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Simplified Semantics for Further Relevant Logics I: Unreduced Semantics for E and Π′

Abstract. This paper shows that the relevant logics **E** and **Π′** are strongly sound and complete with regards to a version of the "simplified" Routley-Meyer semantics. Such a semantics for **E** has been thought impossible. Although it is impossible if an admissible rule of **E** — the rule of restricted assertion or equivalently Ackermann's δ-rule — is solely added as a primitive rule, it is very much possible when **E** is axiomatized in the way Anderson and Belnap did.

The simplified semantics for **E** and **Π′** requires *unreduced* frames. Contra what has been claimed, however, no additional frame component is required over and above what's required to model other relevant logics such as **T** and **R**. It is also shown how to modify the tonicity requirements of the ternary relation so as to allow for the standard truth condition for both fusion — the intensional conjunction $\circ$ — as well as the converse conditional $\leftarrow$.

Keywords: converse conditional; **E**; fusion; γ; relevant logics; simplified Routley-Meyer semantics; unreduced frames

1. Introduction

The purpose of this paper is to ever so slightly expand upon the simplified semantics first set forth in (Priest and Sylvan, 1992) and then extended so as to cover a range of relevant logics (as well as some related non-relevant ones) in (Restall, 1993). Notably lacking from Restall's paper is the characteristic axiom $((A \to A) \wedge (B \to B) \to C) \to C$ of Anderson and Belnap's favorite logic **E** as well as the "γ-rule" $\{A, {\sim}A \vee B\} \Vdash B$ characteristic of Ackermann's closely related logic **Π′**. This paper shows how to model these logical principles within the simplified semantics so

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as to make the semantics strongly sound and complete with regards to the standard relation of derivability.

E has suffered at least two misconceptions which are worth setting aright. These pertain to the very identity of the logic **E** and the complexity involved in its frame semantics — simplified or not.

Identity: It is claimed in (Restall, 1993) that the simplified semantics cannot model **E**'s set of logical theorems. Although the claim actually made is correct, it is a claim which turns out to be about a different logic than Anderson and Belnap's **E**: Restall's **E** has the rule called *restricted assertion*, $\{A\} \Vdash (A \to B) \to B$, as its characteristic principle, which, although *admissible*, fails to be *derivable* in Anderson and Belnap's **E**. This mishap is further laid out in section 3.

Complexity: Priest — one of the original inventors of the simplified semantics — has claimed that there *is* a ternary semantics for **E**, "though of a more complicated kind" (Priest, 2008, p. 202). Priest does not give any clues as to what this complication amounts to, but it seems likely that he either intended to imply that **E** doesn't have a *simplified* semantics altogether — which, as we'll see, it does — or that he alluded to is the frame element P of the original Routley-Meyer semantics for **E**. That version of the Routley-Meyer semantics is found in the first appendix of (Routley et al., 1982), entitled *The semantics of entailment – IV: E,* $\mathbf{\Pi}'$ *and* $\mathbf{\Pi}''$ which I'll refer to as *SE4* in this paper. In SE4, P was utilized so as

> to get round the problems raised by the fact that it is impossible to obtain, [. . .], a prime regular theory T which validates the rule of necessitation. [. . .]
> Admittedly, the presence of P, and its apparent uneliminability, makes the semantics of $\underline{\underline{\mathrm{E}}}$[1] more cumbersome and less attractive than that of some of its relevant rivals such as R and T.
>
> (Routley et al., 1982, p. 407)

The rule of necessitation alluded to here is that which yields $\Box A := (A \to A) \to A$ from A. That rule, like Restall's rule of restricted assertion, is merely admissible in **E**. The notion of derivability most probably alluded to in SE4 simply does not allow derivations from arbitrary premise sets, however. The SE4-calculus, then, merely yields the set of

[1] SE4's logic 'E' is formulated with a propositional constant. $\underline{\underline{\mathrm{E}}}$ is the constant-free fragment of it, and is, modulo minor equivalent details, identical to **E** as specified in this paper.

logical theorems of **E**. Restall's rule of restricted assertion, on the other hand, applies to non-theorems and allows one to derive, then, $\Box p$ from the premise set $\{p\}$, where p is any propositional variable. Since the goal of SE4 is merely to establish *weak* soundness and completeness, the failure to distinguish derivable rules from admissible ones doesn't materialize into technical errors. However, it easily engenders such in the more broader context where derivations from arbitrary assumptions are permitted. When only this is clarified, however, it will be shown that the "simplified" version of the SE4-semantics actually allows for a definable set Z — a variant of that suggested in (Anderson et al., 1992) — which does in fact eliminate the need for Sylvan and Meyer's P. Nothing of additional complexity, then, is needed to model **E**. Not even when the goal is to model the full consequence relation of **E** and not merely its set of logical theorems.

The notion of a *reduced* frame is set forth in subsection 4.1 wherein it is shown that **E** and $\mathbf{\Pi}'$ really require *unreduced* frames. Both the original Routley-Meyer semantics, as well as the simplified version of these, come in two variants. To avoid further confusion regarding **E**, subsection 4.2 give a short explanation of some key aspects which set these apart. Subsection 4.3, then, gives the relevant background of the definition of the set Z.

Sections 5–6 then show that the characteristic axiom of **E** and the characteristic rule of $\mathbf{\Pi}'$ correspond to particular frame requirements. This, then, suffices for proving strong soundness and completeness for any logic with these principles. This paper shows for the first time how to modify tonicity requirements to the ternary relation so as to to deal with fusion and the converse conditional within the simplified semantics. As a corollary of the strong soundness and completeness result, it will follow that the addition of such connectives is strongly conservative.

Section 7 provides a short summary, before the appendix deals with the logic **EM** — **E** augmented by the restricted mingle axiom

$$(A \to B) \to ((A \to B) \to (A \to B)).$$

Relevant logics are often equipped with propositional constants. The sequel to this paper — (Øgaard, 2024) — will deal with their simplified semantics, as well as to show how such constants may be used to define various modal and negation connectives, enthymematical conditionals, and to capture propositions expressing various infinite conjunctions and disjunctions.

2. Initial definitions

DEFN 2.1 (Parenthesis conventions and defined connectives). $\vee$, $\wedge$, and $\circ$ are to bind tighter than $\rightarrow$ and $\leftarrow$, and so I'll usually drop parenthesis enclosing conjunctions and disjunctions whenever possible. Association is otherwise to the left and so $\sim A \wedge B \wedge C \rightarrow D \vee E \vee (F \circ G \circ H)$ is simply shorthand for $((\sim A \wedge B) \wedge C) \rightarrow ((D \vee E) \vee ((F \circ G) \circ H)$.

$$\Box A := (A \rightarrow A) \rightarrow A$$
$$A \xrightarrow{C} B := A \wedge C \rightarrow B$$
$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$$

An axiomatization of a logic will in this paper be thought of as a set of axioms and rules. Any such axiomatization can be used to define a derivability relation. The only derivability relation which will be used in this paper is the "standard" Hilbertian one:

DEFN 2.2 (The Hilbert consequence relation). A HILBERT PROOF of a formula A from a set of formulas Γ in a logic $\boldsymbol{L}$ is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of $\boldsymbol{L}$, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of $\boldsymbol{L}$. The existential claim that there is such a proof is written $\Gamma \vdash_{\boldsymbol{L}} A$.

It is rather trivial that two axiomatizations can beget the same derivability relation. What is less trivial is the criteria of identity for a logic. In the context of relevant logics it is not uncommon to regard two lists of axioms and rules as axiomatizations of the same logic provided they generate the same set of logical theorems. The aim of this paper is a strong soundness and completeness proof. As such it is the full consequence relation which is the proper object of study. The following definition is therefore warranted:

DEFN 2.3 (Identity criteria for logics). Let $\boldsymbol{L}$, $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ be lists of axioms and rules.

- The logic $\boldsymbol{L}$ denotes the set $\{\langle \Gamma, A\rangle \mid \Gamma \vdash_{\boldsymbol{L}} A\}$, where $\Gamma \cup \{A\}$ is any subset of the set of well-formed formulas.[2]

[2] Relevantists often state philosophical reasons for thinking that the Hilbertian consequence relation — even for relevant logics — fails to capture true logical consequence since it fails ensure that premises are relevant to the conclusion: If A is an

- $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are EQUIVALENT AXIOMATIZATIONS if and only if $\{\langle \Gamma, A\rangle \mid \Gamma \vdash_{\boldsymbol{L}_1} A\} = \{\langle \Gamma, A\rangle \mid \Gamma \vdash_{\boldsymbol{L}_2} A\}$.
- $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are THEOREMWISE IDENTICAL if $\{A \mid \varnothing \vdash_{\boldsymbol{L}_1} A\} = \{A \mid \varnothing \vdash_{\boldsymbol{L}_2} A\}$.

DEFN 2.4. For any logic $\boldsymbol{L}$ and axioms/rules $\theta_1, \ldots, \theta_n$: $\boldsymbol{L}[\theta_1, \ldots, \theta_n]$ is the logic obtained by adding $\theta_1, \ldots, \theta_n$ as axioms/rules and expanding the language to include any connective occurring in $\theta_1, \ldots, \theta_n$ which is not already present in $\boldsymbol{L}$.

Unless otherwise specified, I will consider the language of a logic to be determined by the connectives which explicitly occurs in its axioms and rules.

DEFN 2.5 (Derivable vs. admissible rules).
- A rule $\Delta \Vdash A$ is said to be DERIVABLE in a logic $\boldsymbol{L}$ just in case $\Delta^\sigma \vdash_{\boldsymbol{L}} A^\sigma$ for every uniform substitution σ.
- A rule $\Delta \Vdash A$ is said to be ADMISSIBLE in a logic $\boldsymbol{L}$ just in case for every uniform substitution σ, if ($\varnothing \vdash_{\boldsymbol{L}} \delta^\sigma$ for every $\delta \in \Delta$), then $\varnothing \vdash_{\boldsymbol{L}} A^\sigma$.

DEFN 2.6. A logic $\boldsymbol{L}_2$ EXTENDS a logic $\boldsymbol{L}_1$ just in case every well-formed formula (wff) of $\boldsymbol{L}_1$ is a wff of $\boldsymbol{L}_2$ and that for every set of $\boldsymbol{L}_1$-wffs $\Gamma \cup \{A\}$, if $\Gamma \vdash_{\boldsymbol{L}_1} A$, then also $\Gamma \vdash_{\boldsymbol{L}_2} A$.

It follows from the above definitions that if a rule is derivable in a logic, then it is also admissible in it.

Except for A13, R7, the fusion and converse conditional principles, all axioms and rules in the following list are found in (Restall, 1993).[3]

$$
\begin{array}{ll}
\text{(A1)} & A \to A \\
\text{(A2)} & A \to A \vee B \ \text{ and } \ B \to A \vee B \\
\text{(A3)} & A \wedge B \to A \ \text{ and } \ A \wedge B \to B \\
\text{(A4)} & A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)
\end{array}
$$

axiom of the logic of $\boldsymbol{L}$, and p is any propositional variable not occurring in A, then $\{p\} \vdash_{\boldsymbol{L}} A$ will true (cf. Priest, 2015, § 5.4). I have elsewhere argued that the best account of the philosophy of Anderson-Belnap type relevance is a *pluralists* account of logical consequence where the Hilbertian relation is but one legitimate notion of consequence — what they called *enthymematical entailment*. See (Øgaard, 2021b) for more on this.

[3] To be entirely precise, Restall (1993) doesn't consider the pre- and suffixing rules, but rather the interderivable *affixing* rule $\{A \to B, C \to D\} \Vdash (B \to C) \to (A \to D)$.

(A5)	$(A \to B) \land (A \to C) \to (A \to B \land C)$	
(A6)	$(A \to C) \land (B \to C) \to (A \lor B \to C)$	
(A7)	$\sim\sim A \leftrightarrow A$	
(A8)	$(A \to B) \to (\sim B \to \sim A)$	
(A9)	$(A \to B) \to ((C \to A) \to (C \to B))$	
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	
(A11)	$(A \to (A \to B)) \to (A \to B)$	
(A12)	$(A \to \sim A) \to \sim A$	
(A13)	$((A \to A) \land (B \to B) \to C) \to C$	
(A14)	$A \to ((A \to B) \to B)$	
(A15)	$A \lor \sim A$	
(A16)	$A \land (A \to B) \to B$	
(R1)	$\{A, B\} \Vdash A \land B$	adjunction (α)
(R2)	$\{A, A \to B\} \Vdash B$	modus ponens (β)
(R3)	$\{A \to B\} \Vdash (C \to A) \to (C \to B)$	prefixing rule
(R4)	$\{A \to B\} \Vdash (B \to C) \to (A \to C)$	suffixing rule
(R5)	$\{A \to B\} \Vdash \sim B \to \sim A$	contraposition rule
(R6)	$\{A\} \Vdash (A \to B) \to B$	restricted assertion
(R7)	$\{A, \sim A \lor B\} \Vdash B$	disj. syllogism (γ)
(A$\circ$)	$A \to (B \to A \circ B)$	
(R$\circ$)	$\{A \to (B \to C)\} \Vdash A \circ B \to C$	
(A$\leftarrow$)	$A \to ((B \leftarrow A) \to B)$	
(R$\leftarrow$)	$\{A \to (B \to C)\} \Vdash B \to (C \leftarrow A)$	

Every logic considered in this paper will be an extension of the relevant logic **B**, the definition of which, along with some of the more standard relevant logics, is set forth in the following definition:

B	A1–A7; R1–R5
TW	A1–A10; R1–R2
T	**TW**[A11, A12]
E	**T**[A13]
Π′	**E**[R7]
R	**T**[A14]

In logics with both A8 and A14 one can define the fusion connective using $A \circ B := \sim(A \to \sim B)$. In logics weaker than **E**-type logics, however, fusion is often regarded as an optional extra connective. This is also the case in this paper. It is easy to verify that the axiom $A\circ$ is in fact interderivable with the converse of R∘, namely the rule

$$\{A \circ B \to C\} \Vdash A \to (B \to C).$$

These two ∘-rules are often called the *residuation rules* and is the most common way to axiomatize fusion. $\to$ is the *left residual* of ∘, whereas the converse conditional $\leftarrow$ is often called the *right residual*. In logics with A14 it is easily seen that $(A \to B) \leftrightarrow (B \leftarrow A)$ is a logical theorem, and so the two conditionals become notational variants of each other. Like ∘, however, $\leftarrow$ is undefinable in logics like **E**. Although ∘ is often added as a connective to relevant logics, the converse conditional is rarely considered. $\leftarrow$ does play a prominent role in Lambek-type calculi, however, in which slashes are often used instead of arrows with $B/A := A \to B$ and $A\backslash B := B \leftarrow A$ (cf. Ono, 2003). Note, then, that $A \leftarrow A$ fails to be a theorem of $\mathbf{E}^{\mathrm{d}}[\mathrm{A}{\leftarrow}, R{\leftarrow}]$.[4]

That ∘ and $\leftarrow$ are undefinable follows from the fact that **E** augmented with either of these loose, as we shall see, an important property, namely that of being *disjunctive*, which is the topic of the following subsection.

