' consists of the successive constructions of 2, 3 and 5, followed by a construction that adds 2 and 3, followed by a construction that compares the outcome of this addition and 5." Even if atomic proofs involve the construction of numbers and constructions that multiply or add some constructed numbers, these constructions of numbers are different from constructions that amount to proofs of statements, and the comparison of numbers can be seen as an elementary activity.

In the presentation of the BHK-interpretation in [5, p. 154], van Dalen writes that "[i]n order to deal with the quantifiers we assume that some domain D of objects is given." It is not explicitly said that the domain is a domain of constructed elements that grows as the result of presenting proofs over time. Troelstra and van Dalen [22, p. 9] point out that the BHK-interpretation "is quite informal and rests itself on our understanding of the notion of construction and implicitly, the notion of mapping; it is not hard to show that, on a very "classical" interpretation of construction and mapping" the BHK-clauses "justify the principles of two valued (classical) logic." Moreover, they note that the notion of absurdity, $\bot$, is to be regarded as an unexplained primitive notion, that any mapping whatsoever may count as a proof of $\bot \to A$, and that $\bot \to A$ has been rejected by Ingebrigt Johansson [11] as a non-constructive principle.

According to Troelstra and van Dalen, the BHK-clauses nevertheless "suffice to show that certain logical principles should be generally acceptable from a constructive point of view, while some other principles from classical logic are not" [22, p. 10]. Notably, the examples given usually involve negation, one example of a principle that is highly implausible given the BHK-interpretation is the Law of Excluded Middle. The Law of Excluded Middle cannot be strictly refuted on the basis of the BHK-Interpretation, but the interpretation allows one to give a "weak counterexample." A proof of $A \vee {\sim}A$ for any A would amount to a universal method for obtaining either a proof of A or a proof of ${\sim}A$, while there is no reason to assume that such a method exists.

According to Buss, [3, p. 65], it is not difficult to see that the sequent calculus LJ for intuitionistic first-order logic **HQInt** is sound under the BHK interpretation, insofar that any formula provable in it has a proof in the sense of the BHK interpretation. Since the set of $\mathcal{L}$-formulas provable in LJ and **HQInt** coincide, **HQInt** is sound under the BHK interpretation as well. Given certain concerns about how negation is treated in the BHK interpretation, we restrict our attention to $\mathbf{HQInt}^+$ in the language $\mathcal{L}^+$. If we think of derivations from assumptions, a proof of A from premises $A_1, \ldots, A_n$, then is a function f such that if it is applied to proofs π_1, $\ldots$, π_n of $A_1, \ldots, A_n$, one obtains a proof $f(\pi_1, \ldots, \pi_n)$ of A. If such a function exists, the inference from $A_1, \ldots, A_n$ to A is regarded by the intuitionists as constructively valid, and a formula for which the BHK-interpretation ensures the existence of a proof is seen as constructively acceptable and justified.

FACT 1 (Soundness of $\mathbf{HQInt}^+$ w.r.t the BHK-interpretation).
If $\{A_1, \ldots, A_n\} \vdash_{\mathbf{HQInt}^+} A$ *then there exists a function* f *such that for any proofs* $\pi_1, \ldots, \pi_n$ *of* $A_1, \ldots, A_n$, $f(\pi_1, \ldots, \pi_n)$ *is a proof of* A.

PROOF. It is enough to show that there exist proofs of axioms a1–a8, a11, and a12, and that the rules r1–r3 preserve the existence of proofs. $\dashv$

4.3. The López-Escobar interpretation

The BHK-interpretation can be and has been criticized, however, for its treatment of negation as "implies absurdity" not only by Johansson. Edgar López-Escobar reasons as follows [see also 25]:

> For example, if one accepts that there is no construction that proves an absurdity (as do most people) then a salient property of the construction π that proves "not-A" is that when π is applied to a particular non-existent construction (namely a proof of A) it yields another non-existent construction! [13, p. 362f.]

By the BHK-clause for negation, a proof of the intuitionistically valid $\sim(A \wedge \sim A)$ is a function f, which maps each proof π of $A \wedge \sim A$ to a proof $f(\pi)$ of $\bot$, which does not exist. Since $(A \wedge \sim A)$ has no proof, any function (mapping, construction) whatsoever proves $\sim(A \wedge \sim A)$, but this conception may be criticized as being non-constructive, which might have led Troelstra and van Dalen [22, p. 9] to considering hypothetical proofs in their BHK-clause for negation. An early criticism can be found in the work of George Griss [cf. 7], whose criticism of intuitionistic negation results from a certain understanding of mental constructions. As Thomas Ferguson explains (notation adjusted), for Griss

> [c]onstructions serving to witness a conditional act as transformations whose application to constructions of an antecedent yield constructions of the consequent. In this context, the executability of a construction is interpreted as the possibility of successful acts of transformation. In principle the act of applying a function can only be considered successful in case there exists some operand to which the function is applied. Consequently, Griss' reading requires that the possibility of a construction of A serves as a precondition of the possibility of constructions of $A \to B$. [7, p. 3]

This precondition is not satisfied for $\sim(A \wedge \sim A)$ understood intuitionistically as $(A \wedge (A \to \bot)) \to \bot$; the required construction cannot be executed because there is no construction of $(A \wedge (A \to \bot))$ on which it could be performed.

