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KD45 with Propositional Quantifiers

Abstract. Steinsvold (2020) has provided two semantics for the basic modal language enriched with propositional quantifiers $(\forall p)$. We define an extension **EM** of the system $\mathbf{KD45}_\Box$ and prove that **EM** is sound and complete for both semantics. It follows that the two semantics are equivalent.

Keywords: modal logic; $\mathbf{KD45}_\Box$; propositional quantifier

1. Introduction

"Epistemic modesty" refers to the assumption that it is rational to believe that not all of our rational beliefs are correct. Encoding rational belief with the modality $\Box$, Steinsvold (2020) formalized epistemic modesty as

$$\Box(\exists p)(\Box p \wedge \neg p).$$

This is a formula in an extension of the basic modal language by propositional quantifiers. Steinsvold developed two proposals for semantics that satisfy this formula. One approach uses a system of French (2006) that enriches the traditional relational semantics for $\Box$ with bisimulation quantifier semantics for $(\exists p)$. An example of an early survey of propositional quantifiers is the thesis by Fine (1969), where $(\exists p)$ is interpreted in the most principled way, i.e. "there exists a subset of the set of possible worlds", which entails that bisimulations do typically not preserve truth. Many later publications have been in the same line of thinking. However, bisimulation quantifiers had already been studied before French in the nineties (see Ghilardi and Zawadowski, 1995). The other approach uses a system of Steinsvold (2007) that combines the co-derived set semantics

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for $\Box$ with a direct interpretation of $(\exists p)$. The syntactical study of the co-derived set operator dates back to the sixties (see Spira, 1967).

Steinsvold (2020) looked at the theory (set of all valid formulas) of each of the two semantic systems, and raised the question of whether these two theories are equal. In this paper we answer the question affirmatively. Furthermore we provide a sound and complete axiomatization for this theory: a logic of epistemic modesty. The main theorem, Theorem 4.1, summarizes these results.

Outline of the paper. Section 2 reviews the two semantic systems. Section 3 presents the logic of epistemic modesty and proves its soundness for both semantic systems (Lemma 3.3). Section 4 presents a completeness proof that works for both semantic systems, thus entailing that the two semantics are equivalent.

2. Semantics

Definition 2.1 (language). Fix a countably infinite set $\mathbf{P} = \{p_0, p_1, \dots\}$ of proposition letters or propositional variables. For a finite set Q of proposition letters we write $Q \subset_\omega \mathbf{P}$. Define the language L by

$$L ::= \mathbf{P} \mid \top \mid (L \vee L) \mid \neg L \mid \Box L \mid (\forall \mathbf{P})L.$$

We will follow the standard rules for omission of the parentheses. As abbreviations, we include the usual symbols $\bot$, $\wedge$, $\leftarrow$, $\rightarrow$, $\leftrightarrow$, $\Diamond = \neg\Box\neg$ and $(\exists p) = \neg(\forall p)\neg$.

Definition 2.2 (frames). Let $\mathcal{C}^{\mathrm{ste}}$ be the class of serial, transitive, euclidean Kripke frames.[1] Let $\mathcal{C}^{\mathrm{co}}$ be the class of all topological spaces[2] that consist of an infinite universe with the topology of cofinite[3] or empty sets. Let $\mathcal{C}^{\mathrm{cco}} \subseteq \mathcal{C}^{\mathrm{co}}$ be the subclass of spaces in $\mathcal{C}^{\mathrm{co}}$ with a countable universe. (Thus, $\mathcal{C}^{\mathrm{cco}}$ contains only one space up to topological equivalence.)

[1] A Kripke frame (X, R) is serial if $\forall x \in X \colon \exists y \in X \colon xRy$; it is transitive if $\forall x, y, z \in X \colon (xRy \;\&\; yRz) \Rightarrow xRz$; and it is euclidean if $\forall x, y, z \in X \colon (xRy \;\&\; xRz) \Rightarrow yRz$.

[2] A topological space is a pair $(X, \mathcal{J})$ where $\mathcal{J} \subseteq \mathcal{P}(X)$ is closed under taking finite intersections and arbitrary unions and $\emptyset, X \in \mathcal{J}$. X is called the universe of the space and $\mathcal{J}$ the topology.

[3] A cofinite set is a set with a finite complement.

DEFINITION 2.3 (models). Recall that a Kripke frame $F = (X, R)$ with a valuation $V : \mathbf{P} \to \mathcal{P}(X)$ forms a Kripke model $M = (F, V)$ based on F. A $\mathcal{C}^{\text{ste}}$-*model* is a Kripke model based on a frame in $\mathcal{C}^{\text{ste}}$.

A *topo-model* based on a topological space $S = (X, \mathcal{J})$ is a pair $T = (S, V)$ where $V : \mathbf{P} \to \mathcal{P}(X)$ is a valuation. A $\mathcal{C}^{\text{co}}$-*model* (or $\mathcal{C}^{\text{cco}}$-*model*) is a topo-model based on a space in $\mathcal{C}^{\text{co}}$ (or $\mathcal{C}^{\text{cco}}$).

For all these types of models: a *pointed model* is a pair (M, x) where M is a model and x is in the universe of M.

If (X, R) is a Kripke frame and $x \in X$, let $R[x] = \{y \in X : xRy\}$.

DEFINITION 2.4 (bisimulation). Recall that a *bisimulation* between two Kripke models $M = (F, V)$ and $M' = (F', V')$ based on the respective frames $F = (X, R)$ and $F' = (X', R')$ is a relation $Z \subseteq X \times X'$ such that, whenever xZx', we have

- (forth) for all $y \in R[x]$ there is a $y' \in R'[x']$ such that yZy';
- (back) for all $y' \in R'[x']$ there is a $y \in R[x]$ such that yZy';
- for all $p \in \mathbf{P}$: $x \in V(p) \iff x' \in V'(p)$.

If Z is a bisimulation between M and M' and xZx', then we say that the pointed Kripke models (M, x) and (M, x') are *bisimilar*.

LEMMA 2.1. *Every pointed $\mathcal{C}^{\text{ste}}$-model is bisimilar to a pointed $\mathcal{C}^{\text{ste}}$-model whose designated world is irreflexive.*

PROOF. This is not difficult, and has been known for a long time (cf., e.g., Nagle, 1981). $\dashv$

DEFINITION 2.5 (Θ-bisimulation; French, 2006, Def. 2.25). Let $\Theta \subseteq \mathbf{P}$, let $F = (X, R)$ and $F' = (X', R')$ be Kripke frames and $M = (F, V)$ and $M' = (F', V')$ Kripke models. A relation $Z \subseteq X \times X'$ is a Θ-*bisimulation* between M and M' if, whenever xZx', the conditions (forth) and (back) hold and, moreover, for all $p \in \mathbf{P} \setminus \Theta$: $x \in V(p) \iff x' \in V'(p)$.

If Z is a Θ-bisimulation between M and M' and xZx', then we say that (M, x) and (M', x') are Θ-*bisimilar* and write

$$M, x \leftrightarrows_\Theta M', x'.$$

We omit curly brackets if Θ is a singleton.

Remark 2.1. An $\emptyset$-bisimulation is the same thing as a bisimulation. Hence two pointed models are bisimilar iff they are $\emptyset$-bisimilar. $\dashv$

Definition 2.6. If V is a valuation, $p \in \mathbf{P}$ and Y is a set, then $V[p \mapsto Y]$ is the valuation defined by

$$V[p \mapsto Y](q) = \begin{cases} Y & (q = p) \\ V(q) & (q \neq p) \end{cases}$$

for all $q \in \mathbf{P}$. If $M = (F, V)$ is a Kripke model we let $M[p \mapsto Y] = (F, V[p \mapsto Y])$, and similarly for topo-models.

Lemma 2.2. *Let (M, x) be a pointed Kripke model with $M = (F, V)$ and $F = (X, R)$.*

1. *If $\Theta \subseteq \mathbf{P}$ and $V' \colon \mathbf{P} \to \mathcal{P}(X)$ is another valuation that agrees with V on $\mathbf{P} \setminus \Theta$, then $M' = (F, V')$ satisfies $M, x \leftrightarroweq_\Theta M', x$.*
2. *If $p \in \mathbf{P}$ and $Y \subseteq X$, then $M, x \leftrightarroweq_p M[p \mapsto Y], x$.*

Proof. 2 follows from 1. In 1, we simply take the identity relation. ⊣

Definition 2.7 (relational semantics; French, 2006, Sect. 2.1). Let (M, x) be a pointed $\mathcal{C}^{\text{ste}}$-model with $M = (F, V)$ and $F = (X, R)$. If $p \in \mathbf{P}$ and $\phi, \psi \in L$, we define the satisfaction relation:

$$\begin{aligned}
M, x \vDash p &\iff x \in V(p) \\
M, x \vDash \top & \text{ always} \\
M, x \vDash \phi \vee \psi &\iff M, x \vDash \phi \text{ or } M, x \vDash \psi \\
M, x \vDash \neg\phi &\iff \text{not } M, x \vDash \phi \\
M, x \vDash \Box\phi &\iff \text{for all } y \in R[x] \colon M, y \vDash \phi \\
M, x \vDash (\forall p)\phi &\iff \text{for all pointed } \mathcal{C}^{\text{ste}}\text{-models } (M', x'), \\
& \text{ if } M, x \leftrightarroweq_p M', x' \text{ then } M', x' \vDash \phi.
\end{aligned}$$

$\mathcal{C}^{\text{ste}} \vDash \phi$ means that $M, x \vDash \phi$ for all pointed $\mathcal{C}^{\text{ste}}$-models (M, x).