2.1. Disjunctive logics

The simplified Routley-Meyer semantics is a frame semantics in which every point of evaluation is required to make a disjunction $A \vee B$ true if and only it makes either A or B true. The semantics will be set up with a base point g with the notion of truth in a model being defined as being true at g and a rule being regarded as holding in a model if it is truth preserving over g. If, then, a rule $\{A_1, \dots, A_n\} \Vdash B$ is truth preserving over g, then it's disjunctive version will also be truth preserving over g.

defn 2.7. The disjunctive version of a rule $\varDelta \Vdash A$ is the rule

$$\varDelta \vee B \Vdash A \vee B,$$

where $\varDelta \vee B := \{\delta \vee B \mid \delta \in \varDelta\}$. ‘$\mathrm{R}^{\mathrm{d}}i$’ is to be a name for the disjunctive version of any rule Ri.

The proof-theoretic property which corresponds to truth preservance is derivability. In order to obtain a strong completeness theorem it must therefore be the case that the disjunctive version of any derivable rule

[4] A counter-model is found in figure 6. Nor is it a theorem of the logic called *Lambek Associative Calculus (L)* in (Restall, 2000, p. 40), although it is a theorem of LI − L augmented with a propositional constant t satisfying $A \leftrightarrow (t \to A)$. ∘ is fully associative in figure 6, and so the model validates all of L.

is derivable. Note, however, that it is not required that the disjunctive version of every *admissible* rule be admissible. In fact, we shall later see that there are admissible rules of **E** and **Π′** which are such that their disjunctive versions are not admissible.

DEFN 2.8. A consequence relation $\vdash$ — that is a *logic* — is called DISJUNCTIVE provided for every set of formulas $\Delta \cup \{A, B\}$,

$$\Delta \vdash A \Longrightarrow \Delta \vee B \vdash A \vee B.$$

In other words, a logic is disjunctive just in case the disjunctive version of every derivable rule is derivable.

Many logics fail to be disjunctive. Logics between **B** and **TW**[R7] are examples of such: B is in all such obviously derivable from $A \wedge (A \rightarrow B)$, and also from $A \wedge ({\sim}A \vee B)$ if R7 is a derivable rule of the logic. However, $C \vee B$ fails to be derivable from both $C \vee (A \wedge (A \rightarrow B))$ and $C \vee (A \wedge ({\sim}A \vee B))$ in the logic **TW**[R7].[5] The consequence relation of such logics as **TW**[R7] and **B**, therefore, cannot be modeled by the simplified Routley-Meyer semantics set forth in this paper. As shown in (Priest and Sylvan, 1992), however, there is a neat way to strengthen a logic so as to ensure disjunctiveness: by simply adding to the axiomatization the disjunctive version of every primitive rule of the original logic.[6]

DEFN 2.9. The axiomatization $\boldsymbol{L}^{\mathrm{d}}[\theta_1, ..., \theta_n]$ is obtained from $\boldsymbol{L}[\theta_1, ..., \theta_n]$ by adding the disjunctive version of every primitive rule of $\boldsymbol{L}[\theta_1, ..., \theta_n]$.

[5] Figure 1 shows forth a model which verifies this. The model, as is the case for all of the algebraic models displayed in this paper, was found using Slaney's `MaGIC` — an acronym for *Matrix Generator for Implication Connectives* — which is an open source computer program created by John K. Slaney (1995). The Hasse diagram displays the partial order $\preceq$ which conjunction and disjunction are interpreted as, respectively, greatest lower bound and least upper bound over. The other connectives are evaluated according to the displayed matrices. The subset $\mathcal{T}$ is the set of designated elements. A formula is true in such a model just in case it is evaluated to one such designated element, and a rule holds in a model just in case it is truth preserving.

[6] A popular alternative within the relevant literature has been to add so-called "meta-rules." The notion of such a rule was introduced in (Brady, 1984) where the propositional meta-rule is stated as "if $A \Rightarrow B$, then $C \vee A \Rightarrow C \vee B$," where '$\Rightarrow$' is the symbol used to state ordinary rules. Just how to cash out this is at times a bit unclear, but (Brady, 2006, p. 6) — in which the notion of meta-rule is credited to Meyer — states that the idea of such meta-rules is that they allow one to introduce subproofs just as in natural deduction calculi. A precise account of this, one which basically hard-codes reasoning by cases into the notion of a Hilbert derivation, was given in (Øgaard, 2017, df. 2).

$\mathcal{T} = \{x \mid 3 \preceq x\} = \{3\}$
$2 \vee (1 \wedge (1 \to 0)) = 3$
$2 \vee 0 = 2$
$1 \vee (2 \wedge ({\sim}2 \vee 0)) = 3$
$1 \vee 0 = 1$

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] & & 2 \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	1	3	1	3	1
2	1	1	3	3	2
3	1	1	1	3	0

Figure 1. A non-prime **TW**[R7]-model

THEOREM 2.1. *A consequence relation $\vdash$ given by an axiomatization $\boldsymbol{L}$ is disjunctive if and only if $\boldsymbol{L}$ and $\boldsymbol{L}^{\mathrm{d}}$ are equivalent axiomatization; if and only if, then, $\vdash_{\boldsymbol{L}} = \vdash_{\boldsymbol{L}^{\mathrm{d}}}$.*

PROOF. Assume that $\vdash_{\boldsymbol{L}}$ is disjunctive. Then the disjunctive version of every primitive rule of $\boldsymbol{L}$ is derivable, and so $\vdash_{\boldsymbol{L}} = \vdash_{\boldsymbol{L}^{\mathrm{d}}}$.

Assume that $\vdash_{\boldsymbol{L}} = \vdash_{\boldsymbol{L}^{\mathrm{d}}}$, and let $\varDelta \Vdash A$ be a derivable rule of $\boldsymbol{L}$. To complete this proof we must show that $\varDelta \vee B \Vdash A \vee B$ is also a derivable rule of $\boldsymbol{L}$. Let $C_1, \ldots, C_n$ be a derivation of A from $\varDelta$ in $\boldsymbol{L}^{\mathrm{d}}$. The proof, now, is to the effect that $\varDelta \vee B \vdash_{\boldsymbol{L}^{\mathrm{d}}} C_i \vee B$, for every $i \leq n$. For any $i \leq n$, if C_i is either an axiom of $\boldsymbol{L}^{\mathrm{d}}$ or a member of $\varDelta$, then obviously it is the case that $\varDelta \vee B \Vdash C_i \vee B$. Assume that the claim is true for every $j < i$ and that C_i is obtained from some $\varDelta' \subseteq \{C_j \mid j < i\}$ using some rule of $\boldsymbol{L}^{\mathrm{d}}$. Since, then, the disjunctive version of the rule used is a primitive rule of $\boldsymbol{L}^{\mathrm{d}}$, it follows, therefore that $C_i \vee B$ can be obtained from $\varDelta' \vee B$ using this rule, and so $\varDelta \vee B \vdash_{\boldsymbol{L}^{\mathrm{d}}} C_i \vee B$ also in this case. Since every C_i must be obtained in one of these three ways, it follows, therefore that $\varDelta \vee B \vdash_{\boldsymbol{L}^{\mathrm{d}}} A \vee B$. Since $\vdash_{\boldsymbol{L}} = \vdash_{\boldsymbol{L}^{\mathrm{d}}}$, it therefore follows that $\varDelta \vee B \vdash_{\boldsymbol{L}} A \vee B$. $\dashv$

An easy consequence of the above definition of a disjunctive logic is that the “meta-rules” of reasoning by cases and its one-premise sibling hold true of the consequence relation:

COROLLARY 2.1. *If $\boldsymbol{L}$ is disjunctive, then*

- $\{A\} \vdash_{\boldsymbol{L}} C$ *and* $\{B\} \vdash_{\boldsymbol{L}} C$, *then* $\{A \vee B\} \vdash_{\boldsymbol{L}} C$.
- $\{A\} \vdash_{\boldsymbol{L}} B$, *then* $\{A \vee C\} \vdash_{\boldsymbol{L}} B \vee C$.

The weakest logic that will be considered in this paper which is fit for the simplified semantics, then, is $\mathbf{B}^{\mathrm{d}}$. Even though $\mathbf{B}$ must be extended so as to become disjunctive, this is far from always the case. The axiomatizations set forth in both (Priest and Sylvan, 1992) and (Restall,

1993) are all disjunctive, in the sense that the disjunctive version of every primitive rule is yet another primitive rule of the axiomatization. This will not be the case in this paper. The simple reason for this is that logics such as **E** and **Π′** are prime despite being axiomatized using only R1 and R2, as well as R7 in the case of **Π′**. Note, then, that it is the consequence relation generated by an axiomatization which must be disjunctive — that is it is the *logic*, as here defined, which must be disjunctive — in order for the simplified semantics to have a chance of capturing it. The following two lemmas show forth two sets of sufficient criteria for a logic being disjunctive.

LEMMA 2.1. *If $\boldsymbol{L}$ extends* **B**, *has no primitive rule other than R1 and R2, and is such that there is a formula D such that $\varnothing \vdash_{\boldsymbol{L}} D$ and that for every formula A, B, $\varnothing \vdash_{\boldsymbol{L}} A \wedge (A \to B) \overset{D}{\to} B$, then $\boldsymbol{L} = \boldsymbol{L}^{\mathrm{d}}$.*

PROOF. We must show that the disjunctive version of both R1 and R2 are derivable rules of $\boldsymbol{L}$.

(R1): We must show that $\{A \vee C, B \vee C\} \vdash_{\boldsymbol{L}} (A \wedge B) \vee C$. That this is so follows by basic applications of R1 and R2 using axioms A2–A6. Details are left for the reader.

(R2): We must show that $\{A \vee C, (A \to B) \vee C\} \vdash_{\boldsymbol{L}} B \vee C$. By assumption of the lemma, $A \wedge (A \to B) \overset{D}{\to} B$ is a logical theorem. A bit fiddling, then, yields that so is

$$((A \vee C) \wedge ((A \to B) \vee C) \wedge (D \vee C)) \to (B \vee C).$$

Since $D \vee C$ is also a logical theorem, we can use R1 to adjunct the formulas $A \vee C$, $(A \to B) \vee C$ and $D \vee C$ so as to obtain the antecedent of the displayed conditional. R2 does the rest. ⊣

COROLLARY 2.2. *Any logic which has no primitive rule other than* R1 *and* R2 *and extends* **TW**[A16] *is disjunctive. Hence,* **T**, **E**, *and* **R** *are disjunctive.*[7]

PROOF. From lemma 2.1 with $D = p \to p$ for some p. That A16 is a theorem given A11 is well-known, and so the *hence*-claim follows. ⊣

LEMMA 2.2. *If $\boldsymbol{L}$ extends* **B**[A15, R7] *and has no more primitive rules than* R1, R2, *and* R7, *and is such that there is a D such that $\varnothing \vdash_{\boldsymbol{L}} D$ and that for all A, B, $\varnothing \vdash_{\boldsymbol{L}} A \wedge (A \to B) \overset{D}{\to} B$, then $\boldsymbol{L}$ is disjunctive.*

[7] That these logics are disjunctive is an obvious corollary of the proof of reasoning by cases in (Meyer and Dunn, 1969, pp. 461f).