Whether or not the criticism of the BHK-clause for negation is seen as convincingly justifying the rejection of that clause, there is an alternative to it proposed by Edgar López-Escobar [13]. He suggested to supplement the BHK interpretation of positive intuitionistic logic with the primitive notion of *refutation* (or *disproof*) to give an interpretation for negation. As a result, and upon disregarding $\bot$, one obtains a semantics for the four-valued paraconsistent constructive logic with strong negation introduced by David Nelson and his co-author, Ahmad Almukdad, and now known as the system **N4** [see 1, 25, 16]. López-Escobar gives the following disproof interpretation of the intuitionistic connectives $\wedge$, $\vee$,

$\to$, the quantifiers $\forall$ and $\exists$, and the strong negation $\sim$ (notation and presentation adjusted):

(i) the construction c refutes $A \wedge B$ iff c is of the form $\langle i, d\rangle$ with i either 0 or 1 and if $i = 0$, then d refutes A and if $i = 1$ then d refutes B;
(ii) the construction c refutes $A \vee B$ iff c is of the form $\langle d, e\rangle$ and d refutes A and e refutes B;
(iii) the construction c refutes $A \to B$ iff c is of the form $\langle d, e\rangle$ and d proves A and e refutes B;
(iv) the construction c refutes $\forall x A$ iff c is of the form $\langle \mathbf{a}, d\rangle$ and d refutes $A(a/x)$;
(v) the construction c refutes $\exists x A$ iff c is a general method of construction such that given any individual (i.e. construction) from the species under consideration, $c(\mathbf{a})$ (i.e. c applied to $\mathbf{a}$) refutes $A(a/x)$;
(vi) [t]he construction c refutes $\sim A$ iff c proves A.[6]

Definition 9. The López-Escobar interpretation of the logical operations is obtained from the BHK-interpretation by replacing the clause 5 for negation by the above clauses (i)–(vi) and adding the clauses:

(vii) for atomic sentences, we assume that we know intrinsically what a refutation is;
(viii) the construction c is a proof of $\sim A$ iff c refutes A.

We will assume that proofs and disproofs of atomic sentences (closed formulas) are atomic. Whilst the first five disproof clauses specify the form of (canonical) refutations of conjunctions, disjunctions, implications, and quantified formulas, the clause for strong negation is different. It specifies that a construction c is a refutation of $\sim A$ just in case the very same construction c is a proof of A. Similarly, clause (viii) requires that a construction c proves $\sim A$ iff c itself refutes A (and not iff c is a proof of $A \to \bot$ and thus has a specific form).

A fundamental assumption made by López-Escobar is that for no formula A there exists a construction that both proves and disproves A. If it is assumed that for no formula A there exist proofs of both A and

[6] The numbering in [13] is different. There the above clause (v) is numbered (vi) and the above clause (vi) is numbered (viii).

$\sim A$, then *ex contradictione quodlibet* expressed as a formula, $(A \wedge \sim A) \rightarrow B$, becomes provable.[7]

FACT 2 (Soundness of **HQN4** w.r.t. to the López-Escobar interpretation). *If* $\{A_1, \ldots, A_n\} \vdash_{\mathbf{HQN4}} A$ *then there exists a function* f *such that for any proofs* $\pi_1, \ldots, \pi_n$ *of* $A_1, \ldots, A_n$, $f(\pi_1, \ldots, \pi_n)$ *is a proof of* A.

PROOF. It is enough to show that there exist proofs for the axioms of **HQN4** and that the rules r1–r3 preserve the existence of proofs. Axioms a14–a19 require the consideration of disproofs. ⊣

Note that López-Escobar does not draw a distinction between a positive notion of validity as a conversion of proofs and disproofs into a proof and negative validity as a conversion of proofs and disproofs into a disproof. He is justified in avoiding such a bilateralism at the level of derivability insofar as every disproof of A is a proof of $\sim A$ and vice versa. In general, however, from the perspective of proof-theoretic semantics, proof-theoretic bi- and multilateralism understood as a multiplicity of derivability relations have advantages [cf. 31].

5. The connexive López-Escobar interpretation

DEFINITION 10. The connexive López-Escobar interpretation is obtained from the López-Escobar interpretation by replacing clause (iii) by the following clause

(ix) the construction c refutes $A \rightarrow B$ iff c is a function f, which maps each proof π of A to a disproof $f(\pi)$ of B.