Definition 2.8 (topological semantics; Steinsvold, 2007). Let (T, x) be a pointed topo-model with $T = (S, V)$ and $S = (X, \mathcal{J})$. If $p \in \mathbf{P}$ and $\phi, \psi \in L$, we define the satisfaction relation:

$$\begin{aligned}
T, x \vDash^\tau p &\iff x \in V(p) \\
T, x \vDash^\tau \top & \text{ always} \\
T, x \vDash^\tau \phi \vee \psi &\iff T, x \vDash^\tau \phi \text{ or } T, x \vDash^\tau \psi \\
T, x \vDash^\tau \neg\phi &\iff \text{not } T, x \vDash^\tau \phi \\
T, x \vDash^\tau \Box\phi &\iff x \text{ has a neighbourhood } O \in \mathcal{J} \text{ such that} \\
& \text{ for every } y \in O \text{ it holds that } y = x \text{ or } T, y \vDash^\tau \phi
\end{aligned}$$

$$T, x \vDash^{\tau} (\forall p)\phi \iff \text{for all } Y \subseteq X \colon T[p \mapsto Y], x \vDash^{\tau} \phi.$$

$\mathcal{C}^{\mathrm{co}} \vDash^{\tau} \phi$ means that $T, x \vDash^{\tau} \phi$ for all pointed $\mathcal{C}^{\mathrm{co}}$-models (T, x). Define $\mathcal{C}^{\mathrm{cco}} \vDash^{\tau} \phi$ similarly.

Remark 2.2. The semantics in Definition 2.8 can be viewed as a special case of the complete proper filter algebra semantics presented by Ding (2021) where we choose the proper filter to be the set of all cofinite sets in an infinite power set algebra. This specification is essential: there are many formulas ϕ for which $\mathcal{C}^{\mathrm{co}} \vDash^{\tau} \phi$ but ϕ can be falsified on some complete proper filter algebra. (For instance the formulas ($\exists$T), ($\exists$F) and ($\Diamond$2) to be introduced in Definition 3.1 below.) $\dashv$

It turns out that the theories of $\mathcal{C}^{\mathrm{ste}}$, $\mathcal{C}^{\mathrm{co}}$ and $\mathcal{C}^{\mathrm{cco}}$ coincide (Theorem 4.1 below).

Example 2.1 (axiom of epistemic modesty; Steinsvold, 2020).

1. $\mathcal{C}^{\mathrm{ste}} \vDash \Box(\exists p)(\Box p \wedge \neg p)$.
2. $\mathcal{C}^{\mathrm{co}} \vDash^{\tau} \Box(\exists p)(\Box p \wedge \neg p)$.

We gather a few technical tools for working with these semantics.

LEMMA 2.3. *Let (T, x) be a pointed $\mathcal{C}^{\mathrm{co}}$-model with universe X, $x \in X$ and $\phi \in L$. Then:*

1. *$T, x \vDash^{\tau} \Box\phi$ iff there is a cofinite set $C \subseteq X$ such that $T, y \vDash^{\tau} \phi$ for all $y \in C$.*
2. *$T, x \vDash^{\tau} \Diamond\phi$ iff there is an infinite set $I \subseteq X$ such that $T, y \vDash^{\tau} \phi$ for all $y \in I$.* $\dashv$

DEFINITION 2.9 (free variables). For $\phi \in L$, define the set $\mathrm{FV}(\phi)$ of free variables recursively: $\mathrm{FV}(p) = \{p\}$, $\mathrm{FV}(\top) = \emptyset$, $\mathrm{FV}(\phi \vee \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$, $\mathrm{FV}(\neg\phi) = \mathrm{FV}(\phi)$, $\mathrm{FV}(\Box\phi) = \mathrm{FV}(\phi)$, $\mathrm{FV}\big((\forall p)\phi\big) = \mathrm{FV}(\phi) \setminus \{p\}$.

LEMMA 2.4 (French, 2006, Lemma 2.31). *Suppose that $\Theta \subseteq \mathbf{P}$ and $\phi \in L$ such that $\mathrm{FV}(\phi) \cap \Theta = \emptyset$. If (M, x) and (M', x') are pointed $\mathcal{C}^{\mathrm{ste}}$-models with $M, x \leftrightarrows_{\Theta} M', x'$, then: $M, x \vDash \phi \iff M', x' \vDash \phi$.*

LEMMA 2.5. *Let $\phi \in L$.*

1. *If $F = (X, R) \in \mathcal{C}^{\mathrm{ste}}$, $x \in X$ and $V, V' : \mathbf{P} \to \mathcal{P}(X)$ are valuations that agree on $\mathrm{FV}(\phi)$, then $(F, V), x \vDash \phi$ iff $(F, V'), x \vDash \phi$.*
2. *If $S = (X, \mathcal{J})$ is a topological space, $x \in X$ and $V, V' \colon \mathbf{P} \to \mathcal{P}(X)$ are valuations that agree on $\mathrm{FV}(\phi)$, then $(S, V), x \vDash^{\tau} \phi$ iff $(S, V'), x \vDash^{\tau} \phi$.*

Proof. 1 follows from Lemmas 2.2(1) and2.4.

2 follows essentially from Lemma 4.11 in (Steinsvold, 2020).[4] ⊣

3. Axiomatization

Before we introduce a proof system in the language L, we explain our notation for proofs.

Let **Bool** be a natural deduction system for propositional logic in the language L. Recall that in natural deduction, each stated formula is either an assumption, an axiom or a formula derived from earlier formulas. We keep track of which assumptions we used: each stated formula ϕ is accompanied by a set a of open assumptions. This can be read as $\wedge a \to \phi$. An assumption can be closed by the introduction rule for implication: if we derive ϕ under some set a of open assumptions, then we can next obtain the formula $\psi \to \phi$ under the set $a \setminus \psi$ of open assumptions.

We write natural deduction proofs in Lemmon style. That is: the first column in a formal proof indicates the set of open assumptions for each derived formula, usually by listing the line numbers in which the formulas were assumed. The last column justifies the asserted formulas, by either saying "A" (new assumption) or referring to previously derived formulas, to rules, Lemmas etc. However, we shall often suppress reference to **Bool** in the justifications.

Let $\mathbf{K}_\Box$ be the extension of **Bool** by the axiom

$$(\mathrm{K}_\Box) \quad \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

and the necessitation rule

$$(\Box\mathrm{I}) \quad \frac{\phi}{\Box\phi} \qquad \text{(if no open assumptions)},$$

for all $\phi, \psi \in L$. The side condition in the rule $(\Box\mathrm{I})$ means that it can only be applied if ϕ was obtained under the empty set of open assumptions.

We shall skip some of the details of derivations in $\mathbf{K}_\Box$. For example, we may justify a formula ϕ by simply referring to some previously derived formula ψ and "$\mathbf{K}_\Box$", which would mean that there exists a derivation in $\mathbf{K}_\Box$ of ϕ from ψ.

[4] Steinsvold (2020) assumed that $S \in \mathcal{C}^{\mathrm{cco}}$, but the proof is the same for uncountable spaces.

Let $\mathbf{KD45}_\Box$ be the extension of $\mathbf{K}_\Box$ by the standard axioms of belief:[5]

$$\begin{array}{ll} (\mathrm{D}_\Box) & \Diamond(\phi \vee \neg\phi) \\ (4_\Box) & \Box\phi \to \Box\Box\phi \\ (5_\Box) & \Diamond\phi \to \Box\Diamond\phi \end{array}$$

DEFINITION 3.1 (proof system for epistemic modesty). Let **EM** be the system obtained by extending $\mathbf{KD45}_\Box$ with the axioms

$$\begin{array}{lll} (\exists\mathrm{T}) & (\exists p)\big(p \wedge \Box(p \leftrightarrow \phi)\big) & \big(\text{if } p \notin \mathrm{FV}(\phi)\big) \\ (\exists\mathrm{F}) & (\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \phi)\big) & \big(\text{if } p \notin \mathrm{FV}(\phi)\big) \\ (\Diamond 2) & \Diamond\phi \to (\exists p)\big(\Diamond(\phi \wedge p) \wedge \Diamond(\phi \wedge \neg p)\big) & \big(\text{if } p \notin \mathrm{FV}(\phi)\big) \\ (\mathrm{K}_\forall) & \big((\forall p)(\phi \to \psi)\big) \to \big((\forall p)\phi \to (\forall p)\psi\big) & \end{array}$$

and the rule

$$(\forall\mathrm{I}) \quad \frac{\phi}{(\forall p)\phi} \qquad \text{(if } p \text{ is not free in any open assumption).}$$

Remark 3.1. The logic **EM** is not closed under uniform substitution. For example, if $p, q \in \mathbf{P}$ are distinct, the formula $(\exists p)\big(p \wedge \Box(p \leftrightarrow \neg q)\big)$ is an instance of the axiom ($\exists$T), but the formula $\phi = (\exists p)\big(p \wedge \Box(p \leftrightarrow \neg p)\big)$ is not, and **EM** does not entail ϕ. (This follows from Lemma 3.3 below because $\mathcal{C}^{\mathrm{ste}} \models \neg\phi$.) $\dashv$

DEFINITION 3.2 (substitution). For $p, q \in \mathbf{P}$, denote by $[q/p]$ the operation on formulas in L that replaces every occurence of p by q (and every occurrence of $(\forall p)$ by $(\forall q)$).

LEMMA 3.1. *Let $p, q \in \mathbf{P}$ and $\phi \in L$. If q does not occur in ϕ, and nor does $(\forall q)$, then* $\mathbf{EM} \vdash \phi$ *implies* $\mathbf{EM} \vdash [q/p]\phi$.

