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Nathaniel Gan

On Some Meta-Theoretic Topological Features of the Region Connection Calculus

Abstract. This paper examines several intended topological features of the Region Connection Calculus (RCC) and argues that they are either underdetermined by the formal theory or given by the complement axiom. Conditions are identified under which the axioms of RCC are satisfied in topological models under various set restrictions. The results generalise previous results in the literature to non-strict topological models and across possible interpretations of connection. It is shown that the intended interpretation of connection and the alignment of self-connection with topological connection are underdetermined by the axioms of RCC, which suggests that additional axioms are necessary to secure these features. It is also argued that the complement axiom gives RCC models much of their topological structure. In particular, the incompatibility of RCC with interiors is argued to be given by the complement axiom.

Keywords: region connection calculus; mereotopology; topology; mereology

1. Introduction

The Region Connection Calculus (henceforth RCC), initially proposed in [33, 34, 35] and subsequently developed in [3, 9, 12, 13, 14, 15, 16, 17, 24, 26], is a mereotopological theory that aims to model spatial relations between regions in topological terms. RCC has been used as a formal theory for Qualitative Spatial Reasoning [10, 11, 12, 15], which has applications to, among other things, document classification [22], geographic informational systems [4, 21, 39, 40], natural language processing [1, 2, 8, 28], robot navigation [27], and visual representation [23].

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RCC is typically associated with several meta-theoretic features. Some of these features distinguish RCC from other mereotopological theories: for instance, it is sometimes thought that RCC, contra the theories of Clarke [6, 7] and Whitehead [38], is inconsistent with a definition of interiors [1, 35], and consequently does not countenance distinctions between open, semi-open, and closed regions [13, 24, 34]. Other meta-theoretic features are thought to capture our intuitions regarding spatial reasoning: for instance, the intended interpretation of the connection predicate is typically given as intersection of closures, which is sometimes thought to agree with the way ordinary objects come into contact [16, 33, 34, 35]. Yet other features would align RCC with related theories: for instance, the self-connection predicate in RCC is presumably intended to align with the property of topological connection in topological models of RCC. Insofar as these meta-theoretic features are determined by RCC's formal theory, this would validate RCC as a theory of spatial reasoning over alternatives.

However, it will be argued in this paper, some of RCC's intended features are underdetermined by its axioms, such as the intended interpretation of connection and the alignment of self-connection with topological connection, which suggests that supplementary axioms might be required to secure such features. It will also be argued that in many cases where the intended features are determined by the axioms of RCC, such as meta-theoretic properties involving interiors, these features are determined only by a particular axiom regarding complements. This suggests that the complement axiom plays a crucial role in determining the topological structure of RCC's models.

The argument in this paper will involve examining topological models of RCC. Previous work has shown that set-theoretic topological spaces can provide such models. Gotts [25] showed that the non-empty regular closed sets of any connected regular space are a model of RCC. Düntsch and Winter [20], as well as Li and Ying [29], later showed that the regularity property can be replaced with weaker stipulations (to be discussed below). Recent work on models of RCC has moved away from a topological perspective and focused more on the algebraic structure of the models in question [29, 30, 31, 32, 36, 37, 41].[1] This is mainly because RCC was intended to model spatial relations between regions without the need for

[1] For a study on various ways of representing region-based theories like RCC, see [18, 19].

reference to points, so some aspects of point-set topology would seem to be at odds with the theory's motivation. Nevertheless, given that many meta-theoretic properties associated with RCC models are typically stated in topological terms (with examples above), there remain reasons to investigate models of RCC from a topological perspective.

Previous work on models of RCC has also largely focused on *strict* models, in which the coincidence relation aligns with identity. As will be seen, investigations into topological models of RCC typically adopt an interpretation of parthood that assumes the models in question to be strict [25, 30, 36]. Indeed, Düntsch and Winter [20] in their formulation of RCC included an axiom requiring that models be strict. From a practical perspective, this focus on strict models is justified since the applicability of non-strict models is somewhat doubtful. Nevertheless, it has been observed that non-strict models of RCC are possible [29, 36]. With meta-theoretic considerations in view, non-strict topological models might be worth investigating, to see how much room the formal aspect of RCC leaves for non-standard topological features.

Section 2 reviews the axioms and definitions of RCC, and relevant ideas from set-theoretic topology. Section 3 identifies, for a range of possible interpretations of connection, the implied topological interpretation of the parthood relation. This will be done without assuming the topological models in question to be strict, and without assuming the intended interpretation of connection. Section 4 investigates topological models of RCC comprising the non-empty regular closed sets of a topological space. It will be shown that a relatively simple set of constraints on the interpretation of connection are necessary and sufficient for the non-empty regular closed sets of any topological space to be a model of RCC less the complement axiom. Additional necessary and sufficient conditions will also be identified for the complement axiom to be satisfied in these models. Section 5 discusses the significance of the results in Section 4, noting in particular that the interpretation of connection is underdetermined by RCC's axioms. Section 6 investigates topological models comprising the non-empty regular open sets of a topological space and observes that despite their formal similarity to the models considered in Section 4, a significant meta-theoretic difference exists between the two classes of models. Section 7 turns to topological models comprising all the non-empty sets of a topological space. It will be observed that although this class of models do not provide non-trivial models of RCC, they can provide models of RCC less part of the complement axiom. This observation suggests the notion of a deep topological property, which (roughly) is a property that depends heavily on RCC's topological structure. Section 8 argues that the incompatibility of RCC with interiors is a deep property.

2. RCC and topology

RCC has as its only primitive the connection relation C, a binary relation between regions stipulated to be reflexive and symmetric:

(1) $C(x, x)$ (Reflexivity)
(2) $C(x, y) \to C(y, x)$ (Symmetry)

In terms of C, other relations between regions are defined, including

$P(x, y) := \forall z(C(x, z) \to C(y, z))$ (Parthood)
$PP(x, y) := P(x, y) \land \neg P(y, x)$ (Proper parthood)
$EQ(x, y) := P(x, y) \land P(y, x)$ (Coincidence)
$O(x, y) := \exists z(P(z, x) \land P(z, y))$ (Overlap)
$DR(x, y) := \neg O(x, y)$ (Discreteness)
$EC(x, y) := C(x, y) \land \neg O(x, y)$ (External connection)
$NTPP(x, y) := PP(x, y) \land \neg \exists z(EC(z, x) \land EC(z, y))$
(Non-tangential proper part)

RCC posits that there is a universal region u, to which every region is connected:

(3) $C(x, u)$ (Universe)

For every non-universal region x, RCC posits the existence of x's complement, denoted $\mathrm{compl}(x)$. Complements are governed by the following axioms:

(4a) $\neg EQ(x, u) \to (C(y, \mathrm{compl}(x)) \leftrightarrow \neg NTPP(y, x))$
(Complement connection)
(4b) $\neg EQ(x, u) \to (O(y, \mathrm{compl}(x)) \leftrightarrow \neg P(y, x))$
(Complement overlap)

In the discussion to come, axioms (4a) and (4b) will at some points be considered separately, though formulations of RCC typically give these axioms in conjunction. Henceforth, 'axiom (4)' will refer to the conjunction of axioms (4a) and (4b).