FACT 3 (Soundness of **HQC** w.r.t. the connexive López-Escobar interpretation). *If* $\{A_1, \ldots, A_n\} \vdash_{\mathbf{HQC}} A$ *then there is a function* f *such that for any proofs* $\pi_1, \ldots, \pi_n$ *of* $A_1, \ldots, A_n$, $f(\pi_1, \ldots, \pi_n)$ *is a proof of* A.

PROOF. It suffices to show that there exist proofs for the axioms of **HQC** and that the rules r1–r3 preserve the existence of proofs. Axiom a20 is taken care of by clause (ix). ⊣

The following weakening clause for derivations is sound:

[7] Note that the notion of refutation in the López-Escobar interpretation is to be distinguished from the notion of refutation in the theory of refutation calculi [cf. 10]. The latter theory is concerned with the axiomatization of the non-theorems of given logics.

if A is derivable from $\{A_1, \ldots, A_n\}$, then A is derivable from $\{A_1, \ldots, A_n, B\}$ for any formula B.

From a proof π of A from $\{A_1, \ldots, A_n\}$ one can define the $(n+1)$-place function π' that for any formula B maps any proof of B and any proofs $\pi_1, \ldots, \pi_n$ of $A_1, \ldots, A_n$, respectively, to $\pi(\pi_1, \ldots, \pi_n)$. Therefore, one and the same construction can be a proof of formulas $(A \to (B \to A))$ and $(A \to (C \to A))$. Also, any proof π of a formula A is also a proof of $A \vee B$, for any $\mathcal{L}$-formula B. According to the BHK-interpretation thus one and the same construction proves infinitely many syntactically different $\mathcal{L}$-formulas. According to the López-Escobar and the connexive López-Escobar interpretation, it may happen that two syntactically distinct formulas have identical proofs or identical disproofs. Any formulas A and $\sim\sim A$, for example, have both identical proofs and identical disproofs, i.e., no proofs, respectively disproofs, of A and $\sim\sim A$ are distinct.[8] In view of such a situation, one may find the following property desirable.

DEFINITION 11. *Proof/disproof parity* is satisfied whenever for every $\mathcal{L}$-formula A, the following equivalence holds: A has the same proofs as an $\mathcal{L}$-formula B iff A and B have the same disproofs.

Observation 1. The López-Escobar interpretation does not enjoy proof/disproof parity.

PROOF. We have

$$\begin{array}{ll} & \pi \text{ is a proof of } \sim(A \to B) \\ \text{iff} & \pi \text{ is a disproof of } A \to B \\ \text{iff} & \pi \text{ is a pair } \langle \pi_1, \pi_2 \rangle \text{ such that } \pi_1 \text{ is proof of } A \text{ and } \pi_2 \text{ is disproof of } B \\ \text{iff} & \pi \text{ is a pair } \langle \pi_1, \pi_2 \rangle \text{ such that } \pi_1 \text{ is proof of } A \text{ and } \pi_2 \text{ is proof of } \sim B \\ \text{iff} & \pi \text{ is a proof of } A \wedge \sim B. \end{array}$$

Now, every disproof of $\sim(A \to B)$ is a proof of $A \to B$, but a proof of $A \to B$ is different from a disproof of $A \wedge \sim B$ because every disproof of $A \wedge \sim B$ is a proof of $\sim A \vee B$, and a proof of $\sim A \vee B$ is different from a function f, which maps each proof π of A to a proof $f(\pi)$ of B. ⊣

[8] In [30], this identification of derivations of syntactically different formulas is used to define a notion of inherited identity of derivations in a bilateral cut-free sequent calculus for propositional **N4** that makes use of two kinds of sequent arrows, one standing for proofs (verifications) and the other for disproofs (falsifications). Based on that concept of inherited identity, a proof-theoretic notion of synonymy is defined.

Observation 2. Suppose proof/disproof parity holds for literals. Then it holds for the connexive proof/disproof interpretation, i.e., the connexive López-Escobar interpretation.

Proof. The proof is by induction on the weight $w(A)$ of $\mathcal{L}$-formulas A. If $w(A) = 0$, the claim holds by assumption. Let A be a conjunction $A_1 \wedge A_2$, so that $w(A) = w(A_1) + w(A_2) + 1$. Suppose A and B have the same proofs. Then

for every π, π is a proof of $A_1 \wedge A_2$ iff π is a proof of B
iff for every π, $\pi = \langle \pi_1, \pi_2 \rangle$, where π_1 is a proof of A_1 and π_2 is a proof of A_2 iff $\pi = \langle \pi_1, \pi_2 \rangle$ is a proof of B, where $B = B_1 \wedge B_2$, π_1 is a proof of B_1, and π_2 is a proof of B_2.