PROOF. Let Π be a proof of ϕ in **EM**. Let $r \in \mathbf{P}$ be a proposition letter that does not occur in Π (and nor does $(\forall r)$). Let Π' be the result of replacing each formula ψ stated in Π by $[r/q]\psi$. Then Π' is still a proof in **EM**. (This can be checked formally, but the idea is that we only changed the name of the propositional variable q into r.) The last formula of Π' is $[r/q]\phi = \phi$. Then q and $(\forall q)$ do not occur in Π'. Again we let Π'' be the result of replacing each formula ψ stated in Π' by $[q/p]\psi$, and Π'' is another proof in **EM**. The last formula of Π'' is $[q/p]\phi$. $\dashv$

[5] Here in fact it suffices to take only quantifier-free instances of these three axioms, since we only use these axioms to prove Lemma 3.2. But for the sake of homogeneity we also include their instances with quantifiers.

Lemma 3.2 (adequacy of $\mathbf{KD45}_\Box$). *Let $\phi \in L$ be quantifier-free. Then the following are equivalent:*

(i) $\mathbf{KD45}_\Box \vdash \phi$,
(ii) $\mathcal{C}^{\mathrm{ste}} \vDash \phi$,
(iii) $\mathcal{C}^{\mathrm{co}} \vDash^\tau \phi$,
(iv) $\mathcal{C}^{\mathrm{cco}} \vDash^\tau \phi$.

Proof. (i) $\Rightarrow$ (ii): It is well known that the axioms and rules of $\mathbf{KD45}_\Box$ in the basic modal language are sound for the relational semantics on $\mathcal{C}^{\mathrm{ste}}$-models. Since the relational semantics is compositional, they are also sound in the full language L.

(ii) $\Rightarrow$ (i): There is a traditional canonical model for $\mathbf{KD45}_\Box$ that proves its completeness for the relational semantics. See for instance (Chagrov and Zakharyaschev, 1997, Chapter 5).

(i) $\Rightarrow$ (iii): The reader can check that all axioms and rules of $\mathbf{KD45}_\Box$ are sound for the topological semantics.

(iv) $\Rightarrow$ (i): This follows from Theorem 4.8 in (Steinsvold, 2020).

(iii) $\Rightarrow$ (iv): It is trivial. ⊣

Lemma 3.3 (soundness of $\mathbf{EM}$). *For all $\phi \in L$,*

$$\mathbf{EM} \vdash \phi \quad \Rightarrow \quad \mathcal{C}^{\mathrm{ste}} \vDash \phi \;\;\&\;\; \mathcal{C}^{\mathrm{co}} \vDash^\tau \phi.$$

Proof. As mentioned in the proof of Lemma 3.2, all axioms and rules of $\mathbf{KD45}_\Box$ are sound for both semantics. It suffices to show that the new axioms ($\exists$T), ($\exists$F), ($\Diamond$2), ($\mathrm{K}_\forall$) and the new rule ($\forall$I) are sound. We start with the topological semantics. Throughout, let (T, x) be an arbitrary pointed $\mathcal{C}^{\mathrm{co}}$-model with universe X.

For ($\exists$T) and ($\exists$F): Assume that $p \notin \mathrm{FV}(\phi)$. Let $Y' = \{y \in X : T, y \vDash^\tau \phi\}$, $Y_T = Y' \cup \{x\}$ and $Y_F = Y' \setminus \{x\}$. By Lemma 2.5(2), we have $T[p \mapsto Y], y \vDash^\tau \phi$ for any $Y \in \{Y_T, Y_F\}$ and $y \in Y_F$. Similarly, $T[p \mapsto Y], y \vDash^\tau \neg\phi$ for any $Y \in \{Y_T, Y_F\}$ and $y \in X \setminus Y_T$. Furthermore, $T[p \mapsto Y], y \vDash^\tau p$ for any $y \in Y_F$, and $T[p \mapsto Y], y \vDash^\tau \neg p$ for any $y \in X \setminus Y_T$. Thus $T[p \mapsto Y], y \vDash^\tau p \leftrightarrow \phi$ for all $y \in X \setminus \{x\}$. Therefore $T[p \mapsto Y], x \vDash^\tau \Box(p \leftrightarrow \phi)$ (Lemma 2.3(1)). On the other hand $T[p \mapsto Y_T], x \vDash^\tau p$ and $T[p \mapsto Y_F], x \vDash^\tau \neg p$. So $T, x \vDash^\tau (\exists p)\big(p \wedge \Box(p \leftrightarrow \phi)\big)$ and $T, x \vDash^\tau (\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \phi)\big)$.

For ($\Diamond$2): Assume $T, x \vDash^\tau \Diamond\phi$. Then by Lemma 2.3(2) there is an infinite set $I \subseteq X$ such that $T, y \vDash^\tau \phi$ for all $y \in I$. Split $I = Y \sqcup I'$ in two infinite sets.[6] By Lemma 2.5(2), we have $T[p \mapsto Y], y \vDash^\tau \phi \wedge p$ for all $y \in Y$, and $T[p \mapsto Y], y \vDash^\tau \phi \wedge \neg p$ for all $y \in I'$. So $T[p \mapsto Y], x \vDash^\tau \Diamond(\phi \wedge p) \wedge \Diamond(\phi \wedge \neg p)$ by Lemma 2.3(2). Thus,

$$T, x \vDash^\tau (\exists p)\big(\Diamond(\phi \wedge p) \wedge \Diamond(\phi \wedge \neg p)\big).$$

For ($\mathrm{K}_\forall$): Assume $T, x \vDash^\tau (\forall p)(\phi \to \psi)$. Then $T[p \mapsto Y], x \vDash^\tau \phi \to \psi$ for all $Y \subseteq X$. In order to check $T, x \vDash^\tau (\forall p)\phi \to (\forall p)\psi$, suppose that $T, x \vDash^\tau (\forall p)\phi$. Then $T[p \mapsto Y], x \vDash^\tau \phi$ for all $Y \subseteq X$. Hence $T[p \mapsto Y], x \vDash^\tau \psi$ for all $Y \subseteq X$. So $T, x \vDash^\tau (\forall p)\psi$.

For ($\forall$I): Suppose that in some derivation in **EM** we have derived ϕ under the assumptions $\alpha_1, \ldots, \alpha_n$, and $p \in \mathbf{P} \setminus \big(\mathrm{FV}(\alpha_1) \cup \cdots \cup \mathrm{FV}(\alpha_n)\big)$. We have to show that, if the derivation is sound up until this point, it is still sound after we apply the rule ($\forall$I). Hence, it suffices to show that

$$\mathcal{C}^{\mathrm{co}} \vDash^\tau \alpha_1 \wedge \cdots \wedge \alpha_n \to \phi \tag{1}$$

implies

$$\mathcal{C}^{\mathrm{co}} \vDash^\tau \alpha_1 \wedge \cdots \wedge \alpha_n \to (\forall p)\phi. \tag{2}$$

Assume (1). Suppose that $T, x \vDash^\tau \alpha_i$ for each i. Then by Lemma 2.5(2), we have $T[p \mapsto Y], x \vDash^\tau \alpha_i$ for all i and $Y \subseteq X$. So $T[p \mapsto Y], x \vDash^\tau \phi$ for all $Y \subseteq X$. Thus $T, x \vDash^\tau (\forall p)\phi$, which proves (2).

To show soundness for the relational semantics, let (M, x) be a pointed $\mathcal{C}^{\mathrm{ste}}$-model with $M = (F, V)$ and $F = (X, R)$.

For ($\exists$T) and ($\exists$F): By Lemmas 2.1 and 2.4 and Remark 2.1, we may assume w.l.o.g. that not xRx. Assume that $p \notin \mathrm{FV}(\phi)$. Let $Y' = \{y \in X : M, y \vDash \phi\}$, $Y_T = Y' \cup \{x\}$ and $Y_F = Y' \setminus \{x\}$. For each $Y \in \{Y_T, Y_F\}$ it holds $M, x \leftrightarrows_p M[p \mapsto Y], x$ by Lemma 2.2(2). By Lemma 2.5(1), we have $M[p \mapsto Y], y \vDash \phi$ for any $Y \in \{Y_T, Y_F\}$ and $y \in Y_F$. Similarly, $M[p \mapsto Y], y \vDash \neg\phi$ for any $y \in X \setminus Y_T$. Furthermore, $M[p \mapsto Y], y \vDash p$ for any $y \in Y_F$, and $M[p \mapsto Y], y \vDash \neg p$ for any $y \in X \setminus Y_T$. Thus $M[p \mapsto Y], y \vDash p \leftrightarrow \phi$ for all $y \in X \setminus \{x\}$. In particular this holds for all $y \in R[x]$. Therefore, $M[p \mapsto Y], x \vDash \Box(p \leftrightarrow \phi)$. On the other hand $M[p \mapsto Y_T], x \vDash p$ and $M[p \mapsto Y_F], x \vDash \neg p$. So $M, x \vDash (\exists p)\big(p \wedge \Box(p \leftrightarrow \phi)\big)$ and $M, x \vDash (\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \phi)\big)$.

For ($\Diamond$2): Assume $M, x \vDash \Diamond\phi$. Then there is a $y \in R[x]$ such that $M, y \vDash \phi$. Let $y' \notin X$ be a fresh object, $X' = X \sqcup \{y'\}$, and let $\pi \colon X' \to X$ be the map which sends y' to y and other points to themselves.

[6] We write $A \sqcup B := A \cup B$, provided A and B are disjoint sets.