PROOF. The proof is an extension of the above, and so I'll only consider $\mathrm{R}^{\mathrm{d}}7$. We need, then, to show that $\{A \vee C, (\sim A \vee B) \vee C\} \vdash_{\boldsymbol{L}} B \vee C$. The premises adjunct together to form $(A \vee C) \wedge ((\sim A \vee B) \vee C)$. Fiddling yields $(A \wedge (\sim A \vee B)) \vee C$ and then $(A \wedge \sim A) \vee ((A \wedge B) \vee C)$. Since A15, i.e. excluded middle, is an axiom, $\sim(A \wedge \sim A)$ is a logical theorem, and so $(A \wedge B) \vee C$ follows using R7. $B \vee C$ follows, then, by fiddling. ⊣

COROLLARY 2.3. *Any logic which has no more primitive rules than* R1, R2*, and* R7 *and extends* **TW**[A15, A16, R7] *is disjunctive. Hence,* **T**[R7], $\mathbf{\Pi}'(= \mathbf{E}[\mathrm{R7}])$*, and* **R**[R7] *are disjunctive.*

PROOF. Same as above corollary with the additional note that it is well-known that A15 is a logical theorem given A12. ⊣

Notice also that if $\boldsymbol{L}$ is any logic extending **B** and contained in $\mathbf{\Pi}'$, then $\boldsymbol{L}[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$ fails to be disjunctive: any logic with the $\circ$-principles yield that

$$\{A \wedge (A \to (B \to (C \to D)))\} \vdash_{\boldsymbol{L}} B \circ C \to D$$

but the $\mathbf{\Pi}'[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$-model depicted in figure 2 shows that

$$\{\sim A \vee (A \wedge (A \to (B \to (C \to D))))\} \nvdash_{\boldsymbol{L}} \sim A \vee (B \circ C \to D).$$

Similarly, any such logic yield that

$$\{B \wedge (A \to (B \to C))\} \vdash_{\boldsymbol{L}} C \leftarrow A,$$

but the $\mathbf{\Pi}'[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$-model depicted in figure 2 shows that

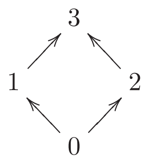
$$\{A \vee (B \wedge (A \to (B \to C)))\} \nvdash_{\boldsymbol{L}} A \vee (C \leftarrow A).$$

Generally, adding an axiom to a disjunctive logic will result in a disjunctive logic, but adding a new rule, as is the case in the examples above, need not.

2.2. Two easy lemmas

In order to show that the suggested frame condition for A13 truly does capture this axiom, we need to have ready two easy derivational facts regarding the axiom A13.

LEMMA 2.3. *The rule* $\{\Box A, \Box B\} \Vdash \Box(A \wedge B)$ *is derivable in* **B**[A13].



$\rightarrow$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	3	0	3	2
2	1	1	3	3	1
3	0	1	0	3	0

$\leftarrow$	0	1	2	3
0	3	2	0	0
1	3	3	1	1
2	3	2	2	2
3	3	3	3	3

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	1	2	3
3	0	1	2	3

$$\mathcal{T} = \{x \mid 3 \preceq x\} = \{3\}$$

$$\sim 1 \vee (1 \wedge (1 \rightarrow (2 \rightarrow (1 \rightarrow 0)))) = 3$$
$$\sim 1 \vee (2 \circ 1 \rightarrow 0) = 2$$

$$2 \vee (1 \wedge (2 \rightarrow (1 \rightarrow 0))) = 3$$
$$2 \vee (0 \leftarrow 2) = 2$$

Figure 2. A non-prime $\mathbf{\Pi}'[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\!\leftarrow, \mathrm{R}\!\leftarrow]$-model

Proof.

(1)	$(A \rightarrow A) \rightarrow A$	assumption
(2)	$(B \rightarrow B) \rightarrow B$	assumption
(3)	$(A \rightarrow A) \wedge (B \rightarrow B) \rightarrow A \wedge B$	1, 2, fiddling
(4)	$(A \wedge B \rightarrow A \wedge B) \rightarrow$ $((A \rightarrow A) \wedge (B \rightarrow B) \rightarrow A \wedge B)$	3, suffixing rule (R4)
(5)	$((A \rightarrow A) \wedge (B \rightarrow B) \rightarrow A \wedge B) \rightarrow A \wedge B$	A13
(6)	$(A \wedge B \rightarrow A \wedge B) \rightarrow A \wedge B$	4, 5, transitivity $\dashv$

Lemma 2.4. *$\{A \rightarrow B\} \Vdash \Box(A \rightarrow B)$ is a derivable rule in* $\mathbf{B}$[A13].

Proof. Let $C := A \rightarrow B$.

(1)	$A \rightarrow B$	assumption
(2)	$(B \rightarrow B) \rightarrow (A \rightarrow B)$	1, R4
(3)	$(A \rightarrow A) \wedge (B \rightarrow B) \rightarrow (A \rightarrow B)$	2, fiddling
(4)	$(C \rightarrow C) \rightarrow ((A \rightarrow A) \wedge (B \rightarrow B) \rightarrow C)$	3, R4
(5)	$((A \rightarrow A) \wedge (B \rightarrow B) \rightarrow C) \rightarrow C$	A13
(6)	$(C \rightarrow C) \rightarrow C$	4, 5, transitivity $\dashv$

The next task is to set forth the semantic machinery. Before doing so, however, a digression is given on a rather unfortunate feature of **E**, namely that several distinct logics have been thus named.

3. Several logics named 'E'

Great confusion can arise when distinct objects go by the same name. The is the case with **E** and has resulted in the impression that **E** fails to be even weakly complete with regards to the simplified semantics. This section gives an account of why the confusion arose to begin with. The section is divided into two subsection. The first shows that certain suggestions for how to axiomatize **E** are too weak — even when the target notion is restricted to logical theoremhood — whereas the second subsection deals with axiomatizations which utilize primitive rules which fail to be derivable in **E**.

3.1. Too weak axiomatizations

Meyer (1970, fn. 3) noted that "E has been around long enough to have picked up several alternative sets of axioms." Anderson and Belnap's first axiomatization of **E** used A13 as the characteristic axiom of **E** (cf. Anderson and Belnap, 1958). In a quest to separate the purely conditional features of **E** from its conjunctive features, however, they later came to prefer an axiomatization which replaces A13 by the following two axioms (cf. Anderson and Belnap, 1975, § 21.1):

$$\text{(A17)} \quad \Box A \wedge \Box B \to \Box(A \wedge B)$$
$$\text{(A18)} \quad ((A \to A) \to B) \to B$$

It is well-known that both A17 and A18 are logical theorems of **E** and that A13 is a logical theorem of **T**[A17, A18]. Note, however, that A17 fails in general to be derivable in logics with $A13$, but without the pre- and suffixing axioms A9 and A10. A counter-model for $\mathbf{B}^{\mathrm{d}}$[A13] is displayed in figure 3.

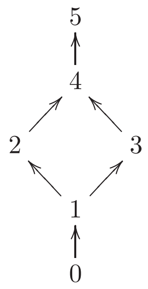
There is, however, a way of separating the implicational and conjunctive parts of A13 which works also for logics without the pre- and suffixing axioms:

Theorem 3.1. A13 *is a logical theorem in any logic extending* **B** *if and only if* A18 *is a logical theorem and* A17r *below is an admissible rule.*

$$\text{(A17r)} \quad \{\Box A, \Box B\} \Vdash \Box(A \wedge B)$$

Proof. A18 is evidently a theorem if A13 is, and lemma 2.3 showed that A17r is in fact *derivable* given A13 which entails that it is also admissible. To complete the proof it therefore suffices to show that A13

$\mathcal{T} = \{x \mid 2 \preceq x\} = \{2, 4, 5\}$
$\Box 2 \wedge \Box 3 \to \Box(2 \wedge 3) = 0$



$\to$	0	1	2	3	4	5	$\sim$	$\Box$
0	4	4	4	4	4	4	5	0
1	0	4	4	4	4	4	4	0
2	0	1	2	1	4	4	3	2
3	0	0	0	2	4	4	2	1
4	0	0	0	1	4	4	1	4
5	0	0	0	0	0	4	0	4

Figure 3. A $\mathbf{B}^{\mathrm{d}}$[A13] counter-model to A17

is a logical theorem given the admissibility of A17r and the theoremhood of A18. Let D in the following be $(A \to A) \wedge (B \to B)$

(1)	$\Box(A \to A)$	A18, def. of $\Box$
(2)	$\Box(B \to B)$	A18, def. of $\Box$
(3)	$(D \to D) \to D$	1, 2, A17r, def. of $\Box$
(4)	$(D \to C) \to ((D \to D) \to C)$	3, suffixing rule
(5)	$((D \to D) \to C) \to C$	A18
(6)	$(D \to C) \to C$	4, 5, transitivity $\dashv$

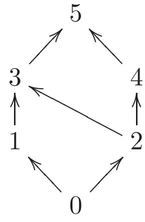
Belnap noted in his PhD thesis that the logic **P**, which amounts to **T**[A18], is "apparently inadequate" for deriving A17 (cf. Belnap, 1959, 1960, ch. 6.1). From the above results it follows that A17 is a logical theorem of **P** if and only if A17r is admissible in **P** if and only if A13 is a logical theorem of **P**, if and only if **P** is the same logic as **E**. That **P** is a proper sublogic of **E** follows, then, from the fact that the **P**-model displayed in figure 4 verifies the **P**-theorems $\Box(A \to A)$ and $\Box(B \to B)$ for every formula A and B, but fails to make true $\Box((A \to A) \wedge (B \to B))$.[8]

THEOREM 3.2. *The logical theorems of* **P** *is a proper subset of the logical theorems of* **E**.

This, then, shows that the logic called '**E**' in the supplement to (Mares, 2024b) is rather the logic **P**. Furthermore, the logic called '**E**' in (Mares, 2024a) is axiomatized in a way which amounts to **T**[A19], where

[8] Note also that the logic **T** was on occasion referred to using the letter 'P' (see, e.g., Kron, 1973) although this naming convention is long since obsolete.

$\mathcal{T} = \{x \mid 2 \preceq x\} = \{2, 3, 4, 5\}$
$\Box((1 \to 1) \wedge (2 \to 2)) = 0 \notin \mathcal{T}$



$\to$	0	1	2	3	4	5	$\sim$	$\Box$
0	5	5	5	5	5	5	5	0
1	0	4	0	5	0	5	4	0
2	0	0	3	3	5	5	3	0
3	0	0	0	3	0	5	2	3
4	0	0	0	0	4	5	1	4
5	0	0	0	0	0	5	0	5

Figure 4. A **P**-model in which A17r fails

A19 — called *specialised assertion* therein (cf. Mares, 2024a, § 9.9) — is the axiom

$$\text{(A19)} \quad (A \to B) \to (((A \to B) \to C) \to C).$$

T[A19] is, as the following theorem shows, identical to **P**, and therefore a proper sublogic of Anderson and Belnap's **E**.[9]

Theorem 3.3. **P** *and* **T**[A19] *are the same logic.*

Proof. It is evident that A18 is a logical theorem of **T**[A19]. That A19 is a logical theorem of **P** is seen from the following derivation, where for readability α is used as a stand-in for $A \to B$:

(1)	$\alpha \to ((B \to B) \to \alpha)$	A10
(2)	$((B \to B) \to \alpha) \to ((\alpha \to C) \to ((B \to B) \to C))$	A10
(3)	$((B \to B) \to C) \to C$	A18
(4)	$\big((\alpha \to C) \to ((B \to B) \to C)\big) \to ((\alpha \to C) \to C)$	3, R3
(5)	$\alpha \to ((\alpha \to C) \to C)$	3–5, trans. $\dashv$

This subsection started with Meyer's note regarding **E**'s many axiomatizations. He himself made use

$$\text{(A20)} \quad (A \to (B^{\Box} \to C)) \to (B^{\Box} \to (A \to C))$$

rather than A13, where $B^{\Box}$ is any formula on the form $D \to E$ or conjunctions thereof.[10] It is easy to see that this yields A13 as a logical

[9] Mares has pointed out to me in private communication that both the semantics and the natural deduction calculus set forth in (Mares, 2024a) do yield A17. The best interpretation, then, is that the axiomatizations provided in (Mares, 2024a,b) fail to yield the intended target and thus ought to be supplemented by A17.

[10] Note that if the "conjunctions thereof" part is left out, the resultant axiom is

$$\begin{aligned} \mathcal{T} = \{x \mid 1 \preceq x\} &= \{1, 2\} \\ 0 \to ((0 \to 2) \to 0) &= 1 \in \mathcal{T} \\ (0 \to 2) \to (0 \to 0) &= 0 \notin \mathcal{T} \\ (0 \to 2) \to ((0 \to 2) \to 1) \to 1) &= 0 \notin \mathcal{T} \end{aligned}$$

$$\begin{array}{c} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 0 \end{array}$$

$\to$	0	1	2	$\sim$
0	1	1	2	2
1	0	1	1	1
2	0	0	1	0

Figure 5. A $\mathbf{B}^{\mathrm{d}}$[A13]-model in which A19 and A20r fail

theorem. A20, although a logical theorem of **E**, fails to be derivable in many logics with A13. The model displayed in figure 5 validates all of **B**[A13], but fails to make even the rule version of Meyer's axiom truth preserving, nor does it validate Mares' specialised assertion axiom. Whether there are frame conditions which capture these axioms in any ternary semantics is, to my knowledge, an open question.

This subsection has shown that Anderson and Belnap's separation of the implicational and conjunctive features of A13 can also be achieved in weaker logics. However, since the frame conditions corresponding to A17r, A17 and A18 are at the time of writing unknown — if indeed such even exist — the current choice of an axiomatization of **E** will retain A13 as the characteristic axiom. We have also seen that **P** is indeed a proper sublogic of **E**, as indeed was hypothesized in Belnap's PhD thesis. The next section looks at axiomatizations which beget logics which are properly stronger than **E**. As we shall see, the main underlying issue here is that these axiomatizations beget logics which are theorem-wise identical, but which have different derivability relations.

3.2. Too strong axiomatizations — the δ-rule and the rule of restricted assertion

Ackermann stated his logic $\mathbf{\Pi}'$ with R1, R2, and R7 as its primitive rules which he named 'α,' 'β,' and 'γ,' respectively. However, rather than Anderson and Belnap's axiom A13, Ackermann made use of a forth primitive rule, namely the "δ-rule":

$$(\delta) \quad \{B, A \to (B \to C)\} \Vdash A \to C.$$

a logical theorem of **P**. This is, however, the characteristic axiom of **E** as set forth in both (Dunn, 1966, p. 4) and (Galatos et al., 2007, p. 104) which shows, then, that also these axiomatizations of **E** are too weak.