By the induction hypothesis, A_1 and B_1, respectively A_2 and B_2, have the same proofs iff they have the same disproofs. Therefore, $\sim A_1 \vee \sim A_2$ and $\sim B_1 \vee \sim B_2$ have the same proofs, and thus $A_1 \wedge A_2$ and $B_1 \wedge B_2$ have the same disproofs.
Suppose A and B have the same disproofs. Then

for every π, π is a disproof of $A_1 \wedge A_2$ iff π is a disproof of B
iff for every π, $\pi = \langle 0, \pi' \rangle$ and π' is a disproof of A_1 or $\pi = \langle 1, \pi' \rangle$ and π' is a disproof of A_2 iff π is a disproof of B, where $B = B_1 \wedge B_2$, π' is a disproof of B_1, or π' is a disproof of B_2.

By the induction hypothesis, A_1 and B_1, respectively A_2 and B_2, have the same proofs iff they have the same disproofs. Therefore, $\sim A_1 \vee \sim A_2$ and $\sim B_1 \vee \sim B_2$ have the same disproofs, and thus A and B have the same proofs.

The cases where A is a disjunction $A_1 \vee A_2$ is analogous.

Let A be an implication $A_1 \to A_2$, so that $w(A) = w(A_1) + w(A_2) + 1$. Suppose A and B have the same proofs. Then

for every π, π is a proof of $A_1 \to A_2$ iff π is a proof of B
iff for every π, π is a function f, which maps each proof π_1 of A_1 to a proof $f(\pi_1)$ of A_2 iff π is a proof of B, where $B = B_1 \to B_2$ and f maps each proof π_2 of B_1 to a proof $f(\pi_2)$ of B_2.

By the induction hypothesis, A_1 and B_1, respectively A_2 and B_2, have the same proofs iff they have the same disproofs. Therefore, $A_1 \to \sim A_2$ and $B_1 \to \sim B_2$ have the same proofs, and thus $A_1 \to A_2$ and $B_1 \to B_2$ have the same disproofs.

Suppose A and B have the same disproofs. Then

for every π, π is a disproof of $A_1 \to A_2$ iff π is a disproof of B
iff for every π, π is a function f, which maps each proof π_1 of A_1 to a disproof $f(\pi_1)$ of A_2 iff π is a disproof of B, where $B = B_1 \to B_2$ and f maps each proof π_2 of B_1 to a disproof $f(\pi_2)$ of B_2.

By the induction hypothesis, A_1 and B_1, respectively A_2 and B_2, have the same proofs iff they have the same disproofs. Therefore

for every π, π is a disproof of $A_1 \to A_2$ iff π is a disproof of B
iff for every π, π is a function f, which maps each proof π_1 of A_1 to a proof $f(\pi_1)$ of $\sim A_2$ iff π is a disproof of B, where $B = B_1 \to B_2$ and f maps each proof π_2 of B_1 to a proof $f(\pi_2)$ of $\sim B_2$.

and hence $A_1 \to \sim A_2$ and $B_1 \to \sim B_2$ have the same disproofs, and thus $A_1 \to A_2$ and $B_1 \to B_2$ have the same proofs.

Let A be of the form $\forall x C$, so that $w(A) = w(C(a/x)) + 1$. Suppose A and B have the same proofs. Then

for every π, π is a proof of $\forall x C$ iff π is a proof of B
iff for every π, π is a function f, which maps each point $\mathbf{a}$ of the domain of definition to a proof $f(\mathbf{a})$ of $C(a/x)$ iff π is a proof of B, where $B = \forall y D$ and π is a function f, which maps each point $\mathbf{a}$ of the domain of definition to a proof $f(\mathbf{a})$ of $D(a/y)$.

By the induction hypothesis, for each point $\mathbf{a}$ of the domain of definition, $C(a/x)$ and $D(a/y)$ have the same proofs iff they have the same disproofs. Therefore, $\exists x {\sim} C$ and $\exists y {\sim} D$ have the same proofs, and thus for every π, π is a pair $\langle \mathbf{a}, \pi \rangle$, where $\mathbf{a}$ is a point of the domain of definition and π is a disproof of $C(a/x)$ iff π is a pair $\langle \mathbf{a}, \pi \rangle$ where $\mathbf{a}$ is a point of the domain of definition and π is a disproof of $D(a/x)$. That is, $\forall x C$ and $\forall y D$ have the same disproofs.

Suppose A and B have the same disproofs. Then

for every π, π is a disproof of $\forall x C$ iff π is a disproof of B
iff for every π, π is of the form $\langle \mathbf{a}, \pi' \rangle$ and π' refutes $C(a/x)$ iff π is a disproof of B, where $B = \forall y D$ and π is of the form $\langle \mathbf{a}, \pi' \rangle$ and π' refutes $D(a/y)$.

