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Local Applications of Logics via Model-Theoretic Interpretations

Abstract. This paper analyses the notion of 'interpretation', which is often tied to the semantic approach to logic, where it is used when referring to truth-value assignments, for instance. There are, however, other uses of the notion that raise interesting problems. These are the cases in which interpreting a logic is closely related to its justification for a given application. The paper aims to present an understanding of interpretations that supports the model-theoretic characterization of validity to the detriment of the proof-theoretic one. This is done by making use of the hierarchy of **ST**-related logics. Finally, a localist conception of logic is defended as the natural view stemming from the model-theoretic approach.

Keywords: interpretations; model-theoretic semantics; proof-theoretic semantics; localism, ST-hierarchy

1. Introduction

The notion of 'interpretation' is a widely used one in the logical literature, but it is employed in various related but different senses. Most certainly, it is a concept that is obviously tied to the semantic approach to logic, where we use it when referring to truth-value assignments, for instance. This is one of the technical uses of the notion that will not be the main focus of the paper.

There are, however, other uses that raise interesting problems, or so I will argue. These are the cases in which interpreting a logic is closely related to its justification for a given application. Thus, I will focus on that kind of interpretation and, more concretely, I will try to show how

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certain understanding of interpretations supports the model-theoretic characterization of validity to the detriment of the proof-theoretic one.

In doing so, my argument will rely on a novel identity criterion for logical systems, developed by E. Barrio, F. Pailos and D. Szmuc. The authors argue, for reasons that will be made clear later on, that a logic is to be identified with an infinite hierarchy of inferences and they illustrate such a hierarchy by constructing the **ST**-collection. I will show, then, how this novel identity criterion necessitates a model-theoretic approach if the logical systems are to be justified for any application at all.

I will conclude with some remarks on the kind of conception of logic that these issues favour. In particular, I claim that the model-theoretic characterization fosters localism in logic, which we will distinguish from the standard conception of pluralism championed by Beall and Restall [2006].

The strategy, then, will be the following. In Section 2, I present the conceptual framework for the problems that we are going to deal with. That is, I clarify the notions of interpretation and treat the issue of the applications of logics. In Section 3, I summarize the debate between model-theorists and proof-theorists and frame it within the more general discussion between inferentialism and representationalism. Then, in Section 4, I present the novel identity criterion and develop my objection to proof-theoretic semantics. Finally, in Section 5, I advance my localist thesis based on the model-theoretic characterization of validity. Let us now turn to the conceptual framework.

2. Conceptual framework

2.1. What do we mean by 'interpretation'?

To my mind, there are at least two senses of 'interpretation' that should be kept apart within the discussion that is relevant to us. We can broadly refer to them as the *internal* and the *external* interpretations. On the one hand, internal interpretations are the truth-makers of the atomic formulas of a language, L. This is the notion that we all learn when presented with a formal semantics in introductory logic courses. So understood, 'interpretation' is a concept that belongs to the semantics and that can take the form of an assignment of truth values to each and every propositional letter, of a structure in which atomic formulas are true or false (or something else), etc.

On the other hand, internal interpretations also have to do with the meaning of the logical vocabulary. In this sense, when we interpret a logic, we specify the meanings of the logical symbols. Now, I believe that this already is an interesting notion, the reason being that it often plays a central role in accounting for validity and logical consequence and that it is disputable how to better explain the meaning specification of the logical symbols.

On top of these interpretations of vocabulary (logical and non-logical) there is another kind of interpretation that we shall call 'external'. This is a notion of interpretation that has to do with the philosophical justification of a logic; i.e., with an explanation of why our meaning specification of the logical vocabulary is such that it makes some inferences valid and not others. Moreover, this justification is typically done in the light of a concrete application of the logic. So, for instance, one might interpret a modal operator so as to capture the concept of metaphysical necessity, legal permissibility, knowledge, and so on. Or one can give a justification of the sense in which a third truth-value is understood in terms of accounting for empty names, or in terms of an epistemic reading of paraconsistency, or whatnot. Thus, to interpret a logic, in this sense, is a broader task than the previous two, since it implies giving a justification of the logic, as a whole, for a given application.[1]

Now, I have been speaking loosely about the applications of a logic, but it is not a settled issue at all what the applications of a logic might be, or whether it makes sense to distinguish between applied and pure logics. But, certainly, these are problems that affect our views on interpretations.