Every pair of regions x and y has a sum, product, and (relative) difference, denoted $\mathrm{sum}(x, y)$, $\mathrm{prod}(x, y)$, and $\mathrm{diff}(x, y)$ respectively. These are governed by the following axioms:

$$(5)\quad C(z, \mathrm{sum}(x, y)) \leftrightarrow C(z, x) \vee C(z, y) \qquad \text{(Sum)}$$
$$(6)\quad C(z, \mathrm{prod}(x, y)) \leftrightarrow \exists w(P(w, x) \wedge P(w, y) \wedge C(z, w)) \qquad \text{(Product)}$$
$$(7)\quad C(z, \mathrm{diff}(x, y)) \leftrightarrow C(z, \mathrm{prod}(x, \mathrm{compl}(y))) \qquad \text{(Difference)}$$

The characterisations in axioms (3)–(7) only determine u, complements, sums, products, and differences up to coincidence—if two regions x and y are universal, for instance, then $EQ(x, y)$. In strict models, where coincidence is identity, u is unique, and $\mathrm{compl}(x)$, $\mathrm{sum}(x, y)$, $\mathrm{prod}(x, y)$, and $\mathrm{diff}(x, y)$ are functions. However, this may not be the case in non-strict models—for instance, u is not unique in Example 3.1 below.[2]

In terms of C and sum, the property CON, corresponding to self-connection, is defined:

$$\mathrm{CON}(x) := \forall y \forall z(EQ(x, \mathrm{sum}(y, z)) \to C(y, z)) \qquad \text{(Self-connection)}$$

The product mapping is partial on the domain of regions, in that a product of two regions is not always a region. The mapping can be made total if its range is permitted to include non-regions. RCC defines the NULL predicate to indicate that a member of the domain, extended to be closed under prod, is not a region. The following axiom asserts conditions under which a product of regions is a non-region:

$$(8)\quad \mathrm{NULL}(\mathrm{prod}(x, y)) \leftrightarrow DR(x, y) \qquad \text{(Non-Overlap)}$$

RCC is the theory comprising axioms (1)–(8). Early formulations of RCC included a Non-Atomicity axiom saying that all regions have a non-tangential proper part

$$\exists y\ NTPP(y, x) \qquad \text{(Non-Atomicity)}$$

It was later found that Non-Atomicity follows from the above axioms.

Theorem 2.1 ([12, 29, 36]). *Axioms (3) and (4) imply Non-Atomicity.*

Proof. (3) implies that every region overlaps u, hence no region is externally connected to u and every proper part of u is non-tangential. If a non-universal x lacks non-tangential proper parts, (4a) implies that every region is connected to $\mathrm{compl}(x)$. In particular, every region

[2] Thanks to a reviewer for clarification on this point.

connected to x is connected to compl(x), so x is a part of compl(x) and the two overlap. But this violates (4b) because x is a part of itself. $\dashv$

Another consequence of axiom (4a) is that Non-Atomicity is equivalent to

$$\neg EQ(x,u) \to \exists y \; \neg C(y, \mathrm{compl}(x)) \qquad \text{(Non-Atomicity*)}$$

In later sections, Non-Atomicity* will feature in some results.

Since this paper is interested in topological models of RCC, it might be helpful also to review a few definitions from set-theoretic topology. A *topology* on a set of points X is a collection $\mathcal{T}$ of subsets of X that includes $\emptyset$ and X, and that is closed under finite intersection and arbitrary union. A set with an associated topology $(X, \mathcal{T})$ is a *topological space*, in which the members of $\mathcal{T}$ are the *open* sets of $(X, \mathcal{T})$ and their set-theoretic complements are the *closed* sets of $(X, \mathcal{T})$. The *interior* of a set is the union of all open subsets of that set, and the *closure* of a set is the intersection of all closed supersets of that set. The topological interior and closure of a set A will be denoted $Int(A)$ and $\overline{A}$ respectively. A set x is *regular closed* if $x = \overline{Int(x)}$ and *regular open* if $x = Int(\overline{x})$. A set x is *topologically connected* if it cannot be partitioned into two subsets $y \cup z = x$ with $y \cap \overline{z} = \overline{y} \cap z = \emptyset$.[3]

Properties may also be stipulated of topological spaces entailing that some sets in those spaces are, in some sense, separated or non-separated by open sets. A topological space $(X, \mathcal{T})$ is *connected* if it cannot be partitioned into two disjoint open subsets Y and Z with $Y \cup Z = X$. A topological space is *regular* if, for every open set U and $p \in U$, there is an open set V with $p \in V \subseteq \overline{V} \subseteq U$. Notably, the characteristic property of regular topological spaces makes reference to points, which for reasons noted earlier is somewhat at tension with RCC's motivation. Düntsch and Winter [20] defined a similar weak regularity property that avoids reference to points. A topological space is *weakly regular* if, for every non-empty open set U, there is a non-empty open set V with $\overline{V} \subseteq U$.[4] It is known that all regular topological spaces are weakly regular, but not conversely.

[3] The typical term for this property is 'connected', modified here to avoid ambiguity.

[4] Also see [18] for the use of the weak regularity property. Relatedly, Li and Ying [29] defined the *inexhaustibility* property for Boolean algebras, which in topological terms entails that every non-empty open set U admits a non-empty open set V such that $(X - \overline{V}) \cup U = X$. In topological models, inexhaustibility is equivalent to weak regularity.

3. Interpreting parthood

Gotts [25] showed that the non-empty regular closed (henceforth *NERC*) sets of any connected regular topological space are a model of RCC under the standard interpretation of connection as set-theoretic intersection; that is, $C(x, y) \equiv x \cap y \neq \emptyset$. In regular topological spaces, the interpretation of parthood entailed by this interpretation of connection is $P(x, y) \equiv x \subseteq y$. This interpretation of parthood is typical: previous investigations into topological models of RCC have also made stipulations that entail the same interpretation of parthood [30, 36]. However, this interpretation constrains consideration to strict topological models, in which the coincidence relation EQ aligns with $=$. At the same time, the standard interpretation of connection admits non-strict topological models comprising the NERC sets of a topological space, in which parthood diverges from $\subseteq$. The following is an example.

Example 3.1. Let $X = \{1, 2, 3\}$ and $\mathcal{T} = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}\}$. Then $(X, \mathcal{T})$ is a topological space in which the NERC sets are $\{1, 2\}$, $\{2, 3\}$, and X. Any NERC set in $(X, \mathcal{T})$ intersecting $\{1, 2\}$ also intersects $\{2, 3\}$, and vice versa. $\{1, 2\}$ is thus a part of $\{2, 3\}$ under the standard interpretation of connection; but $\{1, 2\} \nsubseteq \{2, 3\}$. Moreover, $\{1, 2\}$ and $\{2, 3\}$ are coincident despite being non-identical.

The goal of this section is to identify the topological interpretation of parthood entailed by the standard interpretation of connection, without making any stipulation about the topological spaces in question. The resulting interpretation will thus hold even in non-strict topological models. Moreover, toward a generalisation over interpretations of connection, the interpretation of parthood to be identified will follow from only a few properties of the standard interpretation of connection. We note that connection, when interpreted as set-theoretic intersection, has the following properties:

(i) C is reflexive and symmetric
(ii) $x \subseteq y \rightarrow P(x, y)$
(iii) $C(x, y \cup z) \rightarrow C(x, y) \vee C(x, z)$

Any interpretation of connection bearing these properties will entail an interpretation of parthood similar to the one to be identified below. The focus for now will be on topological models comprising the NERC sets of a topological space; sections 6, 7 will consider how the results to follow translate to topological models given by other kinds of set restrictions.

We first note that the coincidence relation $EQ(x, y)$, which is equivalent to $\forall z(C(x, z) \leftrightarrow C(y, z))$, is an equivalence relation. Under this relation, the equivalence class containing x is the class of all NERC sets connected to the same NERC sets as x. Let $[x]$ denote the smallest closed set containing all members of this equivalence class. Namely,

$$[x] := \overline{\bigcup\{y : EQ(y, x)\}}$$

Since the closures of arbitrary unions of NERC sets are NERC, $[x]$ is NERC whenever x is. In strict models, where EQ is identity, $[x] = x$. In the general case, coincident regions are in the same equivalence class, so the topological interpretation of $EQ(x, y)$ is $[x] = [y]$.