Take the binary relation $uR'v \Leftrightarrow \pi(u)R\pi(v)$ on X' to form the frame $F' = (X', R')$. It is easy to check that $F' \in \mathcal{C}^{\mathrm{ste}}$. Define a valuation V' by

$$V'(q) = \begin{cases} V(p) \cup \{y\} & (q = p) \\ \pi^{-1}[V(q)] & (q \neq p). \end{cases}$$

Then we claim that $M' = (F', V')$ satisfies $Z\colon M', x \leftrightarrows_p M, x$, where $Z \subseteq X' \times X$ is the graph of the map π. To check this, suppose that uZw. For any $q \in \mathbf{P} \setminus \{p\}$, we have $u \in V'(q)$ iff $w = \pi(u) \in V(q)$, so the propositional condition holds. Whenever $uR'v$, we have $w = \pi(u)R\pi(v)$ and $vZ\pi(v)$, so the forth condition holds. Whenever wRz, we have $\pi(u) = wRz = \pi(z)$ so $uR'z$ and zZz, so the back condition holds. This proves that Z is a p-bisimulation.

So by Lemma 2.4, we have $M', y \vDash \phi$ and $M', y' \vDash \phi$. Thus $M', y \vDash \phi \wedge p$ and $M', y' \vDash \phi \wedge \neg p$. Therefore, $M', x \vDash \Diamond(\phi \wedge p) \wedge \Diamond(\phi \wedge \neg p)$. Hence, $M, x \vDash (\exists p)\big(\Diamond(\phi \wedge p) \wedge \Diamond(\phi \wedge \neg p)\big)$.

For ($\mathrm{K}_\forall$): analogous to the proof of its soundness for the topological semantics.

For ($\forall$I): analogous to the proof of its soundness for the topological semantics. ⊣

The rest of this section contains some easy lemmas about **EM**.

Remark 3.2. The language L can be viewed as the multimodal language whose set of (unary) modalities is $\mu = \{\Box, (\forall p_0), (\forall p_1), \dots\}$. Observe that for any $p \in \mathbf{P}$, the necessitation rule

$$\frac{\phi}{(\forall p)\phi} \qquad \text{(if no open assumptions)},$$

is a special instance of ($\forall$I). Since we also included the normality axioms ($\mathrm{K}_\forall$) and the system $\mathbf{K}_\Box$ in **EM**, the logic **EM** extends the minimal normal modal logic for μ. Lemma 3.4 follows. ⊣

Lemma 3.4. *For all $\phi, \psi \in L$,*

$$\mathbf{EM} \vdash (\exists p)\phi \vee (\exists p)\psi \leftrightarrow (\exists p)(\phi \vee \psi).$$

Lemma 3.5. **EM** *entails the following natural deduction rule*:

$$\frac{\phi \to \psi}{(\exists p)\phi \to (\exists p)\psi} \qquad (\textit{if } p \textit{ is not free in any open assumption}).$$

PROOF. Suppose that in some natural deduction we derived $\phi \to \psi$ under a certain set a of assumptions whose free variables do not contain p. Recall that $(\exists p) = \neg(\forall p)\neg$.

Continue in **EM**:

a	(1)	$\phi \to \psi$	
a	(2)	$\neg\psi \to \neg\phi$	1
a	(3)	$(\forall p)(\neg\psi \to \neg\phi)$	2, $(\forall\mathrm{I})$
a	(4)	$(\forall p)\neg\psi \to (\forall p)\neg\phi$	3, $(\mathrm{K}_\forall)$
5	(5)	$\neg(\exists p)\psi$	A
5	(6)	$(\forall p)\neg\psi$	5
$a, 5$	(7)	$(\forall p)\neg\phi$	4, 6
$a, 5$	(8)	$\neg(\exists p)\phi$	7
a	(9)	$\neg(\exists p)\psi \to \neg(\exists p)\phi$	8
a	(10)	$(\exists p)\phi \to (\exists p)\psi$	9 $\dashv$

Example 3.1. $(\exists\mathrm{F})$ can be seen as a generalization of the axiom of epistemic modesty (Example 2.1).

To see this, derive in **EM**:

(1)	$(\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \top)\big)$	$(\exists\mathrm{F})$
(2)	$\big(\neg p \wedge \Box(p \leftrightarrow \top)\big) \to (\neg p \wedge \Box p)$	$\mathbf{K}_\Box$
(3)	$(\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \top)\big) \to (\exists p)(\neg p \wedge \Box p)$	2, Lemma 3.5
(4)	$(\exists p)(\neg p \wedge \Box p)$	1, 3
(5)	$\Box(\exists p)(\neg p \wedge \Box p)$	4, $(\Box\mathrm{I})$ $\dashv$

LEMMA 3.6. *Let* $\phi, \psi \in L$ *and* $p \in \mathbf{P} \setminus \mathrm{FV}(\phi \vee \psi)$. *Then*

$$\mathbf{EM} \vdash (\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big).$$

PROOF.

In **EM**:

1	(1)	ϕ	A

1	(2)	$p \wedge \Box(p \leftrightarrow \psi) \to \big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	1, **Bool**
1	(3)	$(\exists p)\big(p \wedge \Box(p \leftrightarrow \psi)\big) \to$ $(\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	2, Lemma 3.5
1	(4)	$(\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	3, $(\exists\mathrm{T})$
5	(5)	$\neg\phi$	A
5	(6)	$\neg p \wedge \Box(p \leftrightarrow \psi) \to \big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	5
5	(7)	$(\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \psi)\big)$ $\to (\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	6, Lemma 3.5
5	(8)	$(\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	7, $(\exists\mathrm{F})$
	(9)	$(\exists p)\big((p \leftrightarrow \phi) \wedge \Box(p \leftrightarrow \psi)\big)$	4, 8 $\dashv$

LEMMA 3.7. *If $\phi \in L$ and $p \in \mathbf{P} \setminus \mathrm{FV}(\phi)$, then*

$$\mathbf{EM} \vdash (\exists p)\phi \to \phi.$$

PROOF.

In **EM**:

1	(1)	$\neg\phi$	A
1	(2)	$(\forall p)\neg\phi$	1, $(\forall\mathrm{I})$
1	(3)	$\neg(\exists p)\phi$	2
	(4)	$\neg\phi \to \neg(\exists p)\phi$	3
	(5)	$(\exists p)\phi \to \phi$	4 $\dashv$

LEMMA 3.8. *Let $p \in \mathbf{P}$.*

1. $\mathbf{EM} \vdash (\exists p)p$.
2. $\mathbf{EM} \vdash (\exists p)\neg p$.
3. $\mathbf{EM} \vdash (\exists p)\Box(p \leftrightarrow \phi)$ *for all $\phi \in L$ satisfying $p \notin \mathrm{FV}(\phi)$.*

PROOF. **Bool** entails the formulas

$$p \wedge \Box(p \leftrightarrow \phi) \to p,$$
$$\neg p \wedge \Box(p \leftrightarrow \phi) \to \neg p,$$
$$p \wedge \Box(p \leftrightarrow \phi) \to \Box(p \leftrightarrow \phi).$$

EM extends **Bool**, so it entails the formulas

$$(\exists p)\big(p \wedge \Box(p \leftrightarrow \phi)\big) \to (\exists p)p,$$
$$(\exists p)\big(\neg p \wedge \Box(p \leftrightarrow \phi)\big) \to (\exists p)\neg p,$$
$$(\exists p)\big(p \wedge \Box(p \leftrightarrow \phi)\big) \to (\exists p)\big(\Box(p \leftrightarrow \phi)\big),$$

by Lemma 3.5. So ($\exists$T) and ($\exists$F) give us the desired formulas. $\dashv$

4. Completeness

We show that **EM** is complete for $\vDash$ and $\vDash^{\tau}$ (Lemma 4.6). The key observation is that the full language with boxes $\Box$ and quantifiers $(\forall p)$ is no more expressive than its quantifier-free fragment (Lemma 4.2). We show that, given any formula, **EM** proves its equivalence with some quantifier-free formula (Lemma 4.5). Next, completeness of **EM** will follow from the known completeness of **KD45**$_{\Box}$ (Lemma 3.2).

DEFINITION 4.1. Consider $Q \subset_{\omega} \mathbf{P}$. Define

$$\mathcal{S}(Q) = \mathcal{P}\big(\mathcal{P}(Q)\big) \setminus \{\emptyset\}$$

and

$$\mathcal{T}(Q) = \mathcal{P}(Q) \times \mathcal{S}(Q).$$

Define associated formulas. For $a \in \mathcal{P}(Q)$ let

$$\chi(Q, a) = \Big(\bigwedge_{q \in a} q\Big) \wedge \Big(\bigwedge_{q \in Q \setminus a} \neg q\Big).$$

For $s \in \mathcal{S}(Q)$ let

$$\chi(Q, s) = \Big(\bigwedge_{a \in s} \Diamond\chi(Q, a)\Big) \wedge \Big(\bigwedge_{a \in \mathcal{P}(Q) \setminus s} \neg\Diamond\chi(Q, a)\Big).$$

For $t = (a, s) \in \mathcal{T}(Q)$ let

$$\chi(Q, t) = \chi(Q, a) \wedge \chi(Q, s).$$

For $\mathbb{T} \subseteq \mathcal{T}(Q)$ let

$$\chi(Q, \mathbb{T}) = \bigvee_{t \in \mathbb{T}} \chi(Q, t).$$

Remark 4.1. 1. We took $\chi(Q, *)$ to mean four different things. Which of the four we are thinking of should always be clear from the letter that we use for the argument $*$.