$\mathcal{T} = \{x \mid 1 \preceq x\} = \{1, 3\}$
$(1 \to 1) \to 1 = 0 \notin \mathcal{T}$
$2 \leftarrow 2 = 2 \notin \mathcal{T}$

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] & & 2 \arrow[ul] \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	3	0	3	2
2	0	0	3	3	1
3	0	0	0	3	0

$\leftarrow$	0	1	2	3
0	3	0	0	0
1	3	1	1	1
2	3	2	2	2
3	3	3	3	3

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

Figure 6. A $\mathbf{\Pi'}^{\mathrm{d}}[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$-model in which δ and restricted assertion fail

Ackermann explicitly stated that the first three rules are applicable also in the context of non-logical axioms, but argued that that cannot be the case with the δ-rule, since it would allow one to infer that A is necessary from the assumption that it is merely true (cf. Ackermann, 1956, pp. 125f). Ackermann therefore restricted the rule to be applicable only if B is a logical theorem.[11] Although noted countless times before, it is worth reminding that the unrestricted δ-rule is interderivable with R6, and that it is the case that for $\boldsymbol{L} \in \{\mathbf{E}, \mathbf{\Pi'}\}$, that if $\varnothing \vdash_{\boldsymbol{L}} B$, then $\{A \to (B \to C)\} \vdash_{\boldsymbol{L}} A \to C$. A13 is obviously a logical theorem given such a restricted δ-rule, and so it seems a fair judgement that the current Anderson-Belnap type axiomatization of $\mathbf{\Pi'}$ captures Ackermann's intended logic. Since it only involves one type of rule, it is also the preferable axiomatization of it.

Restall (1993) showed how to adequately model restricted assertion within the simplified semantics. Although not a new observation, it is for present purposes important to stress that that rule is not derivable in either $\mathbf{E}$ or $\mathbf{\Pi'}$. Figure 6 shows forth a model for the logic $\mathbf{\Pi'}$ as here defined in which the δ-rule and the interderivable restricted assertion rule fail to hold. To obtain a strongly sound and complete semantics for $\mathbf{E}$ and $\mathbf{\Pi'}$, then, the correct frame condition for the characteristic axiom of $\mathbf{E}$ — axiom A13 — must be obtained.

[11] The same type of restriction is intended with regards to Ackermann's ε-rule which, unlike rules α, β, and γ, is stated using the notion of provability: "If $A \to B$ and $(A \to B) \wedge C \to \curlywedge$ are both provable, then so is $C \to \curlywedge$" (Ackermann, 1956, p. 124) (my own translation). '$\curlywedge$' was Ackermann's symbol for his propositional constant.

Despite that the above remarks are not new, **E** is sometimes thought of as including the δ-rule or the interderivable rule of restricted assertion. One such example is found in Slaney's computer program `MaGIC` in which the logic `E` is defined as **T**[R6] (with two sets of propositional constants). If one is only interested in logical theorems, this is fine since R6 is admissible in **E**. If the semantics is to be strongly complete, however, it cannot treat a non-derivable rule as being truth preserving.[12]

All rules dealt with in SE4 — wherein the original Routley-Meyer semantics for **E** was first laid out — are formulated using the locution "from ___to infer …". It is worth underlining, then, that the notion of derivability intended in SE4 is that specified in (Routley et al., 1982, pp. 286f) which does not allow derivations from arbitrary premises.[13] Their claim, then, that **E** as defined here, but augmented with a propositional constant t axiomatized using $t \to (A \to A)$ and $(t \to A) \to A$ as axioms, is such as to make it the case that "the rule of Necessitation (NR): From A to infer t $\to$ A, is a derived rule of E" (Routley et al., 1982, p. 408) is true. It is so, however, because of their restrictive notion of derivability which makes the rules applicable only to logical theorems. That the necessity rule is *admissible* in **E** augmented with t is well-known. So is

[12] Let me stress that Slaney seems quite aware of this as he notes that one can model Anderson and Belnap's **E** by loading `T` and manually add to it the characteristic axiom of **E** (cf. Slaney, 1995, p. 54).

`MaGIC` is a brilliant tool. What makes it truly wonderful is the fact that Slaney has even given instructions for how to hard-code further axioms and rules into the program. One can, therefore, write ones one test routines which generally will speed up the processing time significantly. A test-routine fit for A13 is the following "group-3 test" (`MaGIC` is written in C):

```
/***   (((a -> a) & (b -> b)) -> c) -> c   ***/
boolean characterAxE_test(trs T)
{
  int a, b, c;

  FORaLL(a) FORaLL(b) FORaLL(c)
    if ( !ord[C[K[C[a][a]][C[b][b]]][c]][c] ) {
      Ref(impindex[a][a], T);
      Ref(impindex[b][b], T);
      Ref(impindex[K[C[a][a]][C[b][b]]][c], T);
      return false;
    }
  return true;
}
```

[13] SE4 is found in (Routley et al., 1982, pp. 407–424).

its nonderivability.[14] I should also like to note that SE4 lacks a definition of semantic consequence which would correspond to the notion of derivability used in this paper. The goal of SE4, then, is proof-theoretically and semantically only to specify logical theoremhood and logical truth along with the corresponding result of weak soundness and completeness.

I should stress that the **E**-claims pointed to are not incorrect. However, by using different definitions of **E**, or different accounts of consequence, what is claimed is easily misunderstood. This paper is intended to ever so slightly expand upon the results in (Restall, 1993). First of all, then, it is pertinent to note that although no formal definition of derivability is therein set forth, it is rather evident that his "provability relation" is what is captured by the current definition of a Hilbert proof (cf. Restall, 1993, fn. 1). Like `MaGIC`, Restall (1993) also defines **E** as **T**[R6]. Rather than discussing whether this is correct, however, Restall points to the fact that **T**[R6] fails to be disjunctive:

> It is to be noted that the simplified semantics given can only model *disjunctive systems*. That is, systems such that the disjunctive form of every truth preserving rule is truth preserving. Not every logic satisfies this criterion — a notable candidate is **E**, for the disjunctive form of its characteristic rule, from α to $(\alpha \to \beta) \to \beta$, fails to be truth preserving. The reason for this is that $\alpha \vee \neg\alpha$ is a theorem of **E**, but $\neg\alpha \vee ((\alpha \to \beta) \to \beta)$ is a non-theorem. (Restall, 1993, p. 482)

This, then, is correct provided **E** is identified as **T**[R6]. That **E** as defined in this paper, however, is disjunctive, was shown in lemma 2.2.[15] Note that Restall's justification for why disjunctive restricted assertion isn't truth preserving yields more information that what is strictly speaking

[14] For a model verifying this, let $[\![t]\!] = 3$ in the model in figure 6. It is then easy to verify that the model also makes the t-axioms true, but that the rule $\{A\} \Vdash t \to A$ fails to preserve truth.

[15] Restall's claim that **E** fails to be disjunctive is repeated in (Restall, 2000, p. 305) in which **E** is identified as **T** augmented by a truth constant t governed by the axiom $(t \to A) \to A$ and the rules $\{A\} \Vdash t \to A$ and $\{t \to A\} \Vdash A$ (as well as rules for the residuation for the fusion connective) (cf. ibid. p. 40). It is easy to show that restricted assertion is a derivable rule of such a logic, but that the resultant logic fails to be disjunctive, essentially due to the primitive t- and $\circ$-rules.

I should also note that it is pointed out in (Anderson et al., 1992, p. 172) that the frame condition which is adequate for the δ-rule — $Raga$ for every a in the type-2 setting where g, then, is the base point of the frame (cf. sect. 4.2) — is too strong to yield completeness in the case of **E** (as therein and herein defined), since it "would verify the nontheorem $\sim A \vee (A \to A \to A)$." The point is therein credited to Meyer.

necessary. To show that the rule isn't truth preserving it is necessary to show that $((A \to B) \to B) \vee C$ isn't derivable from $A \vee C$, something which follows from a strongly sound semantics in which it is possible to make the premise true while the conclusion not true. Instead of this, however, Restall points to the fact that there are even instances in which the premise is a *logical* truth, while the conclusion is not true. This, obviously, suffices, but it also shows more: Anderson and Belnap's axiomatization of **E** has restricted assertion as an *admissible* rule, and so is theorem-wise identical to Restall's **T**[R6]. His example, then, shows that there are admissible rules of **E** such that the disjunctive version of the rule fails to be admissible. What is emphatically not claimed in the above quote, then, is that the simplified semantics can only be given for logics which are such that the disjunctive version of every admissible rule is admissible. The justification given for Restall's claim simply shows *more* than is needed to support it.

I have shown forth three instances in which **E** has either been identified as a strictly stronger logic than **E** as defined in this paper, or defined with a restricted notion of derivability which simply disallows non-logical premises. Although the latter is not uncommon (see, e.g., Mares, 2000; Mares and Standefer, 2017) and in some cases makes for a smoother proof theory, it also makes it indeterminate what the intended unrestricted consequence relation for the logic is.

That **T**[R6] isn't disjunctive and therefore cannot be modeled using the simplified semantics is interesting. So is the fact that **T**[R6]'s set of logical theorems is properly contained in that of $\mathbf{T}^{\mathrm{d}}$[R6]. The more philosophically interesting question, however, is whether restricted assertion ought to be a derivable rule of a logic of entailment, which, after all, is the philosophically most interesting notion pertaining to **E**. Note, then, that **E** and $\mathbf{\Pi'}$ were from their very conceptions thought of as *modal* logics. Anderson and Belnap stressed that **E**'s conditional should be thought of as expressing the modal notion of entailment. Ackermann, on the other hand, stressed that a necessity operator could be defined in his logic if it was but extended with the propositional constant $\curlywedge$. Anderson and Belnap (1959) showed that such a constant isn't needed for this purpose, and that the current definition of $\Box$ has **S4**-properties in both **E** and $\mathbf{\Pi'}$. As emphasized in both (Ackermann, 1956) and in Anderson and Belnap's discussion of the δ-rule (cf. Anderson and Belnap, 1975, § 8.2), this modal feature is thwarted if the unrestricted δ-rule is added. One might, of course, hope to find a different definition of $\Box$ in **T**[R6] for which the necessity rule is merely admissible. Thus Ackermann's goal might be satisfied in such a logic. Reading a conditional for which the δ-rule is derivable as expressing entailment, however, seems out of the question since it yields that any contingent truth A is entailed by the necessary truth that A entails itself — $(A \to A) \to A$ follows from A in **T**[R6]. Neither **T**[R6] nor its disjunctive extension $\mathbf{T}^{\mathrm{d}}$[R6], then, are good replacements for **E** given Anderson and Belnap's programmatic ideas of what the conditional is meant to express.

Restricted assertion is admissible in **E**, whereas its disjunctive version fails to be. The referee pointed to the possibility of interpreting the notion of truth-preservation found in (Restall, 1993) as covering not only *derivable* rules but also *admissible*, and so to the possibility of taking Restall as therein claiming that the disjunctive version of merely admissible rules must also be admissible for the simplified semantics to be able to model the logic, and that restricted assertion is to be interpreted as a merely admissible rule in Restall's axiomatization of **E**. This would indeed make Restall's axiomatization of **E** equivalent to that of Anderson and Belnap. First of all, Restall's (1993) wording of every rule is uniform, and so there is nothing explicit which indicates that the rule is to be interpreted differently. This contrasts, then, to what is the case in (Ackermann, 1956). Secondly, when Restall's frame condition for restricted assertion is in place, his semantics yields that $((p \to p) \to p) \vee q$ is true in any model, if $\{p \vee q\}$ is, and so strong completeness would be lost if the proof theory only sanctions using the rules upon logical theorems. It is, in any case, a fact that **E** augmented with admissible disjunctive restricted assertion — $\mathbf{E}[\mathrm{R}^{\mathrm{d}}_{\mathrm{a}}6]$, let's say — does properly extend Anderson and Belnap's **E** even theoremwise, seeing as $\sim A \vee \Box A$ is a logical theorem of $\mathbf{E}[\mathrm{R}^{\mathrm{d}}_{\mathrm{a}}6]$, but not of **E**. No "decent modal logic," as Restall (2000, p. 305) put it, has such a logical theorem, and thus even though $\mathbf{E}[\mathrm{R}^{\mathrm{d}}_{\mathrm{a}}6]$ might be interesting to investigate, it is the purpose of this paper to show that the simplified semantics can indeed model at least two decent modal logics — **E** and $\mathbf{\Pi'}$. Furthermore, it is not merely the logical theorems of these logics which can be captured using the semantics; their full consequence relations are captured if the semantics is set up in the way detailed in the next section.

To sum up: There are several logics that have gone by the name of '**E**.' One of these — **T**[R6] — fails to be even weakly complete with regard to the simplified semantics. Nor is it disjunctive. However, Anderson and Belnap's version of **E** — the only logic that will be so called going forward — is disjunctive. That there is a version of the simplified semantics

E is strongly sound and complete with regard to is shown in the following three sections. From this result it follows, then, that a logic can have *admissible* rules the disjunctive version of which fail to be admissible, yet be adequately modeled using the ternary semantics.

4. Interpretations

The following accounts for the semantics as it is presented in (Restall, 1993). The only significant difference is the latter three requirements in def. 4.1(8) which are needed to allow the standard truth conditions for the converse conditions (condition *iii*) and fusion (condition *iv*) to work.[16] There are three subsection trailing the account of the simplified semantics. The first such accounts for the notion of a reduced frame within the current version of the simplified semantics. The second gives a short presentation of the two main differences regrding how the Routley-Meyer semantics is set up before the third subsection accounts for the origins of the definition of the set Z which will be used to capture the characteristic axiom of **E**.