We will show that $P(y, x)$ may be interpreted $[y] \subseteq [x]$. Toward this result, we prove the following lemma.

LEMMA 3.1. *If connection bears properties* (i)–(iii), *then* $P(y, x) \leftrightarrow EQ(y \cup x, x)$.

PROOF. (ii) implies that x is always a part of $y \cup x$, so it suffices to show that y is a part of x iff $y \cup x$ is a part of x. Assume that y is a part of x and let z be connected to $y \cup x$. (iii) implies that z is connected to either y or x. In the former case, $P(y, x)$ implies that z is connected to x, so either way, z is connected to x, which shows that $y \cup x$ is a part of x. Conversely, if y is not a part of x, some region is connected to y but not x. (ii) implies that this region is also connected to $y \cup x$, so $y \cup x$ is not a part of x. ⊣

The interpretation of parthood now follows:

THEOREM 3.1. *If connection bears properties* (i)–(iii), *then* $P(y, x) \equiv [y] \subseteq [x]$.

PROOF. We have $P(y, x) \leftrightarrow EQ(y \cup x, x) \leftrightarrow [y \cup x] = [x] \leftrightarrow [y] \subseteq [x]$. ⊣

Several corollaries of this result will be helpful in the proofs to come. First, given the interpretation of parthood, the interpretation of overlap follows:

COROLLARY 3.1. *If connection bears properties* (i)–(iii), $O(x, y) \equiv Int([x]) \cap Int([y]) \neq \emptyset$.

Second, we note that $[x]$ is the union across all equivalence classes containing NERC subsets of x.

Corollary 3.2. *If connection bears properties* (i)–(iii)*, then* $[x] = \bigcup\{[y] : y \subseteq x\}$

Proof. The left-to-right inclusion follows from the observation that $x \subseteq x$. For the converse: $y \subseteq x$ implies $P(y, x)$ by property (ii), which implies $[y] \subseteq [x]$ by Theorem 3.1. ⊣

Given this perspective on $[x]$, parthood can be shown to be equivalent to another topological relation.

Corollary 3.3. *If connection bears properties* (i)–(iii)*, then* $P(y, x) \leftrightarrow y \subseteq [x]$.

Proof. If y is a part of x, then $y \subseteq [y] \subseteq [x]$ by Theorem 3.1. Conversely, if y is not a part of x, by Corollary 3.2, we have $\bigcup\{[z] : z \subseteq y\} \nsubseteq \bigcup\{[z] : z \subseteq x\}$. Then $\bigcup\{[z] : z \subseteq y\} - \bigcup\{[z] : z \subseteq x\}$ is a non-empty union, disjoint from $[x]$, of equivalence classes containing subsets of y. Those subsets of y are disjoint from $[x]$, implying $y \nsubseteq [x]$. ⊣

An implication of Corollary 3.3 is that since $[x] \subseteq [x]$, $[x]$ is a part of x. Indeed, since $x \subseteq [x]$, we have coincidence.

Corollary 3.4. *If connection bears properties* (i)–(iii)*, then* $EQ([x], x)$.

4. NERC models

We may now investigate the conditions under which the NERC sets of a topological space are a model of RCC. The main result to be shown in this section is the following:

Theorem 4.1. *If connection bears properties* (i)–(iii)*, the NERC sets of any topological space*

(a) *are a model of RCC less axiom* (4)*,*

(b) *are a model of axiom* (4b) *iff Non-Atomicity* is satisfied, and*

(c) *given Non-Atomicity*, are a model of axiom* 4a) *iff* $C(x, \text{compl}(x))$ *holds for non-universal regions.*

We first show (a). Let a topological space $(X, \mathcal{T})$ be given, and assume that connection bears properties (i)–(iii). Axioms (1) and (2) follow immediately from property (i). For axiom (3), we interpret u as X and note that all NERC sets in $(X, \mathcal{T})$ are subsets of X. Axiom (3) then follows from (ii) and the reflexivity of C.

Turning to axioms (5)–(8), we interpret the complement, sum, product, and difference mappings like so:

$\text{compl}(x) \equiv \overline{X - [x]}$
$\text{sum}(x, y) \equiv [x] \cup [y]$
$\text{prod}(x, y) \equiv \overline{Int([x] \cap [y])}$
$\text{diff}(x, y) \equiv \overline{Int([x] \cap \overline{X - [y]})}$

Since the finite union of NERC sets is NERC, $\text{sum}(x, y)$ is NERC whenever x and y are. And since the closure of any non-empty open set is NERC, $\text{compl}(x)$, $\text{prod}(x, y)$, $\text{diff}(x, y)$ are also NERC whenever x and y are. Then, for axiom (5), we note that (ii) is equivalent to the converse of (iii).[5] Hence:

$$\begin{aligned} C(z, \text{sum}(x, y)) &\leftrightarrow C(z, [x] \cup [y]) \\ &\leftrightarrow C(z, [x]) \vee C(z, [y]) \\ &\leftrightarrow C(z, x) \vee C(z, y) \end{aligned}$$

The last equivalence follows from Corollary 3.4.

For axiom (6), we note that $\text{prod}(x, y) \equiv \overline{Int([x] \cap [y])} \subseteq \overline{[x] \cap [y]} = [x] \cap [y] \subseteq [x]$, and likewise $\text{prod}(x, y) \subseteq [y]$. So by Corollary 3.3, any region connected to $\text{prod}(x, y)$ is thereby connected to a part of both x and y. Conversely, if $P(w, x) \wedge P(w, y) \wedge C(z, w)$ for some NERC w, then $[w]$ is a subset of both $[x]$ and $[y]$ by Theorem 3.1. Hence $[w] = \overline{Int([w])} \subseteq \overline{Int([x] \cap [y])} = \text{prod}(x, y)$ and $C(z, w)$ implies $C(z, \text{prod}(x, y))$.

Axiom (7) follows immediately from the observation that $\text{diff}(x, y) = \text{prod}(x, \text{compl}(y))$. For axiom (8), we interpret $\text{NULL}(x)$ as $Int(x) = \emptyset$ and have

$$\begin{aligned} DR(x, y) &\leftrightarrow \neg O(x, y) \\ &\leftrightarrow Int([x]) \cap Int([y]) = \emptyset \\ &\leftrightarrow Int([x] \cap [y]) = \emptyset \\ &\leftrightarrow Int(\overline{Int([x] \cap [y])}) = \emptyset \\ &\leftrightarrow \text{NULL}(\text{prod}(x, y)) \end{aligned}$$

Thus, we have shown that whenever connection bears properties (i)–(iii), the NERC sets of any topological space are a model of RCC less axiom (4). Furthermore, properties (i)–(iii) are all necessary: an interpretation

[5] This follows from the observation that $x = x \cup y$ iff $y \subseteq x$. The equivalence between properties (ii) and (iii) was also proved in [36].

of connection lacking property (i) would not satisfy axiom (1), and an interpretation lacking either property (ii) or (iii) would not satisfy (5).

We note the following corollary of (a), which will be used in the proof of (b).

COROLLARY 4.1. *If connection bears properties (i)–(iii), then* $[[x] \cap [y]] = [x] \cap [y]$.

PROOF. The right-to-left inclusion is trivial, so we show just the converse. Let z be coincident with $[x] \cap [y]$ and let w be connected to z and hence $[x] \cap [y]$. Now w is connected to $[x]$ by axiom (6) and x by Corollary 3.4; likewise, w is connected to y. z is hence a part of both x and y, and $z \subseteq [x] \cap [y]$ follows from Corollary 3.3. ⊣

Turning to (b), we assume Non-Atomicity* and note that regions do not overlap their complements.