2. The formulas $\chi(Q, a)$ defined in Definition 4.1 are purely propositional.
3. All formulas $\chi(Q, *)$ defined in Definition 4.1 are quantifier-free.
4. All formulas $\chi(Q, *)$ defined in Definition 4.1 only use propositional variables in Q. $\dashv$

In (Balbiani and Tinchev, 2018), modulo notational differences, subsets of $\{p_0, \ldots, p_{n-1}\}$ are called n-arrows, elements of $\mathcal{S}(\{p_0, \ldots, p_{n-1}\})$ are called n-setarrows and elements of $\mathcal{T}(\{p_0, \ldots, p_{n-1}\})$ are called n-tips. The following lemma summarizes the results that we need from their Sections 4–6.

Lemma 4.1. *Let $Q \subset_\omega \mathbf{P}$. Then:*

1. *$\mathcal{P}(Q) \neq \emptyset$, $\mathcal{S}(Q) \neq \emptyset$, and $\mathcal{T}(Q) \neq \emptyset$.*
2. *For every pointed Kripke model (M, x) there is a unique $a(Q, M, x) \in \mathcal{P}(Q)$ such that*
$$M, x \vDash \chi(Q, a(Q, M, x)).$$
In fact, $a(Q, M, x) = \{p \in Q : M, x \vDash p\}$.
3. *For every pointed serial Kripke model (M, x) there is a unique $s(Q, M, x) \in \mathcal{S}(Q)$ such that*
$$M, x \vDash \chi(Q, s(Q, M, x)).$$
In fact, $s(Q, M, x) = \{a(Q, M, y) : y \in R[x]\}$, if M is based on the frame (X, R).
4. *For every pointed serial Kripke model (M, x) there is a unique $t(Q, M, x) \in \mathcal{T}(Q)$ such that*
$$M, x \vDash \chi(Q, t(Q, M, x)).$$
In fact, $t(Q, M, x) = (a(Q, M, x), s(Q, M, x))$.
5. *For every $t \in \mathcal{T}(Q)$ there exists a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M, x) such that $t(Q, M, x) = t$.*
We can arrange that the universe of M has size at most $1 + 2^{|Q|}$.
6. *If $(M, x), (M', x')$ are two pointed $\mathcal{C}^{\mathrm{ste}}$-models, we have*
$$M, x \leftrightarrows_{\mathbf{P} \setminus Q} M', x' \quad \Leftrightarrow \quad t(Q, M, x) = t(Q, M', x').$$

Proof. 1: we have $Q \in \mathcal{P}(Q)$, so $\{Q\} \in \mathcal{S}(Q)$ and $(Q, \{Q\}) \in \mathcal{T}(Q)$.
2 follows easily from the definitions.

3. For any $a \subseteq Q$, we have $M, x \vDash \Diamond\chi(Q, a)$ iff there exists a $y \in R[x]$ such that $M, y \vDash \chi(Q, a)$. Let $s(Q, M, x)$ be the set of all such $a \subseteq Q$. Then $R[x] \neq \emptyset$ guarantees that $s(Q, M, x) \neq \emptyset$, so $s(Q, M, x) \in \mathcal{S}(Q)$.

4 follows from 2 and 3.

5. Suppose $t = (a, s)$. Let $X = s \sqcup \{x\}$ where x is a fresh object, $R = X \times s$ and $F = (X, R)$. Then $F \in \mathcal{C}^{\text{ste}}$. (This follows from Lemma 2.2(i,iv) in Pietruszczak (2009), but it is easy to check it directly.)

Define a valuation $V \colon \mathbf{P} \to \mathcal{P}(X)$ by

$$\begin{aligned} x \in V(p) \quad &\Leftrightarrow \quad p \in a \quad \text{and} \\ a' \in V(p) \quad &\Leftrightarrow \quad p \in a' \quad \text{for all } a' \in s. \end{aligned}$$

Set $M = (F, V)$. Then by 2 it holds $a(Q, M, x) = a$ and $a(Q, M, a') = a'$ for all $a' \in s$. The former implies $M, x \vDash \chi(Q, a)$, while the latter implies $s(Q, M, x) = s$ by 3, whence $M, x \vDash \chi(Q, s)$. Thus $M, x \vDash \chi(Q, t)$.

6. $\Leftarrow$ Assume $t(Q, M, x) = t(Q, M', x')$. Suppose that M and M' are based on the frame $F = (X, R)$ and $F' = (X', R')$ respectively. Define $Z \subseteq \big(\{x\} \cup R[x]\big) \times \big(\{x'\} \cup R'[x']\big)$ by yZy' iff y and y' satisfy the same proposition letters in Q. Since $a(Q, M, x) = a(Q, M', x')$ by 4, we have xZx' by 2. Also $s(Q, M, x) = s(Q, M', x')$ by 4; hence 3 and 2 imply that that Z relates every point in $R[x]$ to a point in $R'[x']$ and vice versa. Since F and F' are transitive and euclidean, from this it follows that Z satisfies the back and forth conditions. So Z is a $(\mathbf{P} \setminus Q)$-bisimulation between (M, x) and (M', x').

$\Rightarrow$ Assume $M, x \leftrightarrows_{\mathbf{P} \setminus Q} M', x'$. From Lemma 2.4 and Remark 4.1(4), $M, x \vDash \chi\big(Q, t(Q, M', x')\big)$. By uniqueness in 4, $t(Q, M, x) = t(Q, M', x')$. $\dashv$

What Lemma 4.1 demonstates is that the quantifier-free formulas $\chi(Q, \mathbb{T})$ together already have *maximal* $(\mathbf{P} \setminus Q)$-bisimulation invariant expressive power. On the other hand, the language L *is* $(\mathbf{P} \setminus Q)$-bisimulation invariant (Lemma 2.4). So L should be no more expressive than its quantifier-free fragment, as we confirm in the next lemma. This observation allows us to aim for a reduction style completeness proof for the system **EM**.

LEMMA 4.2. *Let $\phi \in L$ and $\mathrm{FV}(\phi) \subseteq Q \subset_\omega \mathbf{P}$. Then there exists a unique $\mathbb{T}(Q, \phi) \subseteq \mathcal{T}(Q)$ such that*

$$\mathcal{C}^{\text{ste}} \vDash \chi\big(Q, \mathbb{T}(Q, \phi)\big) \leftrightarrow \phi. \tag{3}$$

PROOF. Let

$$\mathbb{T}(Q,\phi) = \{t \in \mathcal{T}(Q) : t = t(Q,M,x) \text{ for some pointed } \mathcal{C}^{\mathrm{ste}}\text{-model}(M,x) \text{ with } M,x \vDash \phi\}.$$

Then $\mathcal{C}^{\mathrm{ste}} \vDash \phi \to \chi(Q,\mathbb{T}(Q,\phi))$, because whenever (M,x) is a pointed $\mathcal{C}^{\mathrm{ste}}$-model with $M,x \vDash \phi$, by Lemma 4.1(4) there is a $t \in \mathbb{T}(Q,\phi)$ such that $M,x \vDash \chi(Q,t)$. For the other implication, suppose that (M,x) is a pointed $\mathcal{C}^{\mathrm{ste}}$-model with $M,x \vDash \chi(Q,\mathbb{T}(Q,\phi))$. By unique existence in Lemma 4.1(4), this implies $t(Q,M,x) \in \mathbb{T}(Q,\phi)$. So there is a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M',x') with $t(Q,M,x) = t(Q,M',x')$ and $M',x' \vDash \phi$. By Lemma 4.1(6), $M,x \leftrightarroweq_{\mathbf{P}\setminus Q} M',x'$. From Lemma 2.4 it follows $M,x \vDash \phi$, as desired.

For uniqueness of $\mathbb{T}(Q,\phi)$, suppose that $\mathbb{T}(Q,\phi), \mathbb{T}'(Q,\phi) \subseteq \mathcal{T}(Q)$ both satisfy (3) and let $t \in \mathbb{T}(Q,\phi)$. By Lemma 4.1(5), there is a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M,x) with $t(Q,M,x) = t$. Hence $M,x \vDash \chi(Q,\mathbb{T}(Q,\phi))$. Now the assumption implies $M,x \vDash \phi$, and so $M,x \vDash \chi(Q,\mathbb{T}'(Q,\phi))$. By the uniqueness of $t(Q,M,x)$ in Lemma 4.1(4), conclude that $t \in \mathbb{T}'(Q,\phi)$. We have shown that $\mathbb{T}(Q,\phi) \subseteq \mathbb{T}'(Q,\phi)$, concluding the proof. ⊣

PROPOSITION 4.1 (finite model property). *If $\phi \in L$ and there is a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M,x) satisfying $M,x \vDash \phi$, then there is a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M',x') with a universe of size at most $1 + 2^{|\phi|}$ such that $M',x' \vDash \phi$.*[7]

PROOF. Write $Q = \mathrm{FV}(\phi)$. By Lemma 4.2, we have

$$\mathcal{C}^{\mathrm{ste}} \vDash \phi \leftrightarrow \chi(Q,\mathbb{T}(Q,\phi)).$$

So since $M,x \vDash \phi$, we have $\mathbb{T}(Q,\phi) \neq \emptyset$. Pick $t \in \mathbb{T}(Q,\phi)$. Clearly, $|Q| \leq |\phi|$. From Lemma 4.1(5), we get a pointed $\mathcal{C}^{\mathrm{ste}}$-model (M',x') with $M',x' \vDash \chi(Q,t)$ whose universe has size at most $1+2^{|Q|}$. Then also $M',x' \vDash \phi$. ⊣

DEFINITION 4.2. Let $Q \subseteq P \subset_\omega \mathbf{P}$. For $a \in \mathcal{P}(P)$, define $a \restriction Q = a \cap Q$.
For $s \in \mathcal{S}(P)$ define $s \restriction Q = \{a \restriction Q : a \in s\}$.
For $t = (a,s) \in \mathcal{T}(P)$ define $t \restriction Q = (a \restriction Q, s \restriction Q)$.
For $\mathbb{T} \subseteq \mathcal{T}(P)$, define $\mathbb{T} \restriction Q = \{t \restriction Q : t \in \mathbb{T}\}$.