DEFN 4.1. A FRAME is a quintuple $\mathcal{F} = \langle g, W, R, *, \sqsubseteq \rangle$ such that for all $a, b, x, y, z \in W$,

1. $g \in W$
2. $R \subseteq W^3$
3. $Rgxy \Leftrightarrow x = y$
4. $*\colon W \to W$[17]
5. $x^{**} = x$[18]

[16] As we shall see, these are needed for the proof of lemma 4.3 to go through. The requirements are new to this paper. The latter, I would like to note, is, however, a restricted version of that set forth in (Routley et al., 1982, pp. 365f) which reads

$$p6. \quad a \leq d \;\&\; Rbca >. \; Rdcd$$

There is an obvious typo in the consequent here, which ought to have been $Rbcd$. Although it is sufficient for strong soundness, the requirement is too strong for completeness within the simplified semantics as it would force $c = d$ if $b = g$ or both $c = g$ and permutation is in force.

[17] $*(a)$ will be written a^* and is called the *star-mate* of a.

[18] As the referee pointed out, this frame condition can be weakened. One possibility would be to rather require $x^{**} \sqsubseteq x$ & $x \sqsubseteq x^{**}$, where the first conjunct would correspond to the axioms $\sim\sim A \to A$ and the latter to $A \to \sim\sim A$. Since $\sqsubseteq$ is in general not antisymmetric, this would amount to a properly weaker demand.

6. $a \sqsubseteq a$
7. $a \sqsubseteq b \,\&\, b \sqsubseteq c \Rightarrow a \sqsubseteq c$
8.

$$a \sqsubseteq b \Rightarrow \left\{\begin{array}{lll} b^* \sqsubseteq a^* & & (i) \\ \hline Rbxy \Rightarrow Raxy & \text{if } a \neq g & (ii.a) \\ Rbxy \Rightarrow x \sqsubseteq y & \text{if } a = g & (ii.b) \\ \hline Rxby \Rightarrow Rxay & \text{if } x \neq g & (iii) \\ \hline Rxya \Rightarrow \exists z(y \sqsubseteq z \,\&\, Rxzb) & \text{if } x \neq g & (iv) \end{array}\right.$$

defn 4.2. A function $v : W \times \mathrm{PropVar} \to \{0,1\}$[19] is an evaluation function for a frame $\mathfrak{I} = \langle g, W, R, *, \sqsubseteq\rangle$ provided it satisfies the condition that

$$a \sqsubseteq b \Rightarrow (v(a,p) = 1 \Rightarrow v(b,p) = 1)$$

for every $p \in \mathrm{PropVar}$.

If v is an evaluation function on a frame $\mathcal{F}$, then $\mathfrak{M} = \langle \mathcal{F}, v\rangle$ is called a model.

defn 4.3. For every model there is a "true at"-relation $\vDash$ generated as follows, where a is any element in W:

$$\begin{array}{clcl} \text{(i)} & a \vDash p & \Leftrightarrow & v(a,p) = 1 \\ \text{(ii)} & a \vDash A \wedge B & \Leftrightarrow & a \vDash A \,\&\, a \vDash B \\ \text{(iii)} & a \vDash A \vee B & \Leftrightarrow & a \vDash A \;OR\; a \vDash B \\ \text{(iv)} & a \vDash {\sim}A & \Leftrightarrow & a^* \nvDash A \\ \text{(v)} & a \vDash A \to B & \Leftrightarrow & \forall x \forall y(Raxy \,\&\, x \vDash A \Rightarrow y \vDash B) \\ \text{(vi)} & a \vDash B \leftarrow A & \Leftrightarrow & \forall x \forall y(Rxay \,\&\, x \vDash A \Rightarrow y \vDash B) \\ \text{(vii)} & a \vDash A \circ B & \Leftrightarrow & \exists x \exists y(Rxya \,\&\, x \vDash A \,\&\, y \vDash B) \end{array}$$

- A formula A is true in a model, just in case $g \vDash A$.
- A rule $\{A_1, \ldots, A_n\} \Vdash B$ preserves truth in a model, just in case it is truth preserving at g: $(\forall i \leq n : g \vDash A_i) \Rightarrow g \vDash B$.
- Semantic consequence in a model: for any set $\Theta \cup \{A\}$,

$$\Theta \vDash_{\mathfrak{M}} A \Leftrightarrow (g \vDash B \text{ for every } B \in \Theta \Rightarrow g \vDash A).$$

It would, however, complicate the issue of deciding when the $\sqsubseteq$-reduct of a frame — $\langle g, W, R, *\rangle$ — suffices such as is the case with for instance $\mathbf{TW}^{\mathrm{d}}$-frames. Also, one can model logics with weaker double negation principles: For instance, by simply requiring that $g^{**} \sqsubseteq g$ ($g \sqsubseteq g^{**}$), one would obtain a semantics fit for the rule $\{{\sim}{\sim}A\} \Vdash A$ ($\{A\} \Vdash {\sim}{\sim}A$). It is beyond the scope of this paper to go further into this issue, however.

[19] That is, v is a propositional assignment function which for every point $w \in W$ and propositional variable p, assigns either the truth value 0 (false) or 1 (true).

- An **L**-MODEL is a model which satisfies all the frame conditions corresponding to the axioms and rules of $\boldsymbol{L}$.
- SEMANTIC CONSEQUENCE for a logic $\boldsymbol{L}$: for any set $\Theta \cup \{A\}$,

$$\Theta \vDash_{\boldsymbol{L}} A \Leftrightarrow \Theta \vDash_{\mathfrak{M}} A \text{ for every } \boldsymbol{L}\text{-model } \mathfrak{M}.$$

LEMMA 4.1. $g \vDash A \to (B \to A \circ B)$

PROOF. Let $Rgab$ with $a \vDash A$. To show that $b \vDash B \to A \circ B$, let c, d be any points such that $Rbcd$ with $c \vDash B$. Since $Rgab$, it follows that $a = b$ and therefore that $d \vDash A \circ B$. ⊣

LEMMA 4.2. $g \vDash A \to ((B \leftarrow A) \to B)$[20]

PROOF. To show that $g \vDash A \to ((B \leftarrow A) \to B)$, let a, b be any points such that $Rgab$ and $a \vDash A$. To show that $b \vDash (B \leftarrow A) \to B$, let c, d be any points such that $Rbcd$ with $c \vDash B \leftarrow A$. We must show that $d \vDash B$. Since $Rgab$, it follows that $a = b$ and therefore that $b \vDash A$. The truth condition for $\leftarrow$ now yields that $d \vDash B$. ⊣

LEMMA 4.3. *For any model $\mathfrak{M}$, with $a, b \in W$ and A any formula,*

$$a \sqsubseteq b \;\&\; a \vDash A \Longrightarrow b \vDash A.$$

PROOF. The proof is an induction on the complexity of formulas. The base case is immediate from the criteria for being an evaluation function.

Inductive hypothesis (IH):

Assume that B and C are any two formulas such that for every $a, b \in W$ and $D \in \{B, C\}$:

$$a \sqsubseteq b \;\&\; a \vDash D \Longrightarrow b \vDash D.$$

($\sim$) If $a \vDash {\sim}D$, then $a^* \nvDash D$, and since $b^* \sqsubseteq a^*$ follows from the assumption that $a \sqsubseteq b$, it follows from IH that $b^* \nvDash D$ and therefore that $b \vDash {\sim}D$.

($\vee$&$\wedge$) Immediate from IH.

($\to$) See (Restall, 1993, thm. 16).

($\leftarrow$) Let $a \sqsubseteq d$ with $a \vDash C \leftarrow B$. To show that $d \vDash C \leftarrow B$, let b, c be such that $Rbdc$ with $b \vDash B$. The proof ends if we can show that $c \vDash C$.

[20] One of the modally relevant properties of **E** and $\mathbf{\Pi}'$ is the so-called *Ackermann property* — that no logical theorem is on the form $A \to (B \to C)$ unless A has a least one subformula $D \to E$ (see Anderson and Belnap, 1975, § 22.1). Note, then, that any logic with either $\circ$ or $\leftarrow$ cannot satisfy the Ackermann property.

- Assume that $b = g$. Since $b \vDash B$, $g \vDash (C \leftarrow B) \rightarrow C$ by lemma 4.2. Since $Rgaa$ and $a \vDash C \leftarrow B$, it follows that $a \vDash C$, and so IH yields that $d \vDash C$. Since $b = g$ and $Rbdc$, it follows that $d = c$ and therefore that $c \vDash C$.
- Assume that $b \neq g$. It follows from def. 4.1(8)(iii) that $Rbac$. Since $a \vDash C \leftarrow B$ and $b \vDash B$, it then follows that $c \vDash C$.

In either case, then, $c \vDash C$. Since b and c were arbitrary, it follows that $d \vDash C \leftarrow B$.

($\circ$) Let $a \sqsubseteq d$ with $a \vDash B \circ C$. Then there are b, c such that $Rbca$ with $b \vDash B$ and $c \vDash C$. From lemma 4.1 above we have that $g \vDash B \rightarrow (C \rightarrow B \circ C)$ and since $Rgbb$, it follows that $b \vDash C \rightarrow B \circ C$.

- Assume first that $b = g$. Then $c = a$, and since $a \sqsubseteq d$ it follows from IH that $d \vDash C$. Since $Rgdd$ it therefore follows that $d \vDash B \circ C$.
- If $b \neq g$, then by def. 4.1(8) there is some e such that $c \sqsubseteq e$ with $Rbed$. It follows from IH that $e \vDash C$ and therefore that $d \vDash B \circ C$. $\dashv$

DEFN 4.4. For any frame $\langle g, W, R, *, \sqsubseteq \rangle$ with $a, b, c, d, \in W$,

1. $Z := \{a \mid \forall x \forall y (Raxy \Rightarrow x \sqsubseteq y)\}$
2. $R^2abcd := \exists x(Rabx \;\&\; Rxcd)$
3. $R^2a(bc)d := \exists x(Rbcx \;\&\; Raxd)$

LEMMA 4.4. *For any frame $\langle g, W, R, *, \sqsubseteq \rangle$ with $a, b, c \in W$,*

1. $g \in Z$
2. $a \in Z \;\&\; Rabc \Rightarrow b \sqsubseteq c$
3. $a \in Z \;\&\; a \sqsubseteq b \Rightarrow b \in Z$

PROOF. 1. If $Rgab$, then $a = b$ by def. 4.1(3), and since $a \sqsubseteq a$ by def. 4.1(6), it follows that $g \in Z$.

2. This follows trivially from the definition of Z.

3. Assume that $a \in Z \;\&\; a \sqsubseteq b$. In order to show that $b \in Z$, let $c, d \in W$ be such that $Rbcd$. Since $a \sqsubseteq b$ and $Rbcd$, it follows from def. 4.1(8) that either $Racd$ or $c \sqsubseteq d$. If the first, then also $c \sqsubseteq d$ since $a \in Z$. The assumption that $Rbcd$, then, yields the conclusion that $c \sqsubseteq d$, and so it follows that $b \in Z$. $\dashv$

Restall (1993) showed that any logic extending the positive fragment of the weak relevant logic **B** with any collection of the axioms A11–A12 and A14–A16 (and many more) as well as the rule of restricted assertion, will be strongly sound and complete with regards to models which

	Frame condition
$\mathcal{F}$(A8)	$Rabc \Rightarrow Rac^*b^*$
$\mathcal{F}$(A9)	$R^2abcd \Rightarrow R^2a(bc)d$
$\mathcal{F}$(A10)	$R^2abcd \Rightarrow R^2b(ac)d$
$\mathcal{F}$(A11)	$Rabc \Rightarrow R^2abbc$
$\mathcal{F}$(A12)	Raa^*a for $a \neq g$, and $g^* \sqsubseteq g$
$\mathcal{F}$(A14)	$Rabc \Rightarrow \exists x(a \sqsubseteq x \;\&\; Rbxc)$
$\mathcal{F}$(A15)	$g^* \sqsubseteq g$
$\mathcal{F}$(A16)	$Raaa$
$\mathcal{F}$(R6)	$Raga$

Table 1. Restall's frame conditions

	Frame condition
$\mathcal{F}$(R7)	$g \sqsubseteq g^*$
$\mathcal{F}$(A13)	$\exists x(x \in Z \;\&\; Raxa)$

Table 2. Frame conditions for R7 and A13

satisfies the corresponding frame conditions, provided, that is, that the logic in question is disjunctive — either because the missing disjunctive rules are added, or that it is provably disjunctive from the get-go. The goal of the rest of this paper is to show that this is also the case with regards to the frame conditions listed in table 2. As easy corollaries, then, it will follow that also logics such as **E** and **Π′** are indeed strongly sound and complete with regards to the simplified semantics.

Restall (1993) defined frames in a slightly different way. For logics such as $\mathbf{B}^{\mathrm{d}}$ and $\mathbf{TW}^{\mathrm{d}}$, Restall used $\sqsubseteq$-free frames, and only added this frame component so as to model certain axioms and rules. The following theorem makes up for the current unnecessary frame clutter:

THEOREM 4.1. 1. *If **L** is a logic without* $\leftarrow$*, then frame condition 8.iii can be dropped.*
2. *If **L** is a logic without* $\circ$*, then frame condition 8.iv can be dropped.*
3. *If **L** is a logic without* $\sim$*, then frame conditions 4, 5, and 8.i can be dropped.*
4. *8.i can be dropped in any frame in which* $\mathcal{F}$(A8) *holds.*
5. *If **L** is a logic which is such that neither of its specific frame conditions (those listed in tables 1&2) do no involve* $\sqsubseteq$*, then the* $\sqsubseteq$*-reduct of any frame suffices for validating every axiom and rule of the logic.*

Proof. By simply inspecting the above proofs it is readily seen that 8.iii is only needed in connection with $\leftarrow$. Similarly for $\circ$ and $\sim$. Regarding $\sqsubseteq$: Note that $=$ satisfies all the basic requirements on $\sqsubseteq$. Thus if no additional requirements are set forth, any frame $\langle g, W, R, *, \sqsubseteq\rangle$ can be replaced by $\langle g, W, R, *, =\rangle$, and since identity is definable for any frame, it follows that the reduct frame $\langle g, W, R, *\rangle$ does the same job as $\langle g, W, R, *, \sqsubseteq\rangle$.[21] $\dashv$

4.1. Reduced frames

The following explains the notion of a *reduced* frame, and thus in part why this paper is named as it is.

defn 4.5. A frame/model in which $Z = \{x \mid g \sqsubseteq x\}$ is called reduced, and unreduced if not.