LEMMA 4.1. *If connection bears properties (i)–(iii) and Non-Atomicity* holds,* $\neg EQ(x, u)$ *implies* $\neg O(x, \mathrm{compl}(x))$.

PROOF. Suppose toward a contradiction that x is non-universal and overlaps $\mathrm{compl}(x)$. Axiom (8) implies that $\mathrm{prod}(x, \mathrm{compl}(x))$ is a (non-universal) region, and Non-Atomicity* implies that some y is not connected to $\mathrm{compl}(\mathrm{prod}(x, \mathrm{compl}(x)))$. Now

$$
\begin{aligned}
\mathrm{compl}(\mathrm{prod}(x, \mathrm{compl}(x))) &= \overline{X - [[\mathrm{compl}(x)] \cap [x]]} \\
&= \overline{X - ([\mathrm{compl}(x)] \cap [x])} \\
&= \overline{X - [\mathrm{compl}(x)]} \cup \overline{X - [x]} \\
&= \mathrm{compl}(\mathrm{compl}(x)) \cup \mathrm{compl}(x)
\end{aligned}
$$

The second equation follows from Corollary 4.1. Then, by (5), y is connected to neither $\mathrm{compl}(\mathrm{compl}(x))$ nor $\mathrm{compl}(x)$. But compl was interpreted in such a way that compl(compl(x)) and [compl(x)] cover X, so all regions are connected to either $(\mathrm{compl}(x))$ or $\mathrm{compl}(x)$, a contradiction. ⊣

Lemma 4.1 implies the following property regarding complements.

LEMMA 4.2. *If connection bears properties* (i)–(iii) *and Non-Atomicity* holds,* $\neg EQ(x, u)$ *implies* $[\mathrm{compl}(x)] = \mathrm{compl}(x)$.

PROOF. The right-to-left inclusion is trivial. Toward the converse, Lemma 4.1 and the interpretation of overlap imply that the interiors of compl(x) and $[x]$ are disjoint. Since $X - [x]$ is the largest open set not intersecting $Int([x])$, we have $Int([\text{compl}(x)]) \subseteq X - [x]$ and $[\text{compl}(x)] \subseteq \overline{X - [x]} = \text{compl}(x)$. ⊣

The proof of axiom (4b) now follows:

$$\begin{aligned} \neg P(y,x) &\leftrightarrow [y] \nsubseteq [x] \\ &\leftrightarrow \exists z([z] \subseteq [y] \wedge [z] \nsubseteq [x]) \\ &\leftrightarrow \exists w([w] \subseteq [y] \cap \text{compl}(x)) \\ &\leftrightarrow \exists w([w] \subseteq [y] \cap [\text{compl}(x)]) \\ &\leftrightarrow \exists w(P(w,y) \wedge P(w, \text{compl}(x))) \\ &\leftrightarrow O(y, \text{compl}(x)) \end{aligned}$$

The second equivalence follows from Corollary 3.2, and the third equivalence holds with $w = z \cap \text{compl}(x)$. For the necessity direction of (b), we note that if Non-Atomicity* fails, there is a NERC set x to whose complement all NERC sets are connected. x would then overlap its complement, violating axiom (4b).

For (c), we first show the sufficiency. If y is a non-tangential proper part of x, any region not overlapping x is not connected to y, otherwise that region would be externally connected to both x and y. In particular since compl(x) does not overlap x (as a consequence of Lemma 4.1), y is not connected to compl(x). And if y is not a non-tangential proper part of x, either some z connected to y does not overlap x, or y is not a proper part of x. In the former case, $z \subseteq \text{compl}(x)$, so $C(y,z)$ implies $C(y, \text{compl}(x))$. And if the former case does not hold, every region connected to y is connected to x, so y is a part of x, and y is not a proper part of x only if x and y coincide. $C(y, \text{compl}(x))$ then follows from $C(x, \text{compl}(x))$, and we have axiom (4a). The necessity direction of (c) follows from the observation that a failure of $C(x, \text{compl}(x))$ would violate axiom (4a) because x is not a (non-tangential) proper part of itself.

(a)–(c) in conjunction identify sufficient conditions for the NERC sets of a topological space to yield a model of RCC. In particular, under the standard interpretation of connection as set-theoretic intersection, a significant class of topological spaces can provide models of RCC.

COROLLARY 4.2 ([20, 29]). *With $C(x,y)$ interpreted as $x \cap y \neq \emptyset$, the NERC sets of a connected weakly regular topological space are a model of RCC.*

PROOF. It is clear that this interpretation of connection bears properties (i)–(iii). The weak regularity property implies Non-Atomicity* under this interpretation: given any non-universal NERC x, weak regularity implies that there is an open y with $\overline{y} \subseteq Int(x)$; $\overline{y}$ is then a NERC set not connected to compl(x). And, the connectedness property implies that non-universal regions are connected to their complements: for any non-universal NERC x, $X - [x]$ and $X - \text{compl}(x)$ are open and disjoint, and hence by connectedness cannot cover X; $[x]$ and compl(x) thus intersect and are connected. ⊣

5. Discussion

We make three remarks on the results shown in Section 4. First, part of the goal of the present investigation is to examine topological models of RCC without assuming that these models are strict. One might thus wonder if the conditions of Theorem 4.1 leave room for non-strict topological models. To see that they do, consider again Example 3.1. The standard interpretation of connection bears properties (i)–(iii), and the additional conditions in (b) and (c) are trivially satisfied because all regions are universal in a model comprising the NERC sets of this topological space. By Theorem 4.1, then, this model is a model of RCC. But it was observed earlier that this model is not strict.

However, there are (even non-pragmatic reasons) reasons to give more attention to strict models than to non-strict ones. For one, non-strict NERC models of RCC can easily be turned into strict models. Corollary 3.4 implies that any RCC relation between x and y is mirrored between $[x]$ and $[y]$—for instance, Pxy implies $P[x][y]$. Also, the interpretations of compl, sum, prod, and diff are such that they are indifferent between x and $[x]$—for instance, $\text{compl}(x) = \text{compl}([x])$. So if the NERC sets of a topological space satisfy the axioms of RCC, so do the class of sets of the form $[x]$ in that space (under the same interpretation of connection). The latter model is strict: if $EQ([x], [y])$, then $EQ(x, y)$, which implies $[x] = [y]$. Hence, given any NERC model of RCC, we can construct a strict model of RCC simply by taking the quotient of $[\cdot]$.[6]

Moreover, under the standard interpretation of connection as set-theoretic intersection, the only NERC models of RCC satisfying the

[6] Thanks to a reviewer for suggesting this point.

conditions of Theorem 4.1 are strict with respect to non-universal regions. To show this, we first prove a lemma.

LEMMA 5.1. *In a topological model satisfying Non-Atomicity*, and with* $C(x, y)$ *interpreted as* $x \cap y \neq \emptyset$, $P(y, x)$ *implies* $Int(y) \subseteq x$ *for non-universal* y.

PROOF. Suppose $Int(y) \nsubseteq x$. Then $Int(y) \cap (X - x)$ is a non-empty open subset of $\overline{y \cap (X - Int(x))}$, which hence has a non-empty interior. Now $z = \overline{Int(y \cap (X - Int(x)))}$ is a NERC set not overlapping x because the interiors of x and z do not intersect, and z is non-universal because it is a subset of y. Non-Atomicity* now implies that some region does not intersect compl$(z) \supseteq x$. But this region would intersect $[z]$ and hence $z \subseteq y$, and so would be connected to y without being connected to x. ⊣

From Lemma 5.1 it follows that the topological models in question are strict for non-universal regions.