By the induction hypothesis, $C(a/x)$ and $D(a/y)$ have the same proofs iff they have the same disproofs. Therefore, $\exists x{\sim}C$ and $\exists y{\sim}D$ have the same disproofs, and thus $\forall xC$ and $\forall yD$ have the same proofs.

The case where A has the form $\exists xC$ is analogous.

Suppose A has the form ${\sim}{\sim}C$, so that $w(A) = w(C) + 1$. Then

for every π, π is a proof of ${\sim}{\sim}C$ iff π is a proof of B
iff for every π, π is a proof of C iff π is a proof of B
iff for every π, π is a disproof of C iff π is a disproof of B (by the induction hypothesis)
iff for every π, π is a disproof of ${\sim}{\sim}C$ iff π is a disproof of B.

Suppose A has the form ${\sim}(C \rightarrow D)$, so that $w(A) = w(C) + w({\sim}D) + 2$. Observe that for every π, (i) π is a proof of ${\sim}(C \rightarrow D)$ iff π is a proof of $C \rightarrow {\sim}D$, by clauses (viii), (ix), and the BHK-clause for conditionals and (ii) π is a disproof of $C \rightarrow {\sim}D$ iff π is a disproof of ${\sim}(C \rightarrow D)$, by clauses (ix), (vi), and the BHK-clause for conditionals. Then

for every π, π is a proof of ${\sim}(C \rightarrow D)$ iff π is a proof of B
iff for every π, π is a proof of $C \rightarrow {\sim}D$ iff π is a proof of B
iff for every π, π is a disproof of $C \rightarrow {\sim}D$ iff π is a disproof of B (by the induction hypothesis)
iff for every π, π is a disproof of ${\sim}(C \rightarrow D)$ iff π is a disproof of B.

The cases where A is a negated disjunction ${\sim}(C \vee D)$ or a negated conjunction ${\sim}(C \wedge D)$ follow an analogous pattern.

Suppose A has the form ${\sim}\exists xC$, so that $w(A) = w({\sim}C(a/x)) + 1$. Suppose A and B have the same proofs. Then

for every π, π is a proof of ${\sim}\exists xC$ iff π is a proof of B
iff for every π, π is a disproof of $\exists xC$ iff π is a proof of B
iff for every π, π is a function f, which maps each point $\mathbf{a}$ of the domain of definition to a proof $f(\mathbf{a})$ of ${\sim}C(a/x)$ iff π is a proof of B, where $B = \forall yD$ and π is a function f, which maps each point $\mathbf{a}$ of the domain of definition to a proof $f(\mathbf{a})$ of ${\sim}D(a/y)$.

By the induction hypothesis, for each point $\mathbf{a}$ of the domain of definition, ${\sim}C(a/x)$ and ${\sim}D(a/y)$ have the same proofs iff they have the same disproofs. Therefore, $\exists x{\sim}C$ and $\exists y{\sim}D$ have the same proofs, and thus for every π, π is a pair $\langle \mathbf{a}, \pi \rangle$, where $\mathbf{a}$ is a point of the domain of definition and π is a disproof of $C(a/x)$ iff π is a pair $\langle \mathbf{a}, \pi \rangle$ where $\mathbf{a}$ is a point of the domain of definition and π is a disproof of $D(a/x)$. That is, $\forall xC$ and $\forall yD$ have the same disproofs.

Suppose A and B have the same disproofs. Then

for every π, π is a disproof of $\forall xC$ iff π is a proof of B
iff for every π, π is of the form $\langle \mathbf{a}, \pi' \rangle$
and π' refutes $C(a/x)$ iff π is a proof of B, where
$B = \forall yD$, π is of the form $\langle \mathbf{a}, \pi' \rangle$ and π' refutes $D(a/y)$.

By the induction hypothesis, $C(a/x)$ and $D(a/y)$ have the same proofs iff they have the same disproofs. Therefore, $\exists x{\sim}C$ and $\exists y{\sim}D$ have the same disproofs, and thus $\forall xC$ and $\forall yD$ have the same proofs.

The case where A has the form $\exists xC$ is analogous. ⊣

One may wonder whether proof/disproof parity as a property of the connexive proof/disproof interpretation implies the congruentiality of a logic with respect to which the connexive proof/disproof interpretation is sound. This is not the case. The two distinct formulas $P(a) \to P(a)$ and $Q(a) \to Q(a)$ are mutually derivable in **QC**, but their negations ${\sim}(P(a) \to P(a))$ and ${\sim}(Q(a) \to Q(a))$ are not mutually derivable in **QC**.