In order to obtain a strong completeness proof for logics like **E** we need to be able to set the semantics up so that A13 holds, but the rule of restricted assertion does not. The frame condition $\mathcal{F}$(A13) is, as we shall see, correct for modeling A13. As the following result shows, this isn't possible if we demand reduced frames.

Proposition 4.1. *$\{A\} \vDash_{\mathfrak{M}} (A \to B) \to B$ in any reduced model $\mathfrak{M}$ in which the underlying frame satisfies $\mathcal{F}$(A13).*

Proof. Assume that $\mathfrak{M}$ is reduced, and let A be any formula such that $g \vDash A$. To show that $g \vDash (A \to B) \to B$ it suffices to show that for any a, if $a \vDash A \to B$, then $a \vDash B$. Assume, then, that $a \vDash A \to B$. $\mathcal{F}$(A13) yields that $Raba$ for some $b \in Z$, and since $\mathfrak{M}$ is reduced it follows that $g \sqsubseteq b$. Lemma 4.3 yields that $b \vDash A$, and so $a \vDash B$. $\dashv$

It follows, then that since restricted assertion isn't a derivable rule of either **E** or $\mathbf{\Pi'}$, that we need to allow for unreduced models in order to obtain a strong completeness proof for these logics.

Note that Z-points do not necessarily verify every logical theorem. We saw in lemma 2.4 that every true $\to$-formula is necessarily true given A13. However, not every logical truth need be validated at Z-points. In **TW**[A11, A13, A15] — **E** but with the "reductio" axiom $(A \to \sim A) \to \sim A$ replaced by the strictly weaker axiom of excluded middle — for instance,

[21] This is a somewhat informal proof, but I trust that it suffices for convincing the reader of the truth of the theorem.

$$\begin{array}{rcl} W & := & \{g, g^*\} \\ R & := & \{\langle g, g, g\rangle, \langle g, g^*, g^*\rangle, \langle g^*, g^*, g^*\rangle\} \\ \sqsubseteq & := & \{\langle g^*, g\rangle\} \\ v(g, p) & := & 1 \text{ for every prop. variable } p \\ v(g^*, p) & := & 0 \text{ for every prop. variable } p \end{array}$$

Figure 7. A **TW**[A11, A13, A15]-model with an irregular Z-point

there can be Z-points at which neither A nor $\sim A$ hold true. In short, Z-points need not be *regular* in the sense of making every logical theorem true. The **TW**[A11, A13, A15]-model displayed in figure 7 is easily seen to yield $Z = W$.[22] However, even though $g^* \in Z$ and excluded middle is a logical axiom, $g^* \not\vDash p \vee \sim p$ for every propositional variable p. In logics like **E**, however, one can prove that the rule $\{A\} \Vdash \Box A$ is admissible.[23] Models for **E** will therefore only have Z-points which verify every logical theorem of **E**, although such points in unreduced models need not verify everything true at the base point g.

As we shall see (lemma 5.2), $g \vDash A \to B$ if and only if $z \vDash A \to B$ for every $z \in Z$. In short, then, any Z-point, regardless of the logic at hand, must validate every true "inference tickets," but need not, as we saw, validate merely true formulas, nor, assuming that the logic isn't too strong, even *logical* truths. This, then, tells against Slaney's argument agains excluded middle by showing that unreduced models allow for the needed place wherein inference tickets can be differentiated from mere material truths:

> Presumably the motivation for excluded middle is semantic, that it comes out true "no matter what"; but logic is supposed to sort out what follows from what, and as such has surely no place for these material tautologies which just sit around being true and are no inference tickets at all. (Slaney, 1984, p. 161)

4.2. Four versions of the Routley-Meyer semantics

The Routley-Meyer semantics can be set up in various ways. The most important difference, however, pertains to how semantical consequence

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[22] I leave it as an exercise to verify that this is in fact a model for **TW**[A11, A13, A15].

[23] For a proof, see for instance (Mares and Standefer, 2017, § 3) or (Øgaard, 2021a, thm. 1).

is defined. Here there are two approaches. The main difference between these is whether a single base point is required — in which case semantic consequence is defined as truth preservation of it — or a non-empty set of regular points all closed under logical consequence is used in which case semantic consequence is defined as truth preservation over all such.[24] The "unsimplified" original Routley-Meyer semantics set forth in (Routley et al., 1982, ch. 4) as well as in (Méndez, 2009), is of the latter kind. This is also the case for the simplified semantics presented in (Priest, 2008, ch. 10). Let's call this the a type-1 semantics. A frame is called "reduced" in the type-1 semantics just in case it contains but a single such normal point.[25]

The definition of semantical consequence in (Restall, 1993) as well as in this paper makes use of a single base point g. Let's call this a type-2 semantics. A reduced type-1 frame, then, is evidently a type-2 frame, although not necessarily a reduced type-2 frame. Although SE4 lacks a definition of semantical consequence corresponding to Hilbert consequence, the one belonging to it is the type-2 version which employs a single base point. This, then, is the reason why the current version of the Routley-Meyer semantics ought to be viewed as SE4's simplified edition.

There is nothing to indicate that the choice between a type-1 and a type-2 set-up has any impact on which logical principles can be modeled. It is worth pointing out, however, that a frame condition adequate for a given logical axiom or rule may need some tinkering when moving between the two types.

4.3. The origins of the definition of Z

In the semantics set forth in SE4, a primitive frame component P is required which is defined to satisfy the property corresponding to frame condition lemma 4.4(2), namely

$$\exists x(Px \,\&\, Rxab) \Rightarrow a \sqsubseteq b,$$

for any points a, b. The true-at condition for the propositional constant t therein used is then specified as $a \vDash t \Leftrightarrow \exists x(Px \,\&\, x \sqsubseteq a)$. Furthermore,

[24] Such points are called *normal* in (Priest, 2008, ch. 10). Note, then, that that term is most often used to refer to points a for which $a = a^*$ — points which are consistent and complete.

[25] That $\sim A \vee ((A \to A) \to A$ holds in reduced type-1 frames was pointed out in (Maksimowa, 1973, p. 20).

the frame condition for the characteristic axiom of **E** — A13 — was, to my knowledge, first set forth in SE4 (cf. Routley et al., 1982, p. 411) and therein specified as

$$\forall x \exists y (y \in P \;\&\; Rxyx).$$

The authors of SE4 — Sylvan (né Routley) and Meyer — state that

> P is required to cope with E in view of the way E introduces necessity as part of the logic of entailment [...]. Admittedly, the presence of P, and its apparent uneliminability, makes the semantics of $\underline{\underline{\text{E}}}$ more cumbersome and less attractive than that of some of its relevant rivals such as R and T. (Routley et al., 1982, p. 407)

A way to define such a set P within the unsimplified type-2 semantics, however, was suggested in (Anderson et al., 1992), where it is noted that

> a semantics can be given for **E** which is based on no elements other than those required for the nonmodal calculus **R**. The following is due to Meyer (unpublished), though we have played with the details. [...] we add a key definition, answering to the modal character of **E**, describing a set-up a as verifying all those entailments verified at the base set-up 0: Za iff, for every x, y, if Raxy then R0xy.
> (Anderson et al., 1992, p. 171)

Defining Z this way within the simplified semantics would yield that $Raxy$ only if $x = y$ for any $a \in Z$. Although this suffices for soundness, the completeness proof does not seem to go through if this is required. Note, however, that the original Routley-Meyer semantics of the above quote makes use of a defined binary relation $a \leq b := R0ab$ rather than the primitive $\sqsubseteq$ of the simplified semantics. It is, then, the intent, rather than the character, of the above quote which is the backdrop of the current definition of Z.

5. Soundness

The following lemma allows for slightly shorter proofs in that in order to establish that $g \vDash A \to B$ it suffices to show that $a \vDash B$ for any $a \in W$ such that $a \vDash A$. I will in the following do so without reference to the lemma obtained given def. 4.1(3) and def. 4.3(v).

Lemma 5.1. $g \vDash A \to B \iff \forall x \in W (x \vDash A \Rightarrow x \vDash B)$.

The above lemma is often glossed as entailment in a model being representable as truth preservation over all points. Note, then, that entailment is also representable as "supertruth" at Z-points:

LEMMA 5.2. *In any model*: $g \vDash A \to B \iff \forall z(z \in Z \Rightarrow z \vDash A \to B)$.

PROOF. ($\Rightarrow$): Assume that $z \in Z$ and that $g \vDash A \to B$. Let $Rzab$ with $a \vDash A$. By definition of Z it follows that $a \sqsubseteq b$, and so $b \vDash A$ by lemma 4.3. Since $Rgbb$ it then follows that $b \vDash B$ and therefore that $z \vDash A \to B$. ($\Leftarrow$): This follows from the fact (lemma 4.4) that $g \in Z$. ⊣

It follows from the above lemma that $A \to A$ holds at every Z-point:

COROLLARY 5.1. *In any model with* $z \in Z$*:* $z \vDash A \to A$.

LEMMA 5.3 ($\mathcal{F}$(R7) ⇝ R7). R7, *as well as its disjunctive version, preserves truth in any model which satisfies* $\mathcal{F}$(R7).

PROOF. Assume that $g \vDash A$ and $g \vDash {\sim}A \vee B$. Then either $g \vDash {\sim}A$ or $g \vDash B$. If $g \vDash {\sim}A$, then it follows form def. 4.3(iv) that $g^* \nvDash A$. However, since $g \vDash A$ and $g \sqsubseteq g^*$ by $\mathcal{F}$(R7), it follows from lemma 4.3 that $g^* \vDash A$. Contradiction. Thus it must be the case that $g \vDash B$.

The case for the disjunctive version of R7, the rule

$$\{A \vee C, ({\sim}A \vee B) \vee C\} \Vdash B \vee C,$$

is similar and left for the reader. ⊣

LEMMA 5.4 ($\mathcal{F}$(A13) ⇝ A13). A13 *holds in any model which satisfies* $\mathcal{F}$(A13).

PROOF. In order to show that $g \vDash (((A \to A) \wedge (B \to B)) \to C) \to C$, assume that $a \vDash ((A \to A) \wedge (B \to B)) \to C$. By $\mathcal{F}$(A13), let b be such that $b \in Z$ and $Raba$. That $b \vDash (A \to A) \wedge (B \to B)$ follows from corollary 5.1, and so by the true-at clause for the conditional it follows that $a \vDash C$. ⊣

LEMMA 5.5. *The axioms and rules of* $\mathbf{B}^{\mathrm{d}}$ *are true / preserve truth in any model. Moreover,* R6, *as well as it's disjunctive version, preserves truth in any model, and* A8–A12 *and* A14–A16 *hold true in any model in which the corresponding frame condition holds.*

PROOF. For A14, see (Restall and Roy, 2009); for the rest, (Restall, 1993). ⊣

LEMMA 5.6. A∘, R∘, *and* R^{d}∘ *hold in any model.*

PROOF. (A∘) See lemma 4.1.

(R$\circ$) Assume that $g \vDash A \to (B \to C)$. Let c be any point such that $c \vDash A \circ B$. Then there are points a, b such that $Rabc$ with $a \vDash A$ and $b \vDash B$. Since $Rgaa$, it follows that $a \vDash B \to C$ from which it follows that $c \vDash C$ and therefore that $g \vDash A \circ B \to C$.

(R$^{\mathrm{d}}\circ$) Similar to the non-disjunct version. $\dashv$

LEMMA 5.7. A$\leftarrow$, R$\leftarrow$, *and* R$^{\mathrm{d}}\leftarrow$ *hold in any model.*

PROOF. (A$\leftarrow$) See lemma 4.2.

(R$\leftarrow$) Assume that $g \vDash A \to (B \to C)$. In order to show that $g \vDash B \to (C \leftarrow A)$, let b be any point such that $b \vDash B$. To show that $b \vDash C \leftarrow A$, let a, c be any points such that $Rabc$ with $a \vDash A$. It follows that $a \vDash B \to C$, and since $b \vDash B$ that $c \vDash C$. It follows, then, that $b \vDash C \leftarrow A$ and therefore that $g \vDash B \to (C \leftarrow A)$.

(R$^{\mathrm{d}}\leftarrow$) Similar to the non-disjunct version. $\dashv$

We have now seen that the axioms and rules all hold true provided the corresponding frame conditions are enforced. As an easy corollary, then, we have the following result:

THEOREM 5.1 (Strong soundness). $\Sigma \vdash_{\boldsymbol{L}} A \Longrightarrow \Sigma \vDash_{\boldsymbol{L}} A$, *where* $\boldsymbol{L}$ *is any disjunctive logic obtainable from* **B** *by adding any number of the axiom and rules* (*along with their disjunctive versions*) *listed in section 2.*

6. Completeness

The goal of this section is to prove that for any set of formulas $\Sigma \cup \{A\}$,

$$\Sigma \vDash_{\boldsymbol{L}} A \Longrightarrow \Sigma \vdash_{\boldsymbol{L}} A,$$

where $\boldsymbol{L}$ is as in the above soundness theorem. Most of what is needed to make the proof go through was provided either in (Routley et al., 1982), Priest and Sylvan (1992), Restall (1993), or in (Restall, 2000). As previous such proofs, the current one will use the assumption that $\Sigma \nvdash_{\boldsymbol{L}} A$ to make a model in which each element of W is a set of formulas.