THEOREM 5.1. *With* $C(x, y)$ *interpreted as* $x \cap y \neq \emptyset$, *in any topological model satisfying Non-Atomicity*,* $EQ(x, y)$ *implies* $x = y$ *whenever* x *is non-universal.*

PROOF. If $EQ(x, y)$ and x is non-universal, y is non-universal. Applying Lemma 5.1 twice now yields $Int(y) \subseteq x$ and $Int(x) \subseteq y$. Then $Int(x) \cup Int(y)$ is an open subset of x, so $Int(x) \cup Int(y) \subseteq Int(x)$ and $Int(y) \subseteq Int(x)$. Symmetrically, $Int(x) \subseteq Int(y)$. Hence $x = \overline{Int(x)} = \overline{Int(y)} = y$. ⊣

Since RCC has Non-Atomicity* as a theorem, Theorem 5.1 implies that all topological models of RCC are strict for non-universal regions under the standard interpretation of connection. So although the formal theory of RCC leaves room for non-strict models, RCC's meta-theoretic intended features, particularly the intended interpretation of connection, implies that the extent to which coincidence can diverge from identity in such models is limited. In addition to the practical reasons typically given in the literature, therefore, there are in fact theoretical reasons for investigations into models of RCC to focus on strict models, insofar as these investigations are interested only in standard features of the theory.

Second remark: the proofs in Sections 3–4 sought to generalise over interpretations of connection by depending on just three properties of the standard interpretation. One might wonder if there are non-standard interpretations satisfying the conditions of Theorem 4.1. Toward a positive answer, call a pair of sets a, b in a topological space *separated by open neighbourhoods* if there are disjoint open sets A, B such that $a \subseteq A$ and $b \subseteq B$. We might interpret connection as follows:

$$C(x, y) \equiv x \text{ and } y \text{ are not separated by open neighbourhoods}$$

The following example shows that this interpretation of connection is not equivalent to the standard one.

Example 5.1. Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{T}$ be generated by arbitrary unions of $\emptyset$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 2, 3\}$, and $\{3, 4, 5\}$. Then $(X, \mathcal{T})$ is a topological space in which the NERC sets are $\{1, 2\}$, $\{4, 5\}$, $\{2, 3, 4\}$, $\{1, 2, 4, 5\}$, $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, and X. $\{1, 2\}$ and $\{4, 5\}$ are disjoint but not separated by open neighbourhoods.

Nevertheless, this alternate interpretation of connection admits models of RCC in the same class of topological spaces as given in Corollary 4.2.[7]

THEOREM 5.2. *With $C(x, y)$ interpreted as non-separation by open neighbourhoods, the NERC sets of a connected weakly regular topological space are a model of RCC.*

PROOF. The interpretation of connection bears properties (i)–(iii). (i) is easily observed. For (ii): if $x \subseteq y$ and z is not connected to y, then the open sets separating z from y also separate z from x. For (iii): if x is not connected to either y or z, there are open sets separating x from y and x from z; the intersection of the two neighbourhoods of x and the union of the neighbourhoods of y and z separate x from $y \cup z$.

In connected topological spaces, NERC sets are connected to their complements under this interpretation: the proof of Corollary 4.2 showed that NERC sets intersect their complements in connected topological spaces, and intersecting regions are not separated by open neighbourhoods. And, weak regularity again implies Non-Atomicity*. Given any NERC x, applying the characteristic property of weakly regular spaces twice yields open sets U and V with $U \subseteq \overline{U} \subseteq V \subseteq \overline{V} \subseteq Int(x)$. The NERC set $\overline{U}$ is then separated from compl(x) by the open sets V and $X - \overline{V}$. The result now follows from Theorem 4.1. ⊣

[7] However, even in connected weakly regular topological spaces, this interpretation of connection is not equivalent to the standard one. *Normal* topological spaces are characterised by the property that disjoint closed sets are separated by open neighbourhoods, and it is known that not all (weakly) regular topological spaces are normal. Thanks to a reviewer for raising this issue.

The fact that models of RCC exist under non-standard interpretations shows the interpretation of connection to be underdetermined by the formal theory of RCC.

Third, the results of Section 4 may be compared with similar previous work. Corollary 4.2 has previously been shown in [20, 29] and, as observed earlier, its conditions entail the sufficient conditions of Theorem 4.1. In fact, under the standard interpretation of connection, the connectedness property of topological spaces is equivalent to the condition that all regions be connected to their complements: a disconnected topological space can be partitioned into two disjoint (and hence not connected) NERC sets, each of which would be the complement of the other.

However, Non-Atomicity* is not equivalent to weak regularity even under the standard interpretation of connection. In the topological space in Example 3.1, for instance, weak regularity fails but Non-Atomicity* (trivially) holds. Indeed, as noted above, the NERC sets of this topological space are a model of RCC, which shows that weak regularity is not a necessary condition for topological models of RCC even under the standard interpretation. Moreover, with non-standard interpretations in view, neither weak regularity nor connectedness is necessary: under a trivial interpretation of connection entailing that all regions are connected, the NERC sets of any topological space are a (trivial) model of RCC even if those spaces are neither connected nor weakly regular. The conditions in Theorem 4.1, on the other hand, were observed in Section 4 to be necessary as well as sufficient under any interpretation of connection. So the result in Theorem 4.1 may be seen as a strengthening of these prior results and a generalisation over interpretations of connection.

Another similar result was shown by Stell [36]. Stell defined a Boolean connection algebra as a Boolean algebra equipped with a binary relation satisfying certain conditions, and showed that all and only models of RCC have the structure of a Boolean connection algebra, under the assumption that the models in question are strict. It is known that the NERC sets of any topological space are a Boolean algebra less the bottom element. In topological terms, and with NERC sets in view, Stell's additional conditions for a Boolean connection algebra are as follows:

(A1) C is reflexive and symmetric
(A2) $\neg EQ(x, u) \to C(x, \mathrm{compl}(x))$
(A3) $C(x, y \cup z) \leftrightarrow C(x, y) \vee C(x, z)$
(A4) $\neg EQ(x, u) \to \exists y\ \neg C(y, x)$

(A1) and (A3) are equivalent to (i)–(iii), and (A2) is identical to the condition identified in (c). (A4) is equivalent to Non-Atomicity* in strict models, but this equivalence does not generally hold. For example, consider again the topological space in Example 5.1. Under the standard interpretation of connection, all of (A1)–(A4) are satisfied. In particular, (A4) is satisfied because the only non-universal NERC sets in this space are $\{1, 2\}$ and $\{4, 5\}$, which are not connected to each other. However, Non-Atomicity* is not satisfied because all NERC sets intersect $\text{compl}(\{1, 2\}) = \{2, 3, 4, 5\}$. Indeed, the topological model comprising the NERC sets of this space does not satisfy axiom (4b) because $\{2, 3, 4, 5\}$, being universal, has $\{1, 2\}$ as a part; hence $\{1, 2\}$ overlaps its complement. An odd feature of Example 5.1, which underlies the divergence between (A4) and Non-Atomicity*, is that the topological model in this example contains regions whose complements coincide with the universe. This feature does not arise in models satisfying Non-Atomicity*, nor does it arise in strict models, in which $\emptyset$ is the only regular closed set whose RCC complement is universal. So the result in Theorem 4.1 may be seen as a generalisation of Stell's result to models that are not necessarily strict.

6. Non-empty regular open sets

This and the next section will consider how the results of Section 4 translate to topological models other than those comprising the NERC sets of a topological space. Perhaps unsurprisingly, similar results hold for models comprising the non-empty regular open (henceforth *NERO*) sets of a topological space. Nevertheless, it will be seen, there are significant meta-theoretic differences between the two kinds of models.