6. Another proof/disproof interpretation

The connexive proof/disproof interpretation is not the only proof/disproof interpretation that enjoys proof/disproof parity. Nissim Francez [8, p. 89] (notation adjusted), refers to

$$(A \to B) \to {\sim}({\sim}A \to B) \text{ and } ({\sim}A \to B) \to {\sim}(A \to B)$$

as “Boethius’ ${\sim}_l$-theses,” (i.e., “Boethius’ negation-left-theses”) and suggests a motivation of these theses in terms intonational stress patterns in English. His natural deduction proof system $\mathcal{N}^{{\sim}_l}$ lays down a falsification condition for conditionals that amounts to replacing the above axiom a19 by the axiom:

a20 $\quad {\sim}(A \to B) \leftrightarrow ({\sim}A \to B)$.

Axiom a21 gives rise to the following disproof clause::

($\star$) π is a disproof of $A \to B$ iff π is a function f which maps each disproof π of A to a proof $f(\pi)$ of B.

Let us refer to the result of replacing the disproof clause for conditionals in the López-Escobar interpretation by clause ($\star$) as "the $\sim_l$-interpretation."

Observation 3. Suppose that proof/disproof parity holds for literals and the clause for negated conditionals in Definition 6 is altered to $w(\sim(A \to B)) = w(\sim A) + w(B) + 1$. Then proof/disproof parity holds for the $\sim_l$-interpretation.

Proof. The case of formulas of the form $\sim(C \to D)$ in the proof of Observation 2 becomes

	for every π, π is a proof of $\sim(C \to D)$ iff π is a proof of B
iff	for every π, π is a proof of $\sim C \to D$ iff π is a proof of B
iff	for every π, π is a disproof of $\sim C \to D$ iff π is a disproof of B (by the induction hypothesis)
iff	for every π, π is a disproof of $\sim(C \to D)$ iff π is a disproof of B. $\dashv$

However, given the drinker truism and the dual drinker truism, the $\sim_l$-interpretation allows one to validate two bewildering principles that, following Smullyan's drinker story, may be called "the nondrinker principle" and "the dual nondrinker principle":

$$\forall x(\sim P(x) \to \forall y {\sim} P(y)) \qquad (\mathbb{NDP})$$

("Everybody is such that if it is false that she drinks, then for everyone it is false that they drink.")

$$\forall x(\exists y {\sim} P(y) \to \sim P(x)) \qquad (\mathbb{DNDP})$$

("Everybody is such that if for someone it is false that she drinks, then it is false that they drink.").

On the $\sim_l$-interpretation

	$\sim\exists x(P(x) \to \sim\exists y P(y))$	is valid
iff	$\forall x {\sim}(P(x) \to \forall y {\sim} P(y))$	is valid
iff	$\forall x(\sim P(x) \to \forall y {\sim} P(y))$	is valid
	$\sim\exists x(\forall y P(y) \to \sim P(x))$	is valid
iff	$\forall x {\sim}(\forall y P(y) \to \sim P(x))$	is valid
iff	$\forall x(\exists y {\sim} P(y) \to \sim P(x))$	is valid.

If the drinker truism and its dual are wanted, we thus have an "abstainer paradox" and a "dual abstainer paradox" that speak against the $\sim_l$-interpretation.

7. Connexive implication within intuitionistic logic

Davide Fazio, Antonio Ledda, and Francesco Paoli [6] have shown that a connexive implication, that we will denote with the symbol '$\rightarrow_c$', can be defined within intuitionistic logic by putting

$$(A \rightarrow_c B) := (A \rightarrow B) \wedge (\sim A \rightarrow \sim B),$$

where it is important to recall that $\sim$ here is intuitionistic negation. That is, the characteristic principles of connexive logic are provable:

Aristotle's theses: $\sim(A \rightarrow_c \sim A)$, $\sim(\sim A \rightarrow_c A)$,
Boethius' theses: $(A \rightarrow_c B) \rightarrow_c \sim(A \rightarrow_c \sim B)$, $(A \rightarrow_c \sim B) \rightarrow_c \sim(A \rightarrow_c B)$,

while $(A \rightarrow_c B)$ is not interderivable with $(A \rightarrow_c B) \wedge (B \rightarrow_c A)$. This is a very remarkable and certainly also surprising observation. It shows that one can get a connexive implication with respect to the intuitionistic *implies-absurdity-negation*.

In a sense then, **QInt** is a constructive connexive logic. However, since the constructible falsity property and the dual of the existence property fail for **QInt**, the point remains that both **QN4** and **QC** can be seen as being more constructive than **QInt**.

8. Summary

The first-order logics **QN4** and **QC** are more constructive than intuitionistic first-order logic, **QInt**, insofar as they enjoy certain established constructivity properties that **QInt** lacks. The non-trivial first-order logic **QC** is unorthodox because it is connexive (and thus contra-classical) and, moreover, not only paraconsistent but even negation inconsistent, see Table 1.