2.2. On the applications of logic

I believe it is largely uncontentious, nowadays, that the *pure/applied* distinction holds when speaking about logics. Notably, Priest [2005] justifies the distinction by drawing an analogy with geometry and arithmetic (an idea due to Łukasiewicz) and appeals to it in order to argue for

[1] This way of interpreting a logic is closely related to a requirement that G. Priest (2005) imposes on (model-theoretic) semantics in order for it to have any philosophical import and, so, to be justified, which is that the semantics be *informative*, as opposed to merely technical. I do not restrict, however, external interpretations to model-theory.

the revisability of logic and, so, for a sort of anti-exceptionalism about logic along Quinean lines.

But, even if one does not want to follow Priest all the way to anti-exceptionalism (personally, I would), the distinction is easy enough to grasp and does justice to the practice of logic. There are many pure logics. We define them, we study their properties, prove results about them, relations between them, etc. These are "well-defined mathematical structure[s] with a proof-theory, model theory, etc." [Priest, 2005, p. 195].

There are, however, other aspects of doing logic that have to do with the application of pure logics into different domains and problems. This is a common practice within philosophical logic, for instance, where pure logics are often applied in order to deal with paradoxes, to systematically account for reasoning about knowledge, necessity, obligation, morality, etc. But, despite the variety of applications that logics have been used for, some people (most notably Priest) argue that there is a privileged application of logic, a *canonical application*, which is the analysis of reasoning. Thus, the main aim of logic would be to systematically determine what follows from what.

Even so, this is not an unanimous view. Van Benthem, for instance, has a more 'liberal' conception of logic and argues that the view of logic as being about consequence relations may have had some sense when it was thought to provide the foundations of mathematics. But, since the 1930s the field has changed and broadened its scope. Logic is now, van Benthem claims, about *definability, computation* and more [van Benthem, 2008, p. 183]. Indeed, van Benthem defends that the main issue of logic is "the variety of *informational* tasks performed by intelligent interacting agents, of which inference is only one among many, involving observation, memory, questions and answers, dialogue, or general communication" [van Benthem, 2008, p. 182].

Notice that this issue of the range of applications of logics, and whether or not there is a canonical one, affects the possible interpretations, in the external sense, of logics. This is, precisely, due to how the conceptual relation between the two senses of interpretation and the pure/applied distinction works.[2] Intuitively, while a pure logic only

[2] In [Barrio and Da Ré, 2018] the authors take a different approach by not distinguishing between internal/external interpretations and allowing both pure and applied logics to be philosophically interpreted. However, it still seems to be the case that the philosophical interpretations work better as justification procedures for the applied case.

consists of an internal interpretation, an applied logic should tell us something more about its adequacy for the domain of application. A pure logic, with a semantics, has interpretations, in the form of truth-value assignments, structures, and so on. It also has some form of specification of the meaning of logical operators by way of truth tables, rules, or inferences of the proof-theory.

But, in order to externally interpret a logic, one needs to fix some application for the logic in such a way that the logic becomes a theory of that domain of application. This involves justifying why the given logic is suitable for the application. The justification requires an explanation of our reasons for reading connectives in some way or another, for introducing more than two truth-values and choosing which ones are designated, for rejecting some inferences and deeming others valid, etc.

Thus, it seems reasonable to suppose that the view one has on what logic is about will shape the range of possible external interpretations. The more applications there are, the more interpretations there will be. So, under a conception of logic such as van Benthem's, with a 'relaxed' criterion of what logic is about, there will be many more possible interpretations than in a view like Priest's, for whom logic has a canonical application. Furthermore, adding his defence of monism to that 'canonicity', we should conclude that there only is one canonical interpretation of logic.

For ease of exposition, I will assume Priest's position on the canonical application of logic, while rejecting the monist thesis. In fact, what I am going to reject, due to substantial philosophical reasons, is his globalist thesis, as it will be made clear in Section 5 below. Instead, I am going to argue in favour of a localist thesis. Hence, I will hold that there are multiple domains of reasoning susceptible of being systematised by different logics and, therefore, multiple canonical applications and interpretations.