Remark 4.2. We took $\restriction$ to mean four different things. Which one we are thinking of should always be clear from the choice of letters. ⊣

[7] Here $|\phi|$ is the number of symbols in the formula ϕ.

LEMMA 4.3. *Let $Q \subseteq P \subset_\omega \mathbf{P}$.*

1. *For any pointed Kripke model (M, x) we have*

$$a(Q, M, x) = a(P, M, x) \upharpoonright Q.$$

2. *For any pointed serial Kripke model (M, x) we have*

$$s(Q, M, x) = s(P, M, x) \upharpoonright Q.$$

3. *For any pointed serial Kripke model (M, x) we have*

$$t(Q, M, x) = t(P, M, x) \upharpoonright Q.$$

4. *For any $a \in \mathcal{P}(P)$ we have*

$$\mathbf{KD45}_\Box \vdash \chi(P, a) \to \chi(Q, a \upharpoonright Q).$$

5. *For any $s \in \mathcal{S}(P)$ we have*

$$\mathbf{KD45}_\Box \vdash \chi(P, s) \to \chi(Q, s \upharpoonright Q).$$

6. *For any $t \in \mathcal{T}(P)$ we have*

$$\mathbf{KD45}_\Box \vdash \chi(P, t) \to \chi(Q, t \upharpoonright Q).$$

7. *For any $\mathbb{T} \subseteq \mathcal{T}(P)$ we have*

$$\mathbf{KD45}_\Box \vdash \chi(P, \mathbb{T}) \to \chi(Q, \mathbb{T} \upharpoonright Q).$$

PROOF. 1, 2 and 3 follow from Lemma 4.1(2,3,4) respectively.

We claim that for $h = a, s, t$:

$$\mathcal{C}^{\text{ste}} \vDash \chi(P, h) \to \chi(Q, h \upharpoonright Q). \tag{4}$$

To check this, let (M, x) be a pointed $\mathcal{C}^{\text{ste}}$-model with $M, x \vDash \chi(P, h)$. Then $h(P, M, x) = h$ by Lemma 4.1-(2/3/4). By 1, 2 or 3 we find $h(Q, M, x) = h \upharpoonright Q$, whence $M, x \vDash \chi(Q, h \upharpoonright Q)$ by Lemma 4.1(2,3,4). This confirms (4).

Now 4, 5 and 6 follow from (4), Lemma 3.2 and Remark 4.1(3).

6 entails 7. ⊣

Remark 4.3. In Lemma 4.3, $\mathbf{KD45}_\Box$ can be replaced by **Bool** in item 4 and by $\mathbf{K}_\Box$ in items 5, 6 and 7. But we don't need this for the completeness proof of **EM**. ⊣

Lemma 4.4. *Let $Q \subset_\omega \mathbf{P}$ and $p \in \mathbf{P} \setminus Q$.*

1. *For $a \in \mathcal{P}(Q \sqcup \{p\})$,*
$$\mathbf{EM} \vdash (\exists p)\chi(Q \sqcup \{p\}, a) \leftrightarrow \chi(Q, a \restriction Q).$$
2. *For $s \in \mathcal{S}(Q \sqcup \{p\})$,*
$$\mathbf{EM} \vdash (\exists p)\chi(Q \sqcup \{p\}, s) \leftrightarrow \chi(Q, s \restriction Q).$$
3. *For $t \in \mathcal{T}(Q \sqcup \{p\})$,*
$$\mathbf{EM} \vdash (\exists p)\chi(Q \sqcup \{p\}, t) \leftrightarrow \chi(Q, t \restriction Q).$$
4. *For $\mathbb{T} \subseteq \mathcal{T}(Q \sqcup \{p\})$,*
$$\mathbf{EM} \vdash (\exists p)\chi(Q \sqcup \{p\}, \mathbb{T}) \leftrightarrow \chi(Q, \mathbb{T} \restriction Q).$$

Proof. 4 follows from 3 by Lemma 3.4.

We first prove "$\rightarrow$" in 1, 2 and 3. Let h denote either a, s or t (it also works for $\mathbb{T}$).

In **EM**:

1	(1)	$(\exists p)\chi(Q \sqcup \{p\}, h)$	A
	(2)	$\chi(Q \sqcup \{p\}, h) \rightarrow \chi(Q, h \restriction Q)$	Lemma 4.3(4, 5,6)
	(3)	$(\exists p)\chi(Q \sqcup \{p\}, h) \rightarrow$ $(\exists p)\chi(Q, h \restriction Q)$	2, Lemma 3.5
1	(4)	$(\exists p)\chi(Q, h \restriction Q)$	1, 3
1	(5)	$\chi(Q, h \restriction Q)$	4, Lemma 3.7, Remark 4.1(4)
	(6)	$(\exists p)\chi(Q \sqcup \{p\}, h) \rightarrow \chi(Q, h \restriction Q)$	5

Next prove "$\leftarrow$" for the items 1, 2 and 3 successively. Choose $q, r \in \mathbf{P} \setminus (Q \sqcup \{p\})$ such that $q \neq r$.

Item 1. Let $a \in \mathcal{P}(Q \sqcup \{p\})$. Let $\pi = p$ if $p \in a$ and $\pi = \neg p$ if $p \notin a$. Then $\chi(Q, a \restriction Q) \wedge \pi$ is equivalent (in **Bool**) to $\chi(Q \sqcup \{p\}, a)$.

In **EM**:

1	(1)	$\chi(Q, a \restriction Q)$	A
1	(2)	$\pi \rightarrow \chi(Q \sqcup \{p\}, a)$	1, **Bool**

1	(3)	$(\exists p)\pi \to (\exists p)\chi(Q \sqcup \{p\}, a)$	2, Lemma 3.5, Remark 4.1(4)
	(4)	$(\exists p)\pi$	Lemma 3.8(1,2)
1	(5)	$(\exists p)\chi(Q \sqcup \{p\}, a)$	3, 4
	(6)	$\chi(Q, a \restriction Q) \to (\exists p)\chi(Q \sqcup \{p\}, a)$	5

Item 2. This part contains the technical heart of the completeness proof. Let $s \in \mathcal{S}(Q \sqcup \{p\})$. Let

$$\begin{aligned} s_{11} &= \{a \in \mathcal{P}(Q) : a \sqcup \{p\} \in s \ni a\}, \\ s_{10} &= \{a \in \mathcal{P}(Q) : a \sqcup \{p\} \in s \not\ni a\}, \\ s_{01} &= \{a \in \mathcal{P}(Q) : a \sqcup \{p\} \notin s \ni a\}, \\ s_{00} &= \{a \in \mathcal{P}(Q) : a \sqcup \{p\} \notin s \not\ni a\}. \end{aligned}$$

We have $s \restriction Q = s_{11} \sqcup s_{10} \sqcup s_{01}$. Proceed by induction on $N = |s_{11}|$. Suppose that $N = 0$. Then

$$s \restriction Q = s_{10} \sqcup s_{01}. \tag{5}$$

Also $\mathcal{P}(Q) = s_{10} \sqcup s_{01} \sqcup s_{00}$, hence

$$\mathcal{P}(Q \sqcup \{p\}) = \{a, a \sqcup \{p\} : a \in s_{10} \sqcup s_{01} \sqcup s_{00}\}. \tag{6}$$

Define

$$\phi = \Box\Big(p \leftrightarrow \bigvee_{a \in s_{10}} \chi(Q, a)\Big).$$

Observe by uniqueness in Lemma 4.1(2) that

$$\mathcal{C}^{\mathrm{ste}} \vDash \chi(Q, a) \to \neg \bigvee_{a' \in s_{10}} \chi(Q, a') \quad \text{for each } a \in s_{01}. \tag{7}$$

In **EM**:

(1)	$\Box(\chi(Q, a) \to$ $\neg \bigvee_{a' \in s_{10}} \chi(Q, a'))$ [for each $a \in s_{01}$]	(7), Lemma 3.2, Remark 4.1(3), ($\Box$I)
(2)	$\phi \to \Box(\chi(Q, a) \to p)$ [for each $a \in s_{10}$]	$\mathbf{K}_\Box$
(3)	$\phi \to \Box(\chi(Q, a) \to \neg p)$ [for each $a \in s_{01}$]	1, $\mathbf{K}_\Box$