The step from **HQN4** to **HQC**, i.e., the replacement of axiom a16 by axiom a19 that results in both connexivity and negation inconsistency, gives one a logic that validates the drinker truism, $\mathbb{DT}$, and the dual drinker truism, $\mathbb{DDT}$. Moreover, the move from **QN4** to **QC** is motivated not only by the desire to validate the principles $\mathbb{DT}$ and $\mathbb{DDT}$, but also by observing that the connexive proof/disproof interpretation enjoys proof/disproof parity (under the assumption that proof/disproof parity holds for literals), see Table 2.

constructivity properties	**QInt**	**QN4**	**QC**
disjunction property	✓	✓	✓
existence property	✓	✓	✓
constructible falsity property	×	✓	✓
dual of the existence property	×	✓	✓
contra-classicality	×	×	✓
negation inconsistency	×	×	✓

Table 1. Constructivity, contra-classicality, and negation inconsistency

proof interpretation Brouwer-Heyting-Kolmogorov	proof/disproof interpretation López-Escobar	connexive proof/disproof interpretation
sound for **QInt**	sound for **QN4**	sound for **QC**
does not validate $\sim\sim A \rightarrow A$ $\sim(A \wedge B) \rightarrow (\sim A \vee \sim B)$ $\sim\forall x P(x) \rightarrow \exists x \sim P(x)$ $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$	validates $\sim\sim A \rightarrow A$ $\sim(A \wedge B) \rightarrow (\sim A \vee \sim B)$ $\sim\forall x P(x) \rightarrow \exists x \sim P(x)$ $\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$	validates $\sim\sim A \rightarrow A$ $\sim(A \wedge B) \rightarrow (\sim A \vee \sim B)$ $\sim\forall x P(x) \rightarrow \exists x \sim P(x)$ $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$
does not validate $\sim\exists x(P(x) \rightarrow \sim\exists y P(y))$ $\sim\exists x(\forall y P(y) \rightarrow \sim P(x))$	does not validate $\sim\exists x(P(x) \rightarrow \sim\exists y P(y))$ $\sim\exists x(\forall y P(y) \rightarrow \sim P(x))$	validates $\sim\exists x(P(x) \rightarrow \sim\exists y P(y))$ $\sim\exists x(\forall y P(y) \rightarrow \sim P(x))$
does not qualify for proof/disproof parity	does not enjoy proof/disproof parity	enjoys proof/disproof parity (if it holds for literals)

Table 2. Proof and proof/disproof interpretations

The logic **QN4** validates double negation elimination, all De Morgan laws, and the interdefinability of the existential and universal quantifiers familiar from classical logic. From the point of view of **QN4** and the López-Escobar interpretation, these principles are constructively acceptable (and double negation elimination does not lead to the validity of the Law of Excluded Middle if the latter is added to **HQN4** or **HQC**). The logic **QC** is *contra-classical* as it is presented in the language of **QCL** and validates axiom a19, which is not a theorem of classical logic. If contra-classicality is not in principle excluded by one's methodology, then it is conceivable that the BHK-interpretation and the López-Escobar interpretation miss some constructively perfectly acceptable principles which are not theorems of classical logic. According to the connexive López-Escobar interpretation, the drinker truism and its dual are valid, which both are not theorems of classical logic. The intuitive plausibility of

$\mathbb{DT}$ and $\mathbb{DDT}$ and proof/disproof parity as a constructiveness feature possessed by the connexive López-Escobar interpretation suggest that the contradictory logic **QC** is even more constructive than **QN4**.

Acknowledgments. I am grateful to Hitoshi Omori, Graham Priest, Satoru Niki, Nils Kürbis, Grigory Olkhovikov, the participants of the 7th Workshop on Connexive Logics, 26th–28th October 2022, Mexico City, and two anonymous referees for their useful comments. This research has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant agreement ERC-2020-ADG, 101018280, ConLog.

References

[1] Almukdad, A., and Nelson, D., "Constructible falsity and inexact predicates", *The Journal of Symbolic Logic* 49 (1984): 231–233. DOI: 10.2307/2274105

[2] Beth, E. W., *Aspects of Modern Logic*, Reidel, Dordrecht, 1970.

[3] Buss, S., "An introduction to proof theory", pages 1–78 in S. Buss (ed.), *Handbook of Proof Theory*, Elsevier, Amsterdam, 1998.

[4] Casari, E., "Intermediate logics", *Atti degli incontri di Logica Matametica* 1 (1982): 243–298.

[5] van Dalen, D., *Logic and Structure*, 4th ed., Springer, Heidelberg, 2004

[6] Fazio, D., A. Ledda, and P. Paoli, "Intuitionistic logic is a connexive logic", *Studia Logica* (2023). DOI: 10.1007/s11225-023-10044-7

[7] Ferguson, T. M., "Negation in negationless intuitionistic mathematics", *Philosophia Mathematica* (2022). DOI: 10.1093/philmat/nkac026

[8] Francez, N., *A View of Connexive Logic*, College Publications, London, 2021.