But let me now make a stop in order to clarify and address an objection made by an anonymous reviewer, namely that there is a methodological flaw on the way I lay down the distinctions internal/external interpretation and pure/applied logic. The problem seems to be that I allow a sort of *petitio principii* in the way I characterize external interpretations and applied logics. With respect to the former, the reviewer argues that a proof-theorist would simply reject that there is anything more to the justification of validity claims than the internal interpretations. Thus, she would not feel compelled by any argument going from external interpretations to a vindication of model-theoretic approaches, since the distinction is not sound, in the first place, for clarifying the concept of validity. External interpretations, the reviewer claims, have to do with *modelling*, which is a matter disconnected from the validity/invalidity of inferences.

I believe, however, that there is nothing fallacious in the way external interpretations are presented, since as I see it they concern equally both proof-theorists and model-theorists. To be more precise, when I claim that external interpretations deal with justifying the validity of inferences I do not mean the justification of the $\mathfrak{L}_i$-*validity* (let $\mathfrak{L}_i$ be the logic you prefer) of this or that inference. That, we agree, is what internal interpretations are for. What I claim, instead, is that external interpretations are meant to justify the validity (without a relativizing prefix) of inferences and so, the correctness of the logic as a whole. Again, this is nothing specific of a model-theoretic and/or realistic account. In fact, it is put forward in different terms depending on the philosophical tastes. One might find a realist justifying why a logic, $\mathfrak{L}_i$, is the proper theory of *validity*, a more moderate one explaining that another logic is the *correct* theory of reasoning, or an instrumentalist defending that yet another logic is the most *fruitful, useful* or *convenient* for certain purposes [Haack, 1978, p. 224].

Hence, it is true that when you externally interpret a logic you are supposed to be modelling some reasoning, and that is why I defended above the idea that applying a logic is a necessary condition for such an interpretation. But this is something that you cannot escape in any case as long as you stick, for instance, to the orthodoxy of philosophical logic and apply pure logics in order to systematically reason about given phenomena. This way of proceeding will certainly require a justification for the choice of the pure logic, thereby making external interpretations indispensable.

A nice example of the need of justifying a proof-theoretic semantics by externally interpreting it comes from the debate around multiple-conclusion arguments. Steinberger [2011], for instance, argues that inferentialists should reject multiple-conclusion systems because they fail to represent our ordinary modes of reasoning. Stating it in our terms, we could say that, even though the rules of inference of a given multiple-conclusion system, $\mathfrak{L}_i$, are $\mathfrak{L}_i$-*valid*, the system lacks a sound external interpretation, which creates a problem with the justification of the correctness of the system. I guess that this sense of 'external interpretation' is perfectly available to both model-theorists and proof-theorists.

To further clarify the point, let me try to state it more precisely. A proof-theoretic semanticist's use of 'validity' is applied only within a logic. In this sense, it is an internal notion that is different from other uses of 'validity' that aim at a concept that, via modelling reasoning, captures which arguments are valid. This latter notion is also defined internally, within a logic, but, by translating natural language arguments into the formal language of our logic, the notion of validity is transferred to natural language arguments. However, the proof-theoretic characterization of validity cannot imply that it has nothing to do with how well a given logic does when modelling certain reasoning. One logic can have perfectly harmonious connectives whose meanings are given by a set of rules, but if one wants to apply this logic in order to solve, say, a sorites paradox and the logic cannot handle it, then one will probably have to modify those meaning-giving rules, in this case taking into account the inferences that allow one to run the paradox. Thus, despite the proof-theoretic justification of validity being, in a sense, internal, via reduction procedures of some inferences to the meaning-giving ones, there is also an external justification of the meaning-giving rules that has to be taken into account, if we want our logic systems to have any successful application at all.

Going to the methodological problem relative to the notion of application, the reviewer argues that the conception I present is an objectual one, which a proof-theorist would never accept. That is, she would never accept a characterization of a logic as applied to a *domain of objects*.

What I say, though, for characterizing applications, is that logic(s) might be applied to different domains, which leaves open whether it is a singular domain of reasoning, multiple domains of reasoning or even a non-canonical domain. But, most importantly, talking about 'domains of application' does not imply that the domains are individuated by the ontological properties of the objects in that domain. They could be characterized by, say, the different inferential behaviours of speakers when reasoning about given topics. This is, I believe, perfectly in accordance with the notion of application that an inferentialist might have when applying different logics to solving semantic paradoxes, for instance.