DEFN 6.1. • For any set Π of formulas, $\Pi_{\to}$ is the set of all members of Π on the form $A \to B$.
- $\Sigma \vdash_{\Pi} A := \Sigma \cup \Pi_{\to} \vdash A$.
- Σ is a Π-theory :=
  - (a) $A, B \in \Sigma \Rightarrow A \wedge B \in \Sigma$
  - (b) $\varnothing \vdash_{\Pi} A \to B \Rightarrow (A \in \Sigma \Rightarrow B \in \Sigma)$

- Σ is prime $:= A \vee B \in \Sigma \Rightarrow (A \in \Sigma$ or $B \in \Sigma)$.
- If X is any set of sets of formulas, then $\overline{R} \subseteq X^3$ is defined thus:

$$\overline{R}\Sigma\Lambda\Theta \Leftrightarrow \forall A \forall B(A \to B \in \Sigma \Rightarrow (A \in \Lambda \Rightarrow B \in \Theta))$$

- Σ is Π-deductively closed $:= \Sigma \vdash_{\Pi} A \Rightarrow A \in \Sigma$.
- Σ is non-trivial $:= A \notin \Sigma$ for some formula A, and $\Sigma \neq \varnothing$.
- Σ is Π-canonical $:= \Sigma$ is a prime and non-trivial Π-theory.

From (Priest and Sylvan, 1992, p. 224) we have:

Lemma 6.1. *If $\Sigma \nvdash A$ then there is a $\Pi \supseteq \Sigma$ such that*:

- *$A \notin \Pi$.*
- *Π is a prime Π-theory.*
- *Π is Π-deductively closed.*

The points of the canonical model will consists of the set of Π-canonical theories. Note that I've required that each such be non-trivial. The relevant lemmas in (Priest and Sylvan, 1992) and (Restall, 1993) support this, but the requirement can for present purposes be lifted.

Although Π itself may in many cases play the role of the base point g, Restall (1993) noted that one needs a copy of Π to make the frame condition corresponding to the contraction axiom A11 hold true in the canonical model. It was pointed out in (Restall and Roy, 2009) that this is still insufficient when the logic in question additionally satisfies permutation-type principles such as

$$A \to ((A \to B) \to B)$$
$$(A \to (B \to C)) \to (B \to (A \to C))$$

I'll follow the second adequate fix suggested in (Restall and Roy, 2009) with, then, one copy of Π, and one copy of Π's star-mate Π^*.

defn 6.2 (The canonical frame & model). If Π is a prime, non-trivial, and deductively closed theory of a logic $\boldsymbol{L}$, then the canonical frame $\mathcal{C}_{\underline{\Pi}} = \langle g, W, R, *, \sqsubseteq \rangle$ is defined as follows:

1. $W := \{\langle \Sigma, 0 \rangle \mid \Sigma$ is Π-canonical$\} \cup \{\langle \Pi, 1 \rangle, \langle \Pi^*, 1 \rangle\}$
2. $g := \langle \Pi, 1 \rangle$
3. $R\langle \Pi, 1 \rangle \langle \Gamma, j \rangle \langle \Delta, k \rangle \Leftrightarrow \langle \Gamma, j \rangle = \langle \Delta, k \rangle$
4. For $\langle \Sigma, i \rangle \neq \langle \Pi, 1 \rangle$:

$$R\langle \Sigma, i \rangle \langle \Gamma, j \rangle \langle \Delta, k \rangle \Leftrightarrow \forall A \forall B(A \to B \in \Sigma \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta))$$

5. $\Sigma^* := \{A \mid {\sim}A \notin \Sigma\}$
6. $\langle \Sigma, i\rangle^* := \langle \Sigma^*, i\rangle$
7. $\langle \Sigma, i\rangle \sqsubseteq \langle \Delta, j\rangle \Leftrightarrow \Sigma \subseteq \Delta$

The CANONICAL MODEL for a canonical frame $\mathfrak{C}_{\underline{\Pi}}$ is given by the evaluation function

$$v(\langle \Sigma, i\rangle, p) = 1 \Leftrightarrow p \in \Sigma.$$

To avoid too much notational clutter, I will simple use uppercase Greek letters to refer to members of W. To avoid confusion, then, I'll write '$\underline{\Pi}$' for $\langle \Pi, 1\rangle$, '$\underline{\Pi}^*$' for $\langle \Pi^*, 1\rangle$.

LEMMA 6.2. *For any Π-canonical sets Σ, Δ, Θ such that $\Sigma \neq \underline{\Pi}$,*

$$\overline{R}\Sigma\Delta\Theta \Leftrightarrow R\Sigma\Delta\Theta$$

LEMMA 6.3. *For any Π-canonical sets Σ, Δ, Θ,*

$$R\Sigma\Delta\Theta \Rightarrow \forall A \forall B(A \to B \in \Sigma \Rightarrow (A \in \Gamma \Rightarrow B \in \Delta))$$

PROOF. By definition in the case where $\Sigma \neq \underline{\Pi}$, and in virtue of $\Delta = \Theta$ being a Π-theory in the case $\Sigma = \underline{\Pi}$. ⊣

THEOREM 6.1. *If Π is a non-trivial, prime and Π-deductively closed Π-theory, then the canonical frame $\mathcal{C}_{\underline{\Pi}}$ is a frame.*

PROOF. We need to check that $\mathcal{C}_{\underline{\Pi}}$ satisfies def. 4.1(1–8).

(1–4) That $\underline{\Pi} \in W$, $R \subseteq W^3$, $R\underline{\Pi}\Delta\Theta \Leftrightarrow \Delta = \Theta$, and that $*$ is a function on W is obvious.

(5) For any $\Delta \in W$, $A \in \Delta \Leftrightarrow {\sim}{\sim}A \in \Delta$ (since $\vdash_\Pi {\sim}{\sim}A \leftrightarrow A$). Furthermore, ${\sim}{\sim}A \in \Delta \Leftrightarrow {\sim}A \notin \Delta^* \Leftrightarrow A \in \Delta^{**}$, and so $\Delta = \Delta^{**}$.

(6–7) Reflexivity and transitivity are basic properties of subsethood.

(8) Assume that $\Gamma, \Delta \in W$ are such that $\Gamma \subseteq \Delta$.

(a) $\Delta^* \subseteq \Gamma^*$. Indeed, assume that $A \in \Delta^*$. By definition of $*$ in $\mathcal{C}_{\underline{\Pi}}$ it follows that ${\sim}A \notin \Delta$. But then since $\Gamma \subseteq \Delta$ also ${\sim}A \notin \Gamma$, and so $A \in \Gamma^*$.

(b) $\Gamma \neq \underline{\Pi} \Rightarrow (R\Delta\Theta\Psi \Rightarrow R\Gamma\Theta\Psi)$. Indeed, assume that $\Gamma \neq \underline{\Pi}$ and let $\Delta, \Theta, \Psi \in W$ be such that $R\Delta\Theta\Psi$. Let A, B be any formulas such that $A \to B \in \Gamma$ and $A \in \Theta$.Since $\Gamma \subseteq \Delta$, it follows that $A \to B \in \Delta$, and since $R\Delta\Theta\Psi$ that $B \in \Psi$ by the definition of R in $\mathcal{C}_{\underline{\Pi}}$. By the same definition it finally follows that $R\Gamma\Theta\Psi$.

(c) $\varGamma = \underline{\varPi} \Rightarrow (R\varDelta\varTheta\varPsi \Rightarrow \varTheta \subseteq \varPsi)$. Indeed, assume that $\varGamma = \underline{\varPi}$ and let $\varTheta, \varPsi \in W$ be such that $R\varDelta\varTheta\varPsi$. Let $A \in \varTheta$. $A \to A \in \underline{\varPi} \subseteq \varDelta$, and so by the definition of R it follows that $A \in \varPsi$. Thus $\varTheta \subseteq \varPsi$.

(d) $\varTheta \neq \underline{\varPi} \Rightarrow (R\varTheta\varDelta\varPsi \Rightarrow R\varTheta\varGamma\varPsi)$. Indeed, assume that $\varTheta \neq \underline{\varPi}$ and that $R\varTheta\varDelta\varPsi$. To show that $R\varTheta\varGamma\varPsi$, let $A \to B \in \varTheta$ and $A \in \varGamma$. Since $\varGamma \subseteq \varDelta$, it follows that $A \in \varDelta$, and since $R\varTheta\varDelta\varPsi$ that $B \in \varPsi$.

(e) $\varPsi \neq \underline{\varPi} \Rightarrow (R\varPsi\varTheta\varGamma \Rightarrow \exists\varXi(\varTheta \subseteq \varXi \;\&\; R\varPsi\varXi\varDelta))$. Indeed, assume that $R\varPsi\varTheta\varGamma$ with $\varPsi \neq \underline{\varPi}$. We have to find a $\varXi$ such that $\varTheta \subseteq \varXi$ and $R\varPsi\varXi\varDelta$. The following definition of $\varXi$ is so as to avoid forcing $\varPsi$ to be identical to $\varDelta$ in case $\varTheta = \underline{\varPi}$ and a frame condition yielding $Ragb \Rightarrow Rgab$ is in effect:

$$\varXi := \begin{cases} \varTheta & \text{if} \quad \varTheta \neq \underline{\varPi} \\ \varPi & \text{if} \quad \varTheta = \underline{\varPi} \end{cases}$$

$\underline{\varPi}$ and $\varPi$ are identical in terms of the subset-relation, and so it will be the case that $\varTheta \subseteq \varXi$ for every $\varTheta$.[26] Let A, B be any formulas such that $A \to B \in \varPsi$ and $A \in \varXi$. Then $A \in \varTheta$, and since $R\varPsi\varTheta\varGamma$ it follows that $A \in \varGamma \subseteq \varDelta$. By the definition of R in the canonical model, then, it follows that $R\varPsi\varXi\varDelta$. ⊣

Lemma 6.4. *For any $\varPi$-canonical theory $\varSigma$ of any canonical model,*

$$A \in \varSigma \Leftrightarrow \varSigma \vDash A.$$

Proof. The base case holds by definition of the valuation function of the canonical model. The inductive case for any connective different from fusion and the converse conditional is covered in (Restall, 1993, thm. 8). Fusion is covered in (Routley et al., 1982, pp. 365f).

($\leftarrow$): Assume that $C \leftarrow B \in \varSigma$. In order to show that $\varSigma \vDash C \leftarrow B$, let $\varDelta, \varGamma$ be any canonical $\varPi$-theories such that $R\varDelta\varSigma\varGamma$ with $\varDelta \vDash B$. We may for inductive hypothesis assume that the claim holds for B and C, and therefore that $B \in \varDelta$. Since $\varDelta$ is a $\varPi$-theory it follows that $(C \leftarrow B) \to C \in \varDelta$ which by definition of R then yields that $C \in \varGamma$. The inductive hypothesis then yields that $\varGamma \vDash C$, and so it follows that $\varSigma \vDash C \leftarrow B$.

Assume that $\varSigma \vDash C \leftarrow B$, and for contradiction that $C \leftarrow B \notin \varSigma$. By (Restall, 2000, lem. 11.29) there are then $\varPi$-canonical theories $\varDelta$ and

[26] Note, then, that one can strengthen the frame requirement of def. 4.1.8(iv) to

$$a \sqsubseteq b \Rightarrow ((x \neq g \;\&\; Rxya) \Rightarrow \exists z(y \equiv z \;\&\; Rxzb)),$$

where $a \equiv b := a \sqsubseteq b \;\&\; b \sqsubseteq a$.

$\mathit{\Gamma}$ such that $R\mathit{\Delta}\mathit{\Sigma}\mathit{\Gamma}$ with $B \in \mathit{\Delta}$ and $C \notin \mathit{\Gamma}$. By the inductive hypothesis we then have that $\mathit{\Delta} \vDash B$ and $\mathit{\Gamma} \nvDash C$. However, this is impossible since $\mathit{\Gamma} \vDash C \leftarrow B$. ⊣

The next task is to prove that the canonical model satisfies the frame conditions provided the logic in question satisfies the corresponding logical principle.

LEMMA 6.5 (R7 $\rightsquigarrow \mathcal{F}$(R7)). *If* R7 *is a derivable rule of the logic, then any canonical frame* $\mathfrak{C}_{\underline{\mathit{\Pi}}}$ *satisfies* $\mathcal{F}$(R7).

PROOF. $\underline{\mathit{\Pi}}$ is by its very construction non-trivial, so let $B \notin \underline{\mathit{\Pi}}$. Suppose that A is any formula such that $A \in \underline{\mathit{\Pi}}$. $A \in \underline{\mathit{\Pi}}^* \Leftrightarrow {\sim}A \notin \underline{\mathit{\Pi}}$ by definition. Assume for contradiction that ${\sim}A \in \underline{\mathit{\Pi}}$. Since $\underline{\mathit{\Pi}}$ is deductively closed it follows that ${\sim}A \vee B \in \underline{\mathit{\Pi}}$, and since R7 is a derivable rule of the logic and, again, $\underline{\mathit{\Pi}}$ is deductively closed, $B \in \underline{\mathit{\Pi}}$. Contradiction. It follows, then, that ${\sim}A \notin \underline{\mathit{\Pi}}$, and therefore that $A \in \underline{\mathit{\Pi}}^*$. It follows, therefore, that $\underline{\mathit{\Pi}} \subseteq \underline{\mathit{\Pi}}^*$. ⊣

LEMMA 6.6 (A13 $\rightsquigarrow \mathcal{F}$(A13)). *If* A13 *is a logical theorem of the logic, then any canonical frame* $\mathcal{C}_{\underline{\mathit{\Pi}}}$ *satisfies* $\mathcal{F}$(A13).