[9] Girard, J.-Y., *Proofs and Types*, translated and with appendices by P. Taylor and Y. Lafont, Cambridge University Press, Cambridge, 1989.

[10] Goranko, V., G. Pulcini, and T. Skura, "Refutation systems: An overview and some applications to philosophical logics", pages 173–197 in F. Liu, H. Ono, and J. Yu (eds.), *Knowledge, Proof and Dynamics. The Fourth Asian Workshop on Philosophical Logic*, Springer, Cham, 2020.

[11] Johansson, I., "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus", *Compositio Mathematica* 4 (1930): 119–136.

[12] Komori, Y., 'Some results on the super-intuitionistic predicate logics", *Reports on Mathematical Logic* 15 (1983): 13–31.

[13] López-Escobar, E. G. K., "Refutability and elementary number theory", *Indigationes Mathematicae* 34 (1972), 362–374. DOI: 10.1016/1385-7258(72)90053-4

[14] McCall, S., "A history of connexivity", pages 415–449 in D. Gabbay, F. Pelletier, and J. Woods (eds.), *Handbook of the History of Logic*, Vol. 11, Elsevier, Amsterdam, 2012.

[15] Nakamura, T., "Disjunction property for some intermediate predicate logics", *Reports on Mathematical Logic* 15 (1983), 33–39.

[16] Odintsov, S. P., *Constructive Negations and Paraconsistency*, Springer, Dordrecht, 2008.

[17] Odintsov, S. P., and H. Wansing, "Inconsistency-tolerant description logic. Motivation and basic systems", pages 301–335 in V. Hendricks and J. Malinowski (eds.), *Trends in Logic. 50 Years of Studia Logica*, Kluwer Academic Publishers, Dordrecht, 2003. DOI: 10.1007/978-94-017-3598-8

[18] Omori, H., and H. Wansing, "An extension of connexive logic **C**", pages 503–522 in S. Negri et al., (eds.), *Advances in Modal Logic*, Vol. 13, College Publications, London, 2020.

[19] Ono, H., "Some problems in intermediate predicate logics", *Reports on Mathematical Logic* 21 (1987): 55–67; Supplement 22 (1988): 117–118.

[20] Prawitz, D., *Natural Deduction: A Proof-theoretical Study*, Almqvist & Wiksell, Stockholm, 1965. Reprinted with Dover Publications, 2006.

[21] Smullyan, R., *What is the Name of this Book? The Riddle of Dracula and Other Logical Puzzles*, Prentice-Hall, Englewood Cliffs, 1978.

[22] Troelstra, A., and D. van Dalen, *Constructivism in Mathematics. An Introduction*, Elsevier, Amsterdam, 1988.

[23] Vredenduin, P. G. J., "The logic of negationless mathematics", *Compositio Mathematica* 11 (1953): 204–270.

[24] von Kutschera, F., "Ein verallgemeinerter Widerlegungsbegriff für Gentzenkalküle", *Archiv für mathematische Logik und Grundlagenforschung* 12 (1969): 104–118. DOI: 10.1007/BF01969697

[25] Wansing, H., *The Logic of Information Structures*, Springer Lecture Notes in AI 681, Springer, Berlin, 1993.

[26] Wansing, H., “Negation”, pages 415–436 in L. Goble (ed.), *The Blackwell Guide to Philosophical Logic*, Basil Blackwell Publishers, Cambridge/MA, 2001.

[27] Wansing, H., “Connexive modal logic”, pages 367–383 in R. Schmidt et al. (eds.), *Advances in Modal Logic*, Vol. 5, King’s College Publications, London, 2005. http://www.aiml.net/volumes/volume5/

[28] Wansing, H., “One heresy and one orthodoxy: On dialetheism, dimathematism, and the non-normativity of logic”, *Erkenntnis* (2022). DOI: 10.1007/s10670-022-00528-8

[29] Wansing, H., ”Connexive logic”, *The Stanford Encyclopedia of Philosophy* (Summer 2023 Edition), E. N. Zalta and U. Nodelman (eds.). https://plato.stanford.edu/archives/sum2023/entries/logic-connexive/

[30] Wansing, H., “A note on synonymy in proof-theoretic semantics”, to appear in T. Piecha and K. Wehmeier (eds.), *Peter Schroeder-Heister on Proof-theoretic Semantics*, Springer, Cham, 2024.

[31] Wansing, H., and S. Ayhan, “Logical mutilateralism”, *Journal of Philosophical Logic*, 52 (2023): 1603–1636. DOI: 10.1007/s10992-023-09720-9

[32] Warren, L., H. Diener, and M. McKubre-Jordens, “The Drinker paradox and its dual”, 2018. DOI: 10.48550/arXiv.1805.06216

Heinrich Wansing
Department of Philosophy I
Ruhr University Bochum
Bochum, Germany
Heinrich.Wansing@rub.de
https://orcid.org/0000-0002-0749-8847