Given an interpretation of connection, let $[x]$ be the smallest NERO set containing all members of the equivalence class under EQ containing x:

$$[x] := Int(\overline{\bigcup\{y : EQ(y, x)\}})$$

and interpret the complement, sum, product, and difference mappings as follows:

$\text{compl}(x) \equiv X - \overline{[x]}$
$\text{sum}(x, y) \equiv Int(\overline{[x] \cup [y]})$
$\text{prod}(x, y) \equiv [x] \cap [y]$
$\text{diff}(x, y) \equiv [x] \cap (X - \overline{[y]})$

Since the interiors of closed sets are regular open, and the intersection of two NERO sets is NERO, $[x]$, compl(x), sum(x, y), prod(x, y), and diff(x, y) are NERO whenever x and y are. It can now be shown, by proofs largely similar to those in Sections 3–4, that if the adopted interpretation of connection is such that

(i) C is reflexive and symmetric

(ii) $x \subseteq y \to P(x, y)$

(iii) $C(x, Int(\overline{y \cup z})) \to C(x, y) \vee C(x, z)$

then results (a)–(c) in Theorem 4.1 hold for the NERO sets of a topological space. Moreover, as in the NERC case, plural non-equivalent interpretations of connection satisfy the sufficient conditions identified in (a)–(c), and thus yield topological models of RCC. One such interpretation is the standard $C(x, y) \equiv \overline{x} \cap \overline{y} \neq \emptyset$, which may be verified to bear properties (i)–(iii). Under this interpretation, it can be shown by proofs similar to that for Corollary 4.2 and Theorem 5.1 that the NERO sets of any connected weakly regular topological space are a model of RCC, and that all such models are strict for non-universal regions.

Toward a non-standard interpretation of connection, call a function between topological spaces *continuous* if the preimage of every open set in the codomain is open in the domain, and call a pair of sets a, b in a topological space $(X, \mathcal{T})$ *separated by a continuous function* if there is a continuous function $f \colon X \to \mathbb{R}$ such that $f(a) \subseteq \{0\}$ and $f(b) \subseteq \{1\}$. Connection might then be interpreted as: $C(x, y)$ iff x and y are not separated by a continuous function.

This interpretation is non-equivalent to the standard one. As an example, consider again the topological space in Example 5.1. The sets $\{1\}$ and $\{5\}$ are NERO in this topological space, and their closures $\{1, 2\}$ and $\{4, 5\}$ are disjoint. But these NERO sets are not separated by a continuous function because a continuous $f \colon X \to \mathbb{R}$ requires

(I) $f(1) = f(3)$ because $\{1, 2\}$ and $\{2, 3\}$ are not open, and

(II) $f(3) = f(5)$ because $\{3, 4\}$ and $\{4, 5\}$ are not open;

but (I) and (II) cannot be satisfied with $f(1) = 0$ and $f(5) = 1$.

This interpretation of connection also admits models of RCC in a significant class of topological spaces. Call a topological space *normal* just in case disjoint closed sets are separated by a continuous function. We then have the following theorem.

THEOREM 6.1. *With $C(x, y)$ interpreted as non-separation by a continuous function, the NERO sets of a connected, weakly regular, and normal topological space are a model of RCC.*

PROOF. We first show that this interpretation of connection bears properties (i)–(iii). (i) and (ii) are easily observed. For (iii): if x is connected to neither y nor z, then there are continuous real-valued functions g and h on X such that $g(y) \cup h(z) \subseteq \{1\}$ and $g(x) \cup h(x) \subseteq \{0\}$. Then $f : X \to \mathbb{R}$ defined by $f(p) = max(g(p), h(p))$ is continuous, with $f(x) \subseteq \{0\}$ and $f(y \cup z) \subseteq \{1\}$. Indeed, because $\{1\}$ is closed in the standard topology of the reals, its preimage under f is closed and includes $\overline{y \cup z} \supseteq Int(\overline{y \cup z})$. So x is not connected to $Int(\overline{y \cup z})$.

In connected topological spaces, NERO sets are connected to their complements under this interpretation because if a NERO set is separated from its complement by a continuous function, so would the closures of these two sets, which would be disjoint and cover the topological space, violating connectedness. If moreover the topological space in question is weakly regular and normal, Non-Atomicity* would be satisfied by its NERO sets. In weakly regular topological spaces, every NERO x admits an open y with $\overline{y} \subseteq x$. This $\overline{y}$ is a closed set disjoint from the closed $X - x$, so if the topological space is additionally stipulated to be normal, $Int(\overline{y})$ would be a NERO set not connected to compl(x). Applying the NERO analogue of Theorem 4.1 now implies that we have a model of RCC. ⊣

The observed similarity between the NERC and NERO cases is perhaps unsurprising since it is known that in any topological space, the class of NERC sets is isomorphic to the class of NERO sets, and *vice versa*. But despite their formal similarity, significant differences arise between the two cases when meta-theoretic considerations are in view, particularly when *self-connection* is considered. Presumably, in topological models of RCC, CON is intended to align with topological connection. This alignment, however, is countenanced more naturally in the NERC case than in the NERO case.

In NERC models, several possible interpretations of connection, including the standard one, align the two definitions. When only NERC sets are in view, $y \cap \overline{z} = \overline{y} \cap z = \emptyset$ is equivalent to $y \cap z = \emptyset$, so topological connection implies self-connection when connection is interpreted as set-theoretic intersection. For the converse, we have the following lemma.

LEMMA 6.1. *If x is NERC and $y \cup z = x$ is a partition of x with $y \cap \overline{z} = \overline{y} \cap z = \emptyset$, then y and z are NERC.*

PROOF. It suffices to show that $y = \overline{Int(y)}$. Since $\overline{Int(y)} \subseteq \overline{Int(x)} = x = y \cup z$, and $\overline{Int(y)} \subseteq \overline{y}$ is disjoint from z, we have $\overline{Int(y)} \subseteq y$. Conversely, let $p \in y$. Since y is disjoint from $\overline{z}$, there is an open set $U \supseteq y \ni p$ disjoint from z. Now suppose toward a contradiction that $p \notin \overline{Int(y)}$. Then there is an open set $V \ni p$ disjoint from $Int(y)$. Since $p \in x = \overline{Int(x)}$, $U \cap V$ intersects $Int(x)$, and $W = U \cap V \cap Int(x)$ is open. Since W is disjoint from z, W is an open subset of both V and y, contradicting that V is disjoint from $Int(y)$. Hence, $y \subseteq \overline{Int(y)}$. ⊣

Lemma 6.1 entails that under the standard interpretation of connection, self-connection implies topological connection. When connection is interpreted non-standardly as non-separation by open neighbourhoods, by a similar argument to the above, the definitions of self-connection and topological connection agree in normal topological spaces.

In NERO models, however, it seems unlikely that any natural interpretation of connection aligns self-connection with topological connection. To see why, consider the Euclidean plane with the standard topology. In this topological space, the two open balls of radius 0.5 centred at $(0.5, 0)$ and $(1.5, 0)$ are NERO and their RCC sum is their union. The union of these open balls is not topologically connected, so any interpretation of connection that aligns self-connection with topological connection will have to entail that the two open balls are not connected. But in the real numbers with the standard topology, the interval $(0, 2) = Int(\overline{(0, 1) \cup (1, 2)})$ is topologically connected, so an interpretation of connection that aligns self-connection with topological connection will have to entail that $(0, 1)$ and $(1, 2)$ are connected. While it might be possible to construct an interpretation of connection satisfying both these conditions, it is unclear that this can be done in a natural non-contrived manner, especially since the intervals $(0, 1)$ and $(1, 2)$ are simply the open sets induced by the open balls in the previous example when the real line is viewed as a topological subspace of the Euclidean plane. Indeed, under both interpretations of connection considered earlier in this section, including the standard interpretation, the union of the open balls is self-connected, which disagrees with topological connection.