PROOF. We need to show that for all $\mathit{\Pi}$-canonical theories $\mathit{\Sigma}$, that there is a $\mathit{\Pi}$-canonical theory $\mathit{\Lambda}_{\mathit{\Sigma}}$ such that $R\mathit{\Sigma}\mathit{\Lambda}_{\mathit{\Sigma}}\mathit{\Sigma}$, and that $\mathit{\Lambda}_{\mathit{\Sigma}}$ satisfies the Z-clause — that for every $\mathit{\Pi}$-canonical theories $\mathit{\Phi}, \mathit{\Theta}$

$$R\mathit{\Lambda}_{\mathit{\Sigma}}\mathit{\Phi}\mathit{\Theta} \Rightarrow \mathit{\Phi} \subseteq \mathit{\Theta}.$$

For $\mathit{\Sigma} = \underline{\mathit{\Pi}}$ we can simply let $\mathit{\Lambda}_{\mathit{\Sigma}} = \underline{\mathit{\Pi}}$. The case for $\mathit{\Pi}$-canonical theories $\mathit{\Sigma} \neq \underline{\mathit{\Pi}}$ is a bit more involved.

Let $\mathit{\Psi} := \{A \mid \Box A \in \underline{\mathit{\Pi}}\}$.[27]

Sublemma 6.6.1. *$\mathit{\Psi}$ is a non-trivial $\mathit{\Pi}$-theory for which $\overline{R}\mathit{\Sigma}\mathit{\Psi}\mathit{\Sigma}$ for every $\mathit{\Pi}$-theory $\mathit{\Sigma}$.*

Proof. Non-triviality: Since $\underline{\mathit{\Pi}}$ is non-trivial, $A \notin \underline{\mathit{\Pi}}$ for some A. Since $\underline{\mathit{\Pi}}$ is $\mathit{\Pi}$-deductively closed, it follows that $\Box A \notin \underline{\mathit{\Pi}}$, and so $A \notin \mathit{\Psi}$.

$\mathit{\Pi}$-theory: From lemma 2.3 we obtain that $\mathit{\Psi}$ is closed under conjunction. Assume that $\varnothing \vdash_{\underline{\mathit{\Pi}}} A \rightarrow B$ and $A \in \mathit{\Psi}$. We must show that $B \in \mathit{\Psi}$. The following derivation shows that $\varnothing \vdash_{\mathit{\Pi}} (B \rightarrow B) \rightarrow B$:

[27] This definition of $\mathit{\Psi}$ is mere variant of the definition used in (Routley et al., 1982, p. 414).

(1)	$A \to B$	assumption
(2)	$(B \to B) \to (A \to B)$	1, suffixing rule (R4)
(3)	$(A \to A) \to A$	assumption ($A \in \Psi$)
(4)	$((A \to A) \wedge (A \to A)) \to A$	3, fiddling
(5)	$(A \to B) \to (((A \to A) \wedge (A \to A)) \to B)$	4, suffixing rule
(6)	$(((A \to A) \wedge (A \to A)) \to B) \to B$	A13
(7)	$(B \to B) \to B$	2, 5, 6, transitivity

Since $\underline{\Pi}$ is deductively closed, it follows that $\Box B \in \underline{\Pi}$, and therefore that $B \in \Psi$.

$\overline{R}\Sigma\Psi\Sigma$: Let Σ be any Π-theory, and assume that $A \to B \in \Sigma$ and $A \in \Psi$. By definition of Ψ, then, $\Box A \in \underline{\Pi}$. The following derivation shows that $\varnothing \vdash_{\Pi} (A \to B) \to B$:

(1)	$(A \to A) \to A$	assumption ($A \in \Psi$)
(2)	$((A \to A) \wedge (A \to A)) \to A$	1, fiddling
(3)	$(A \to B) \to (((A \to A) \wedge (A \to A)) \to B)$	2, suffixing rule
(4)	$(((A \to A) \wedge (A \to A)) \to B) \to B$	A13
(5)	$(A \to B) \to B$	3, 4, transitivity

Since Σ is assumed to be a Π-theory, it follows that $B \in \Sigma$, and therefore that $\overline{R}\Sigma\Psi\Sigma$. ⊣

Ψ need not be prime. However, (Restall, 1993, lem. 4) states that if $\overline{R}\Sigma\Gamma\Delta$, and Δ is a prime Π-theory, then there is a prime Π-theory $\Lambda \supseteq \Gamma$ such that $\overline{R}\Sigma\Lambda\Delta$. Since $\overline{R}\Sigma\Psi\Sigma$ for every Π-theory Σ, it follows that for every Π-canonical theory there is a prime Π-theory $\Lambda_\Sigma \supseteq \Psi$ such that $\overline{R}\Sigma\Lambda_\Sigma\Sigma$. That Λ_Σ is non-trivial is an easy corollary of Restall's lemma. All Λ_Σ are therefore Π-canonical. It now follows from lemma 6.2 that for every Π-canonical theory $\Sigma \neq \underline{\Pi}$ that $R\Sigma\Lambda_\Sigma\Sigma$.

That Λ_Σ satisfies the Z-condition: Let Δ, Θ be any $\underline{\Pi}$-theories such that $R\Lambda_\Sigma\Delta\Theta$. We need to show that $\Delta \subseteq \Theta$, so assume that $A \in \Delta$. Since A13 is an axiom of the logic, it follows from lemma 2.4 that $\vdash_{\Pi} \Box(A \to A)$, and so $A \to A \in \Psi$, and therefore $A \to A \in \Lambda_\Sigma$. By the definition of R it therefore follows that $A \in \Theta$. That $\underline{\Pi}$ satisfies the Z-condition is trivial. ⊣

We have now seen that the frame conditions hold in the canonical model provided the logic in question validates the corresponding logical axiom/rule. As an easy corollary, then, we have the following result:

Theorem 6.2 (Strong completeness). *$\Theta \vDash_{\boldsymbol{L}} A \Longrightarrow \Theta \vdash_{\boldsymbol{L}} A$, where $\boldsymbol{L}$ is any disjunctive logic obtainable from* **B** *by adding any number of the axiom and rules (along with their disjunctive versions) listed in section 2.*

We end by noting the rather obvious corollary of strong completeness due to the fact that no additional requirements are needed to model the "optional" connectives $\circ$ and $\leftarrow$, namely conservativeness:

defn 6.3. A logic $\boldsymbol{L}_2$ is a strong conservative extension of a logic $\boldsymbol{L}_1$ just in case $\boldsymbol{L}_2$ extends $\boldsymbol{L}_1$, and for every set of $\boldsymbol{L}_1$-formulas $\Theta \cup \{A\}$,

$$\Theta \vdash_{\boldsymbol{L}_2} A \Rightarrow \Theta \vdash_{\boldsymbol{L}_1} A.$$

Corollary 6.1. *If $\boldsymbol{L}$ is a logic which is strongly sound and complete with regards to the simplified semantics set forth in this paper, then $\boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\circ, \mathrm{R}\circ]$, $\boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$, and $\boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]$ are strong conservative extension of $\boldsymbol{L}$.*

Proof. Assume that $\Theta \cup \{A\}$ are $\boldsymbol{L}$-formulas and that $\Theta \nvdash_{\boldsymbol{L}} A$. Then $\Theta \nvDash_{\boldsymbol{L}} A$ by strong completeness. The $\boldsymbol{L}$-counter-model allows for the standard truth conditions for $\circ$ and $\leftarrow$, and so extends to a model for all of

$$\boldsymbol{L}' \in \{\boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\circ, \mathrm{R}\circ], \boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\leftarrow, \mathrm{R}\leftarrow], \boldsymbol{L}^{\mathrm{d}}[\mathrm{A}\circ, \mathrm{R}\circ, \mathrm{A}\leftarrow, \mathrm{R}\leftarrow]\}$$

in which A fails to hold, yet all formulas in Θ do. Thus $\Theta \nvDash_{\boldsymbol{L}'} A$, and so by strong soundness for $\boldsymbol{L}'$ it then follows that $\Theta \nvdash_{\boldsymbol{L}'} A$. ⊣

7. Summary

This paper shows that Anderson and Belnap's favorite logic **E** of entailment as well as Ackermann's logic $\mathbf{\Pi}'$ are both strongly sound and complete with regards to the "simplified" version of the Routley-Meyer semantics. This issue has been vex with confusion since **E** has often been regarded as having the so-called δ-rule or the equivalent rule of restricted assertion as primitive rules. However, Anderson and Belnap's axiomatization of **E** only validates these as *admissible* rules.

The semantics of **E** has been accused of being more complicated than the corresponding one for **T** and **R**. It was shown, however, that **E** does not need any extra frame component, compared to what **T** and **R** do. It does, however, require "unreduced" models — models in which there are points at which all the true inference tickets hold but which may fail to be closed under logical consequence.

Finally, it was shown that the simplified semantics could deal with the intensional conjunction known as *fusion* and the converse conditional provided additional tonicity requirements, which are put on the ternary relation. Such requirements are not needed if these connectives are absent. However, since strong soundness and completeness do hold even with these enforced, it shows that the connectives can be added conservatively even in the context of arbitrary theories.

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A. E-mingle

As pointed out in (Robles, 2022), the logic called *E-mingle* (**EM**), obtained from **E** by adding the restricted mingle axiom

$$(\text{Mr}) \quad (A \to B) \to ((A \to B) \to (A \to B))$$

is rather understudied, despite it being mentioned in (Anderson and Belnap, 1975) as one of the interesting neighbors of **E**. One of the main unresolved question with regards to **EM** is whether it satisfies the variable sharing property. What I want to address here, however, is rather the problem raised in both (Méndez et al., 2011, p. 363) and (Robles, 2022), namely of finding the correct frame condition for restricted mingle. Now (Routley et al., 1982, p. 344) does in fact state that the correct condition is

$$Rabc \;\&\; Rbcde \Longrightarrow Rade \;\&\; Rbde.$$

This, however, is incorrect as it stands, but this is probably simply due to a typesetting error. The correct condition is displayed in table 3.

	Frame condition
$\mathcal{F}(\text{Mr})$	$Rabc \;\&\; Rcde \Longrightarrow Rade \;OR\; Rbde$

Table 3. Frame condition for restricted mingle

LEMMA A.1 ($\mathcal{F}(\text{Mr}) \rightsquigarrow \text{Mr}$). *Restricted mingle holds in any model which satisfies* $\mathcal{F}(\text{Mr})$.

PROOF. Assume that a frame satisfies $\mathcal{F}(\text{Mr})$. Assume that $a \vDash A \to B$, and for contradiction that $a \nvDash (A \to B) \to (A \to B)$. Then there are b, c such that $Rabc$ with $b \vDash A \to B$, but $c \nvDash A \to B$. It follows that there are d, e such that $Rcde$ with $d \vDash A$ and $e \nvDash B$. From $\mathcal{F}(\text{Mr})$ it follows that either $Rade$ or $Rbde$, but since $a \vDash A \to B$ and $b \vDash A \to B$ and $d \vDash A$ it must be the case that $e \vDash B$. Contradiction. $\dashv$

Lemma A.2 (Mr $\rightsquigarrow \mathcal{F}$(Mr)). *If* Mr *is an axiom of the logic, then any canonical model* $\mathfrak{C}_{\underline{\Pi}}$ *satisfies* $\mathcal{F}$(Mr)*.*

Proof. Assume that the logic in question has Mr as an axiom. Let Σ_i for $1 \leq i \leq 5$ be prime $\underline{\Pi}$-theories such that $R\Sigma_1\Sigma_2\Sigma_3$ and $R\Sigma_3\Sigma_4\Sigma_5$. We must show that either $R\Sigma_1\Sigma_4\Sigma_5$ or $R\Sigma_2\Sigma_4\Sigma_5$. Assume for contradiction that neither is the case. By definition of R in $\mathfrak{C}_{\underline{\Pi}}$ it follows that there are formulas A, B, C, D such that $A \to B \in \Sigma_1$, $A \in \Sigma_4$, $B \notin \Sigma_5$, and $C \to D \in \Sigma_2$, $C \in \Sigma_4$, $D \notin \Sigma_5$. Since all these Σ's are prime $\underline{\Pi}$-theories it follows that $A \wedge C \in \Sigma_4$ and that $B \vee D \notin \Sigma_5$. Since $(A \to B) \vee (C \to D) \to (A \wedge C \to B \vee D)$ is a logical theorem of **B**, and Σ_1 and Σ_2 are prime $\underline{\Pi}$-theories, it follows that $\alpha \in \Sigma_1$ and $\alpha \in \Sigma_2$, where $\alpha := A \wedge C \to B \vee D$. Since $\alpha \to (\alpha \to \alpha)$ is an axiom, it follows follows that $\alpha \to \alpha \in \Sigma_1$. Since $R\Sigma_1\Sigma_2\Sigma_3$ and $\alpha \in \Sigma_2$ it then follows that $\alpha \in \Sigma_3$. But since $R\Sigma_3\Sigma_4\Sigma_5$ and $A \wedge C \in \Sigma_4$ it then follows that $B \vee D \in \Sigma_5$. Contradiction. $\dashv$

Corollary A.1. $\boldsymbol{L}$[Mr] *is strongly sound and complete with regards to the simplified semantics, where* $\boldsymbol{L}$ *is any disjunctive logic obtainable from* **B** *by adding any number of the axiom and rules (along with their disjunctive versions) listed in section 2.*

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