4	(4)	$\chi(Q, s \upharpoonright Q)$	A
	(5)	$(\exists p)\phi$	Lemma 3.8(3), Remark 4.1(4)
4	(6)	$\bigwedge_{a \in s_{10}} \Diamond\chi(Q, a)$	4, (5)
4	(7)	$\phi \to \Diamond(\chi(Q, a) \wedge p) \wedge$ $\neg\Diamond(\chi(Q, a) \wedge \neg p)$ [for each $a \in s_{10}$]	2, 6, $\mathbf{K}_\Box$
4	(8)	$\phi \to \Diamond\chi(Q \sqcup \{p\}, a \sqcup \{p\}) \wedge$ $\neg\Diamond\chi(Q \sqcup \{p\}, a)$ [for each $a \in s_{10}$]	7
4	(9)	$\bigwedge_{a \in s_{01}} \Diamond\chi(Q, a)$	4, (5)
4	(10)	$\phi \to \Diamond(\chi(Q, a) \wedge \neg p) \wedge$ $\neg\Diamond(\chi(Q, a) \wedge p)$ [for each $a \in s_{01}$]	3, 9, $\mathbf{K}_\Box$
4	(11)	$\phi \to \Diamond\chi(Q \sqcup \{p\}, a) \wedge$ $\neg\Diamond\chi(Q \sqcup \{p\}, a \sqcup \{p\})$ [for each $a \in s_{01}$]	10
4	(12)	$\bigwedge_{a \in s_{00}} \neg\Diamond\chi(Q, a)$	4, (5)
4	(13)	$\phi \to \neg\Diamond(\chi(Q, a) \wedge p) \wedge$ $\neg\Diamond(\chi(Q, a) \wedge \neg p)$ [for each $a \in s_{00}$]	12, $\mathbf{K}_\Box$
4	(14)	$\phi \to \neg\Diamond\chi(Q \sqcup \{p\}, a \sqcup \{p\}) \wedge$ $\neg\Diamond\chi(Q \sqcup \{p\}, a)$ [for each $a \in s_{00}$]	13
4	(15)	$\phi \to \chi(Q \sqcup \{p\}, s)$	8, 11, 14, (6)
4	(16)	$(\exists p)\phi \to (\exists p)\chi(Q \sqcup \{p\}, s)$	15, Lemma 3.5, Remark 4.1(4)
4	(17)	$(\exists p)\chi(Q \sqcup \{p\}, s)$	5, 16
	(18)	$\chi(Q, s \upharpoonright Q) \to (\exists p)\chi(Q \sqcup \{p\}, s)$	17

Next suppose $N > 0$. Choose some $a_0 \in s_{11}$ and let $s' = s \setminus \{a_0\}$. Since $a_0 \sqcup \{p\} \in s$, we have $a_0 \sqcup \{p\} \in s'$, and therefore

$$a_0 \in s' \restriction Q = s \restriction Q. \tag{8}$$

Furthermore, $s'_{11} := \{a \in \mathcal{P}(Q) : a \sqcup \{p\} \in s' \ni a\} = s_{11} \setminus \{a_0\}$; so $|s'_{11}| = N - 1$. By the inductive hypothesis and (8),

$$\mathbf{EM} \vdash \chi(Q, s \restriction Q) \to (\exists p)\chi(Q \sqcup \{p\}, s').$$

Hence by Lemma 3.1 and Remark 4.1(4) also

$$\mathbf{EM} \vdash \chi(Q, s \restriction Q) \to (\exists q)\psi, \tag{9}$$

where $\psi = [q/p]\big(\chi(Q \sqcup \{p\}, s')\big)$.

Furthermore write

$$\begin{gathered} \phi = \Box\big(p \leftrightarrow (r \wedge \chi(Q, a_0)) \vee (q \wedge \neg\chi(Q, a_0))\big), \\ \rho = \Diamond\big(\chi(Q, a_0) \wedge r\big) \wedge \Diamond\big(\chi(Q, a_0) \wedge \neg r\big) \\ \mathcal{A} = \mathcal{P}\big(Q \sqcup \{p\}\big) \setminus \{a_0, a_0 \sqcup \{p\}\}. \end{gathered}$$

So

$$s \cap \mathcal{A} = s' \cap \mathcal{A}. \tag{10}$$

Since $a \restriction Q \neq a_0$ for $a \in \mathcal{A}$, by Lemmas 4.1(2) and 4.3(1), we have that

$$\mathcal{C}^{\mathrm{ste}} \vDash \chi\big(Q \sqcup \{p\}, a\big) \to \neg\chi(Q, a_0) \quad \text{for each } a \in \mathcal{A}.$$

So by Lemma 3.2 and Remark 4.1(3),

$$\mathbf{EM} \vdash \chi\big(Q \sqcup \{p\}, a\big) \to \neg\chi(Q, a_0) \quad \text{for each } a \in \mathcal{A}. \tag{11}$$

By Lemma 3.1 and Remark 4.1(4) it follows

$$\mathbf{EM} \vdash \big([q/p]\chi(Q \sqcup \{p\}, a)\big) \to \neg\chi(Q, a_0) \quad \text{for each } a \in \mathcal{A}. \tag{12}$$

As a last piece of preparation, observe that if λ is a purely propositional formula, then

$$\mathbf{Bool} \vdash (p \leftrightarrow \ell) \to \big([\ell/p]\lambda \leftrightarrow \lambda\big) \quad \text{for } \ell = q, r. \tag{13}$$

By Remark 4.1(2), in particular for each $a \in \mathcal{A}$ we have

$$\mathbf{Bool} \vdash (p \leftrightarrow q) \to \big([q/p]\chi\big(Q \sqcup \{p\}, a\big) \leftrightarrow \chi\big(Q \sqcup \{p\}, a\big)\big). \tag{14}$$

In **EM**:

	(1)	$\Box\big(\chi(Q \sqcup \{p\}, a) \to \neg\chi(Q, a_0)\big)$ [for each $a \in \mathcal{A}$]	(11), ($\Box$I)
	(2)	$\Box\big([q/p](\chi(Q \sqcup \{p\}, a)) \to \neg\chi(Q, a_0)\big)$ [for each $a \in \mathcal{A}$]	(12), ($\Box$I)
	(3)	$\Box\big((p \leftrightarrow q) \to \big([q/p]\chi\big(Q \sqcup \{p\}, a\big) \leftrightarrow \chi(Q \sqcup \{p\}, a)\big)\big)$ [for each $a \in \mathcal{A}$]	(14), ($\Box$I)
4	(4)	$\chi\big(Q, s \restriction Q\big)$	A
4	(5)	$\Diamond\chi(Q, a_0)$	4, (8)
4	(6)	$(\exists r)\rho$	5, ($\Diamond$2)
7	(7)	ρ	A
4	(8)	$(\exists q)\psi$	4, (9)
9	(9)	ψ	A
	(10)	$(\exists p)\phi$	Lemma 3.8(3), Remark 4.1(4)
11	(11)	ϕ	A
11	(12)	$\Box\big(\chi(Q, a_0) \to (p \leftrightarrow r)\big)$	11, $\mathbf{K}_\Box$
7, 11	(13)	$\Diamond\big(\chi(Q, a_0) \wedge p\big) \wedge \Diamond\big(\chi(Q, a_0) \wedge \neg p\big)$	7, 12, $\mathbf{K}_\Box$
7, 11	(14)	$\Diamond\chi\big(Q \sqcup \{p\}, a_0 \sqcup \{p\}\big) \wedge \Diamond\chi\big(Q \sqcup \{p\}, a_0\big)$	13
11	(15)	$\Box\big(\neg\chi(Q, a_0) \to (p \leftrightarrow q)\big)$	11, $\mathbf{K}_\Box$
11	(16)	$\Box\big([q/p]\big(\chi(Q \sqcup \{p\}, a)\big) \leftrightarrow \chi(Q \sqcup \{p\}, a)\big)$ [for each $a \in \mathcal{A}$]	1, 2, 3, 15, $\mathbf{K}_\Box$
7, 9, 11	(17)	$\bigwedge\{\Diamond\chi(Q \sqcup \{p\}, a) : a \in \mathcal{A} \cap s'\} \wedge \bigwedge\{\neg\Diamond\chi(Q \sqcup \{p\}, a) : a \in \mathcal{A} \setminus s'\}$	9, 16, $\mathbf{K}_\Box$

$7, 9, 11$	(18)	$\chi(Q \sqcup \{p\}, s)$	14, 17, (10)
$7, 9$	(19)	$\phi \to \chi(Q \sqcup \{p\}, s)$	18
$7, 9$	(20)	$(\exists p)\phi \to (\exists p)\chi(Q \sqcup \{p\}, s)$	19, Lemma 3.5, Remark 4.1(4)
$7, 9$	(21)	$(\exists p)\chi(Q \sqcup \{p\}, s)$	10, 20
7	(22)	$\psi \to (\exists p)\chi(Q \sqcup \{p\}, s)$	21
7	(23)	$(\exists q)\psi \to (\exists q)(\exists p)\chi(Q \sqcup \{p\}, s)$	22, Lemma 3.5, Remark 4.1(4)
$4, 7$	(24)	$(\exists q)(\exists p)\chi(Q \sqcup \{p\}, s)$	8, 23
4	(25)	$\rho \to (\exists q)(\exists p)\chi(Q \sqcup \{p\}, s)$	24
4	(26)	$(\exists r)\rho \to (\exists r)(\exists q)(\exists p)\chi(Q \sqcup \{p\}, s)$	25, Lemma 3.5, Remark 4.1(4)
4	(27)	$(\exists r)(\exists q)(\exists p)\chi(Q \sqcup \{p\}, s)$	6, 26
4	(28)	$(\exists p)\chi(Q \sqcup \{p\}, s)$	27, Lemma 3.7, Remark 4.1(4)
	(29)	$\chi(Q, s \restriction Q) \to (\exists p)\chi(Q \sqcup \{p\}, s)$	28

We smoothly derive "$\leftarrow$" for item 3 from "$\leftarrow$" for the previous two items. Let $t = (a, s)$. Set $\phi = (p \leftrightarrow r) \wedge \Box(p \leftrightarrow q)$, $\psi = [q/p](\chi(Q \sqcup \{p\}, s))$ and $\rho = [r/p](\chi(Q \sqcup \{p\}, a))$.