Finally, let me end this excursus by adding that I am not proposing a clear-cut distinction between pure/applied logics in terms of the internal/external interpretations. I agree with an anonymous reviewer who suggests that there is no consensus in what is internal or external to a logic, largely because there is no definite criterion for what a logical constant is and, so, neither for 'logical citizenship'. However, we do not need that much in order to accept that there are pure logics dealing with models, worlds, proof-theories, soundness and completeness results, etc. and that there are applied ones which aim to be theories of what follows from what in a given domain of discourse or for a particular phenomenon like vagueness, for instance. We do not need that much either for accepting that it is the applied logic which gets externally interpreted, inasmuch as it is in the domain of application that we find the clues for such an interpretation, and that this interpretation can bestow *further* justification on a logic, especially in contexts of rivalry between logics.

Hopefully, the clarification has served the purpose of making the premises of the discussion more plausible for each side. What I will try to argue for now is that, given that applications and external interpretations are relevant for everyone in the discussion, as long as they aim at the sort of justification that I have referred to, model-theoretic semantics are better suited for that job.

3. Proof-theoretic vs model-theoretic

As I have previously advanced, one of my main concerns in this paper is to analyse the quite notorious dispute of whether logic should be interpreted proof-theoretically or model-theoretically (truth-theoretically). As it is well known, the discussion can be framed within the more general debate, in philosophy of language, between *inferentialism* and *representationalism*, or between the 'meaning as use' and 'meaning as truth-conditions' slogans.

In a nutshell, inferentialism is the view holding that the meaning of linguistic expressions is determined by their role in inference. To put it in a slogan inspired by one of the biggest influences for inferentialism, the later Wittgenstein, 'meaning is use'. That is, the meaning of an expression is nothing more than the rules that govern its use in conversation. Thus, to know the meaning of an expression is to know its correct use.

Representationalism, on the other hand, can be roughly presented as the thesis stating that the meaning of words has to do with the word-object relation; i.e., with how the words stand for the external entities to which they refer. To make it clear, this should not be understood as implying, by default, any metaphysical commitment. The word-object relation is something that has to be fleshed out and clarified, what can potentially be done in different ways according to the ontology of objects. So, for instance, according to whether the object at issue is mind-dependent or mind-independent, whether it is an object to which we refer by a vague concept or a precise concept, whether we attribute objective or non-objective properties to it, etc.

Another way to present this is in terms of truth-conditions. In fact, it is common to find in the literature that the two big opposing theories of meaning are inferentialism and truth-conditional semantics. In this way, the meaning of a statement is considered to be reducible to its truth-conditions, i.e. to how the world is, or could be, in order for it to be true. The early Wittgenstein made one of the most definitory statements of truth-conditional semantics in his *Tractatus*; "To understand a proposition means to know what is the case, if it is true. (One can therefore understand it without knowing whether it is true or not)"[Wittgenstein, 1922, 4.024].

But, again, we should be more careful and not attribute too many metaphysical commitments to the truth-conditional approach. It is a semantic theory that heavily relies on the concept of truth, but not, necessarily, on correspondence truth, as it is often assumed. Thus, there is, *prima facie*, no tension in formulating the concept of truth as 'superassertibility' [Wright, 1992] or 'superwarrant' [Lynch, 2009], both anti-realist and epistemically constrained notions of truth.

Now, these are general theories of meaning about natural language, but what about the meaning of the logical symbols? The answer to this question is a crucial one, since, many times, an argument is logically valid in virtue of the meanings of the logical symbols appearing in the premises and the conclusion, i.e. in virtue of the (internal) interpretation of the logic. Hence, the relevance of the proof-theoretic/model-theoretic semantics dispute.

There are, however, some subtleties to notice before going on to the main argument. It would seem as if the inferentialists' only possible bet is on proof-theoretic semantics, and similarly for representationalists/ truth-conditionalists and model-theoretic semantics. Up to now, I have also, more or less implicitly, assumed that these connections held and, therefore, that the proof-theoretic/model-theoretic debate was a particular and restricted case of the more general debate between inferentialism and truth-conditional semantics. But I believe that this is just an over-simplification of the problems, precisely, for some of the reasons that have already been pointed out.