The upshot is that although NERC and NERO models of RCC share many formal properties, a significant meta-theoretic difference exists:

NERC models more naturally countenance the intended alignment between self-connection and topological connection.

7. Deep properties

This section will consider the extent to which the axioms of RCC can be satisfied by topological models comprising all the non-empty sets of a topological space. It will be observed that although these topological models do not themselves provide non-trivial models of RCC, they can easily provide models of RCC less axiom (4a). This will suggest that roughly (in a sense to be made precise below), axiom (4) is significantly more difficult to satisfy than are the other axioms of RCC.

Given an interpretation of connection, let $[x]$ be the union of the equivalence class containing x:

$$[x] := \bigcup\{y : EQ(y, x)\}$$

and interpret the complement, sum, product, and difference mappings thus

$\mathrm{compl}(x) \equiv X - [x]$
$\mathrm{sum}(x, y) \equiv [x] \cup [y]$
$\mathrm{prod}(x, y) \equiv [x] \cap [y]$
$\mathrm{diff}(x, y) \equiv [x] \cap (X - [y])$

Then it may be verified, by proofs similar to those in Sections 3–4, that if the adopted interpretation of connection is such that

(i) C is reflexive and symmetric
(ii) $x \subseteq y \to P(x, y)$
(iii) $C(x, y \cup z) \to C(x, y) \vee C(x, z)$

then (a) and (b) in Theorem 4.1 hold for the non-empty sets of any topological space. Moreover, some interpretations of connection admit topological spaces that satisfy the conditions of (a) and (b). For instance, under an interpretation of connection as set-theoretic intersection, which was previously observed to bear properties (i)–(iii), non-empty sets are not connected to their complements under the present interpretation of compl, so Non-Atomicity* is satisfied. So under this interpretation, the non-empty sets of any topological space are a model of RCC less (4a).

However, none of these models can non-trivially satisfy axiom (4b). Indeed, under *any* interpretation of connection bearing properties (i)–(iii), the only topological models of RCC comprising all the non-empty subsets of a topological space are ones in which all regions are universal.

THEOREM 7.1. *If connection bears properties* (i)–(iii) *and the non-empty sets of* $(X, \mathcal{T})$ *are a model of axiom* (4), *then* $EQ(x, u)$ *for all* x *in this model.*

PROOF. Suppose x is a non-universal set in such a model. Property (ii) implies that any subset of x is also non-universal. Let p be a singleton subset of x containing just a point. Now if p is not connected to $X - p$, axiom (4a) is violated. But if p is connected to $X - p$, it follows from (ii) that all non-empty sets are connected to $X - p$, which implies that p overlaps its complement, violating axiom (4b). ⊣

So without any further restriction on the sets under consideration (beyond a restriction to non-empty sets), there are no non-trivial topological models of RCC. Furthermore, a restriction to either NERC or NERO sets, without stipulations about the topological spaces in question, would also not constitute sufficient conditions to guarantee non-trivial topological models of RCC. For, given any point-set X, a topology on X in which every subset of X is open is such that every non-empty set is both NERC and NERO. If connection is interpreted in such a way that there is at least one non-universal region, then, a similar argument to the above would show that the NERC or NERO sets of this space cannot satisfy both parts of axiom (4) simultaneously. So while connection can be interpreted in such a way that the non-empty sets of any topological space are a model of RCC less axiom (4a), a non-trivial topological model of RCC requires restrictions on both the sets and the topological spaces under consideration.

These observations suggest that axiom (4) imposes stronger constraints on topological models than do the other axioms of RCC, and gives RCC much of its topological structure. Insofar as the topological features of models of RCC are determined by the formal theory, then, we might expect these features to be determined by axiom (4). An example of such a feature is Non-Atomicity, which was initially identified as a possible 'deep theorem' of RCC [12] and later shown to be consequence of axiom (4). Non-Atomicity is not a theorem of RCC less axiom (4a). With connection interpreted as set-theoretic intersection, the non-empty sets of any connected topological space with at least two points, which are a model of RCC less axiom (4a), do not satisfy Non-Atomicity because singleton non-universal sets have no proper parts. Non-Atomicity as a topological feature of RCC is thus determined only in models satisfying both parts of axiom (4). This reveals one way in which Non-Atomicity might be considered something of a 'deep' theorem of RCC: it is a consequence of the constraints imposed on topological models by all the axioms of RCC; but it might not hold in models that do not satisfy part of (4).

This notion can be defined somewhat more formally. Call a property *deep* if it holds in all topological models of RCC, but not generally in topological models of RCC less either (4a) or (4b). In the next section, it will be shown that some deep properties arise in relation to interiors.

8. Interiors

It is typically thought to be a feature distinguishing RCC from the theories of Clarke that interiors cannot be defined in RCC. An informal argument was given in [35] to the conclusion that interiors are *inconsistent* with RCC (also see [1]). Here, a formal reconstruction of the argument will be given and it will be shown that interiors can in fact be defined consistently, albeit not very meaningfully. It will also be seen that this incompatibility with a meaningful notion of interiors is a deep property of RCC.

The argument in [35] aims to show that the axioms of RCC do not allow for an arbitrary sum to be taken over all the non-tangential proper parts of a region, in accordance with the usual notion of interior. Let $\text{sum}\{y : \phi y\}$ denote the arbitrary sum over all regions bearing property ϕ. According to the argument, arbitrary sums should be governed by the following axiom:

$$C(z, \text{sum}\{y : \phi y\}) \leftrightarrow \exists y(\phi y \wedge C(z, y))$$

This is perhaps a natural extension of axiom (5), and indeed reduces to (5) in the finite case. The argument now proceeds with the observation that RCC countenances the following 'weak supplementation' principle:

$$PP(y, x) \rightarrow \exists z(P(z, x) \wedge \neg O(z, y))$$

To see that this is a theorem of RCC, assume that y is a proper part of x. Axiom (4b) implies that x overlaps the complement of y, so $z = \text{prod}(x, \text{compl}(y))$ is a region. Axiom (6) implies that z is a part of x and axiom (4b) implies that z does not overlap y.

An implication of the supplementation principle and Non-Atomicity (which is also a theorem of RCC) is that regions coincide with their interiors. For, otherwise, sum$\{y : NTPP(y, x)\}$ would be a proper part of x, which by the supplementation principle implies that some part of x does not overlap sum$\{y : NTPP(y, x)\}$. But Non-Atomicity implies that this part of x contains a non-tangential proper part of x, which entails a contradiction.

Now by axiom (4a), any region x, coincident with its interior, is connected to its complement, and the governing axiom for arbitrary sums implies that compl(x) is connected to some non-tangential proper part of x. But this entails that x overlaps its complement, which contradicts axiom (4b). Therefore, it seems, interiors are inconsistent with RCC. Several topological properties typically associated with RCC follow, particularly that RCC, *contra* Clarke's theories, does not distinguish open, closed, and semi-open regions.

A potentially contentious point of the proof above is the suggested governing axiom for arbitrary sums, and an examination of some typical topological models of RCC will suggest reasons against it. Recall that in NERC models of RCC, the finite sum mapping is interpreted thus:

$$\text{sum}(x, y) \equiv [x] \cup [y]$$

A natural extension to the infinite case seems to be:

$$\text{sum}\{y : \phi y\} \equiv \overline{\bigcup\{[y] : \phi y\}}$$

The closure operator here is necessary to guarantee that arbitrary sums are NERC. Arbitrary sums thus interpreted violate the axiom suggested above. For example, in the standard topology of the reals and under the standard interpretation of connection for NERC models, the sum over all closed intervals $[a, b]$ with $0 < a < b < 1$ is $[0, 1]$, which is connected to $[1, 2]$; but $[1, 2]$ is not connected to any $[a, b]$. Similarly, in the NERO case, the natural interpretation of arbitrary sum is

$$\text{sum}\{y : \phi y\} \equiv Int(\overline{\bigcup\{[y] : \phi y\}})$$

Under this interpretation, and with connection interpreted standardly as intersection of closures, $(1, 2)$ is connected to the sum over all open intervals (a, b) with $0 < a < b < 1$, namely $(0, 1)$; but $(1, 2)$ is not connected to any (a, b).