In **EM**:

1	(1)	$\chi(Q, t \restriction Q)$	A
1	(2)	$\chi(Q, a \restriction Q)$	1
1	(3)	$\chi(Q, s \restriction Q)$	1
1	(4)	$(\exists q)\psi$	3, item 2, Lemma 3.1, Remark 4.1(4)
5	(5)	ψ	A
1	(6)	$(\exists r)\rho$	2, item 1, Lemma 3.1, Remark 4.1(4)

7	(7)	ρ	A
	(8)	$(\exists p)\phi$	Lemma 3.6
9	(9)	ϕ	A
9	(10)	$p \leftrightarrow r$	9
7, 9	(11)	$\chi(Q \sqcup \{p\}, a)$	7, 10, (13), Remark 4.1(2)
9	(12)	$\Box(p \leftrightarrow q)$	9
	(13)	$\Box\big((p \leftrightarrow q) \to \big(\chi(Q \sqcup \{p\}, a') \leftrightarrow [q/p]\chi(Q \sqcup \{p\}, a')\big)\big)$ [for each $a' \in \mathcal{P}(Q \sqcup \{p\})$]	(13), Remark 4.1(2), ($\Box$I)
5, 9	(14)	$\chi(Q \sqcup \{p\}, s)$	5, 12, 13, $\mathbf{K}_\Box$
5, 7, 9	(15)	$\chi(Q \sqcup \{p\}, t)$	11, 14
5, 7	(16)	$\phi \to \chi(Q \sqcup \{p\}, t)$	15
5, 7	(17)	$(\exists p)\phi \to (\exists p)\chi(Q \sqcup \{p\}, t)$	16, Lemma 3.5
5, 7	(18)	$(\exists p)\chi(Q \sqcup \{p\}, t)$	8, 17
5	(19)	$\rho \to (\exists p)\chi(Q \sqcup \{p\}, t)$	18
5	(20)	$(\exists r)\rho \to (\exists r)(\exists p)\chi(Q \sqcup \{p\}, t)$	19, Lemma 3.5, Remark 4.1(4)
1, 5	(21)	$(\exists r)(\exists p)\chi(Q \sqcup \{p\}, t)$	6, 20
1	(22)	$\psi \to (\exists r)(\exists p)\chi(Q \sqcup \{p\}, t)$	21
1	(23)	$(\exists q)\psi \to (\exists q)(\exists r)(\exists p)\chi(Q \sqcup \{p\}, t)$	22, Lemma 3.5, Remark 4.1(4)
1	(24)	$(\exists q)(\exists r)(\exists p)\chi(Q \sqcup \{p\}, t)$	4, 23
1	(25)	$(\exists p)\chi(Q \sqcup \{p\}, t)$	24, Lemma 3.7, Remark 4.1(4)
	(26)	$\chi(Q, t{\restriction}Q) \to (\exists p)\chi(Q \sqcup \{p\}, t)$	25 $\dashv$

LEMMA 4.5. *For every* $\phi \in L$ *and* $Q \subset_\omega \mathbf{P}$ *with* $\mathrm{FV}(\phi) \subseteq Q$,

$$\mathbf{EM} \vdash \chi\big(Q, \mathbb{T}(Q,\phi)\big) \leftrightarrow \phi.$$

The formula $\chi\big(Q, \mathbb{T}(Q,\phi)\big)$ *is quantifier-free.*

PROOF. The second statement follows from Remark 4.1(3). We prove the first statement by induction on the complexity of ϕ.

- $\phi \in \mathbf{P} \cup \{\top\}$. Then $\mathbf{KD45}_\Box \vdash \chi\big(Q, \mathbb{T}(Q,\phi)\big) \leftrightarrow \phi$ by Lemmas 3.2 and 4.2 and Remark 4.1(3).
- $\phi = \neg\psi$. Since (by Lemma 4.2)

$$\mathcal{C}^{\mathrm{ste}} \vDash \psi \leftrightarrow \chi\big(Q, \mathbb{T}(Q,\psi)\big)$$

and

$$\mathcal{C}^{\mathrm{ste}} \vDash \phi \leftrightarrow \chi\big(Q, \mathbb{T}(Q,\phi)\big),$$

it follows

$$\mathcal{C}^{\mathrm{ste}} \vDash \chi\big(Q, \mathbb{T}(Q,\phi)\big) \leftrightarrow \neg\chi\big(Q, \mathbb{T}(Q,\psi)\big),$$

whence

$$\mathbf{EM} \vdash \chi\big(Q, \mathbb{T}(Q,\phi)\big) \leftrightarrow \neg\chi\big(Q, \mathbb{T}(Q,\psi)\big)$$

by Lemma 3.2 and Remark 4.1(3). The inductive hypothesis yields

$$\mathbf{EM} \vdash \psi \leftrightarrow \chi\big(Q, \mathbb{T}(Q,\psi)\big),$$

so

$$\mathbf{EM} \vdash \phi \leftrightarrow \chi\big(Q, \mathbb{T}(Q,\phi)\big).$$

- $\phi = \psi \vee \rho$. Similar to the case for negation.
- $\phi = \Box\psi$. Similar to the case for negation.[8]
- $\phi = (\exists p)\psi$. By Lemma 4.4(4),

$$\begin{aligned}\mathbf{EM} \vdash & (\exists p)\chi\big(Q \cup \{p\}, \mathbb{T}(Q \cup \{p\}, \psi)\big) \leftrightarrow \\ & \chi\big(Q \setminus \{p\}, \mathbb{T}(Q \cup \{p\}, \psi) \restriction (Q \setminus \{p\})\big).\end{aligned} \tag{15}$$

By the inductive hypothesis,

$$\mathbf{EM} \vdash \psi \leftrightarrow \chi\big(Q \cup \{p\}, \mathbb{T}(Q \cup \{p\}, \psi)\big),$$

[8] This is the only place in the completeness proof where we use instances of $(\mathrm{K}_\Box)$ and $(\Box\mathrm{I})$ with propositional quantifiers.

so, by Lemma 3.5, we have

$$\mathbf{EM} \vdash \phi \leftrightarrow (\exists p)\chi\big(Q \cup \{p\}, \mathbb{T}(Q \cup \{p\}, \psi)\big).$$

Combining this with (15) and Remark 4.1(3), conclude that there is a quantifier-free formula ρ such that

$$\mathbf{EM} \vdash \phi \leftrightarrow \rho.$$

By Lemmas 3.3 and 4.2 it follows that

$$\mathcal{C}^{\mathrm{ste}} \vDash \rho \leftrightarrow \chi\big(Q, \mathbb{T}(Q, \phi)\big).$$

Lemma 3.2 and Remark 4.1(3) imply

$$\mathbf{EM} \vdash \rho \leftrightarrow \chi\big(Q, \mathbb{T}(Q, \phi)\big).$$

Thus

$$\mathbf{EM} \vdash \phi \leftrightarrow \chi\big(Q, \mathbb{T}(Q, \phi)\big). \qquad \dashv$$

Lemma 4.6 (completeness of **EM**). *For $\phi \in L$, we have*

$$(\mathcal{C}^{\mathrm{ste}} \vDash \phi \;\; \textit{or} \;\; \mathcal{C}^{\mathrm{cco}} \vDash^{\tau} \phi) \quad \Longrightarrow \quad \mathbf{EM} \vdash \phi.$$

Proof. Assume $\mathcal{C}^{\mathrm{ste}} \vDash \phi$ or $\mathcal{C}^{\mathrm{cco}} \vDash^{\tau} \phi$. By Lemma 4.5, there is a quantifier-free formula $\rho \in L$ such that $\mathbf{EM} \vdash \phi \leftrightarrow \rho$. By Lemma 3.3, $\mathcal{C}^{\mathrm{ste}} \vDash \rho$ or $\mathcal{C}^{\mathrm{cco}} \vDash^{\tau} \rho$. By Lemma 3.2, $\mathbf{KD45}_{\Box} \vdash \rho$. So $\mathbf{EM} \vdash \phi$. ⊣

From Lemmas 3.3 and 4.6 we obtain:

Theorem 4.1 (adequacy of **EM**). *For any $\phi \in L$, the following are equivalent*:

(i) $\mathcal{C}^{\mathrm{ste}} \vDash \phi$,
(ii) $\mathbf{EM} \vdash \phi$,
(iii) $\mathcal{C}^{\mathrm{co}} \vDash^{\tau} \phi$,
(iv) $\mathcal{C}^{\mathrm{cco}} \vDash^{\tau} \phi$.

Proposition 4.2 (decidability). 1. *There is an algorithm that decides, given $\phi \in L$ and a finite pointed $\mathcal{C}^{\mathrm{ste}}$-model (M, x), whether $M, x \vDash \phi$.*
2. *There is an algorithm that decides, given $\phi \in L$, whether $\mathbf{EM} \vdash \phi$.*

Proof. Start enumerating the theorems of **EM**. By Lemma 4.5, at some point we derive $\rho \leftrightarrow \phi$, for some quantifier-free formula ρ. Thus both items 1 and 2 follow from the corresponding results for $\mathbf{KD45}_{\Box}$ (see Segerberg, 1968). ⊣

Future research. For some weaker logics than $\mathbf{KD45}_{\Box}$, such as $\mathbf{K5}_{\Box}$ which is modelled by euclidean frames, it seems plausible that the relational semantics with bisimulation quantifiers can be axiomatized in a way similar to **EM**. It is a question whether equivalent topological semantics (perhaps building on (Steinsvold, 2008)) also exist for those logics.

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