These observations suggest that we might explore other possible axioms for governing arbitrary sums. Casati and Varzi [5] formulated a different mereological theory countenancing arbitrary sums, whose definition, in RCC terms, is as follows:

$$\mathrm{sum}\{y : \phi y\} := \iota z \forall y(O(y, z) \leftrightarrow \exists x(\phi x \wedge O(y, x)))$$

This definition suggests the following possible axiom:

$$O(z, \mathrm{sum}\{y : \phi y\}) \leftrightarrow \exists y(\phi y \wedge O(z, y))$$

This axiom aligns with the topological interpretations of arbitrary sums suggested above: while intersection with $\overline{\bigcup\{[y] : \phi y\}}$ does not imply intersection with some $[y]$, intersection with $Int(\overline{\bigcup\{[y] : \phi y\}})$ does. Moreover, this axiom is a consequence of the properties of connection stipulated in the earlier sections, regardless of whether NERC sets, NERO sets, or all non-empty sets are in view. We show this just for the NERC case.

THEOREM 8.1. *If connection bears properties* (i)–(iii), *then in models comprising the NERC sets of a topological space,* $O(z, \mathrm{sum}\{y : \phi y\}) \leftrightarrow \exists y(\phi y \wedge O(z, y))$.

PROOF. By the interpretations of overlap and general sum, it suffices to show that for any z, $Int([z])$ intersects the interior of $\overline{\bigcup\{[y] : \phi y\}}$ iff it intersects the interior of some $[y]$ such that ϕy. First suppose that $Int([z])$ intersects the interior of $\overline{\bigcup\{[y] : \phi y\}}$. Since $Int([z])$ is open, its intersecting $\overline{\bigcup\{[y] : \phi y\}}$ implies that it intersects $\bigcup\{[y] : \phi y\}$, which implies that it intersects some $[y]$ such that ϕy. Again, since $Int([z])$ is open, its intersecting $[y] = \overline{Int([y])}$ implies that it intersects $Int([y])$. The converse follows from the observation that if ϕy, then $[y] \subseteq \overline{\bigcup\{[y] : \phi y\}}$. ⊣

The proofs for the other cases proceed similarly. Theorem 8.1 and its analogues show that the suggested alternative axiom is a theorem of RCC and hence is consistent with the theory. Interiors, therefore, can be defined consistently in RCC.

Nevertheless, a notion of interiors would not be very useful since, as noted above, it is a theorem of RCC that regions coincide with their interiors. This coincidence might not hold, however, in models of RCC less axiom (4b).

Example 8.1. With connection interpreted as $C(x, y) \equiv (x \cap \overline{y}) \cup (\overline{x} \cap y) \neq \emptyset$, the non-empty sets in the standard topology on the reals are a model of RCC less axiom (4b), in which the interior of $[0, 1]$ is $(0, 1)$.

PROOF. We first show that the given model is a model of RCC less axiom (4b). The given interpretation of connection bears properties (i)–(iii) as given in Section 7, so this model satisfies RCC less axiom (4). Toward axiom (4a), we note that this model is such that all non-universal sets are connected to their complements, so the proof of $\neg NTPP(y, x) \rightarrow C(y, \text{compl}(x))$ proceeds as in Section 4. For the converse, if y is connected to $\text{compl}(x)$, then either y intersects $\overline{\text{compl}(x)}$ or $\overline{y}$ intersects $\text{compl}(x)$. In the former case, $Int(\overline{\text{compl}(x)})$ is connected to y without overlapping x; and in the latter, $\overline{y} - x$ is connected to y without overlapping x. Either way, y is not a non-tangential proper part of x.

Next, we show that the interior of $[0, 1]$ is $(0, 1)$. Any open interval in $[0, 1]$ is a non-tangential proper part because any set connected to such an interval intersects, and hence overlaps, $[0, 1]$. Also, 0 and 1 are not in the interior of $[0, 1]$ because any set containing 1 (or 0) is connected to $(1, 2)$ (or $(-1, 0)$), which does not overlap $[0, 1]$. Since $(1, 2)$ is connected to $[0, 1]$ but not $(0, 1)$, the latter two sets do not coincide. ⊣

This example shows it to be a deep property of RCC that regions coincide with their interiors. In this model, furthermore, open, closed, and semi-open regions can be distinguished. In line with an analogous theorem in topology, define the closure of a non-universal region as the complement of its complement's interior. Then, the closure of $\mathbb{R} - [0, 1]$ is $\mathbb{R} - (0, 1)$. To see this, let $x = \mathbb{R} - [0, 1]$. We first note that $[x] = x$ because any set intersecting $[0, 1]$ is connected to $(0, 1)$ and hence not coincident with x. The complement of x is thus $[0, 1]$, whose interior is $(0, 1)$. We now note that any set intersecting $X - (0, 1)$ is not coincident with $(0, 1)$ because any such set is connected to x. The complement of $(0, 1)$ is thus $X - (0, 1)$. Since $(0, 1)$ is connected to $X - (0, 1)$ but not $X - [0, 1]$, the latter two sets are not coincident.

Therefore, in the model in Example 8.1, not only are regions not generally coincident with their interiors, they are also not generally coincident with their closures. Open, closed, and semi-open regions may hence be defined analogously to topology, namely, a region is open if it coincides with its interior, closed if it coincides with its closure, and semi-open if its closure coincides with the closure of an open proper part.

The incompatibility of RCC with a distinction between these three kinds of regions thus constitutes another deep property.

9. Conclusion

This paper investigated several meta-theoretic topological features of RCC and showed that they are either underdetermined by the formal theory of RCC, or would be underdetermined if part of axiom (4) is dropped. The main findings were the following:

1. The topological interpretation of connection is underdetermined by the axioms of RCC for both NERC and NERO models.

2. The alignment of self-connection with topological connection is also underdetermined. Plural interpretations of connection countenance this alignment in NERC models, while it seems that most natural interpretations (including the standard one) do not for NERO models. Incidentally, this also shows that despite the formal similarity between NERC and NERO topological models, meta-theoretic differences exist between the two classes of models.

3. Restrictions on both sets and topological spaces are necessary for topological spaces to provide non-trivial models of RCC, but neither is necessary if part of axiom (4) is dropped. Under some interpretations of connection, models of RCC less axiom (4a) are given by the non-empty sets of any topological space.

4. Interiors cannot be meaningfully defined in RCC, but this is not the case if part of axiom (4) is dropped. Contrary to what is sometimes thought, interiors are formally consistent with RCC, though regions always coincide with their interiors in models of RCC. A meaningful notion of interior, however, is compatible with RCC less axiom (4b).

5. Open, closed, and semi-open regions cannot be distinguished in RCC, but this is not the case if part of axiom (4) is dropped.

The former two observations suggest that if some intended features of RCC are to be determined, additional axioms are necessary. The latter three show that among the current axioms, axiom (4) is deeper than the other axioms of RCC, giving models of RCC much of their topological structure.

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NATHANIEL GAN
Department of Philosophy
National University of Singapore
Singapore
nat_gan@hotmail.com
https://orcid.org/0000-0002-9463-7955