Of course, many philosophers have argued for some of those connections. Dummett [1977] saw the proof-theoretic approach to logic as a better fit for intuitionistic logic due to his anti-realist semantic theses. Lakoff and Jackendoff reject model-theoretic semantics on anti-realistic grounds [see Abbott, 1997]. But these are cases in which there are (more or less successful) philosophical arguments establishing the connection. What we should avoid is the tacit and automatic connection that is often made between model-theory and realism.

There is a warning, then, we should have in mind when putting proof-theory and model-theory against each other; model-theoretic semantic accounts do not necessitate realism, of any kind. Model-theoretic semantics, as well as truth-conditional semantics, are perfectly compatible with other anti-realist conceptions of truth.[3] So, although it may seem more natural to go for proof-theoretic semantics if one defends inferentialism this is not an enforced option. Even more, one could be a representationalist, as a general theory of meaning, but defend proof-theoretic semantics approaches to logic, arguing that the notion of logical consequence is a relation of derivability perfectly captured by this or that proof-theory.

Now, before going on to the strict-tolerant hierarchy and to the argument for model-theory based on it, let me recall the well-known fact that the discussion is a central one in the philosophy of logic and that, therefore, there have been important arguments developed in each side against the other. Among the most relevant ones, one could highlight, on the one hand, Quine's [1970] and Dummett's [1977] arguments against model-theoretic strategies, which try to show the circularity that these exhibit and, on the other hand, Prior's [1960] 'tonk argument', aiming at an objection against the idea that rules of inference can determine connective meaning by themselves.

In this vein, it is also worth mentioning a particularly interesting line of thought that Priest [2005] mentions, but does not develop much, and that comes from substructural logics, i.e., logics that drop one or more of the classical structural rules such as Contraction, Weakening or Cut. In recent years,these rules, and others, have been put into question and, therefore, their justification is now relevant and not so obvious anymore. However, Priest argues that this justification is not possible in purely proof-theoretic terms (although see [Hjortland and Standefer, 2018] for

[3] Abbott [1997] stresses this point too.

a possible response). While Priest does not further develop the objection, I deduce that the reason for the proof-theoretic characterization to be impossible is that structural rules do not depend on connectives nor, *a fortiori*, on the connectives' meaning. Therefore, even the troublesome route of justification from connective meaning to validity would be discarded. Lacking connective meaning and the semantic notion of truth-preservation, it becomes harder to come up with an acceptable proof-theoretic justification of structural rules.

In any case, the argument that I am going to put forward now is of a different sort. Although related to a problem of justification, I will try to exploit a deficiency having to do with a type of justification stemming from external interpretations of logics.

4. An objection based on the strict-tolerant hierarchy

In their paper "A hierarchy of classical and paraconsistent logics", Barrio, Pailos and Szmuc develop a new identity criterion for logics, by analysing the cases of Strict-Tolerant (**ST**) and Classical Logic (**CL**). Here, I will try to briefly present their proposal in order to derive an additional consequence of it that is relevant to our discussion.[4]

The logic **ST** is a logic that was thought to 'classically' deal with some semantic paradoxes that affect **CL**. 'Strict' and 'Tolerant' refer to the two different notions of satisfaction that operate for the premises and conclusion, respectively. More precisely, it is semantically defined by three-valued valuations with connectives defined by Strong Kleene truth-tables. The logical consequence relation is a *mixed* one, hence, the 'Strict' and 'Tolerant' epithets, which refer to the two notions of satisfaction in play. A formula belonging to the set of premises is *strictly-satisfied* if and only if it has value 1, while a formula, appearing as a conclusion, is *tolerantly-satisfied* if and only if $v(\alpha) \neq 0$ (i.e., if and only if $v(\alpha) = 1$ or $v(\alpha) = \frac{1}{2}$). Therefore, in **ST**, an argument, $\mathit{\Gamma} \Rightarrow \alpha$[5], is *valid* if and only if there is no assignment of truth-values making the premises true and the conclusion false. And α is a *logical consequence* of $\mathit{\Gamma}$ if and only if $\mathit{\Gamma} \Rightarrow \alpha$ is valid.

[4] For a full presentation of the hierarchy see [Barrio et al., 2018, 2019, 2020] and for **ST** see [Cobreros et al., 2012].

[5] We stick to the notation that Barrio et al. [2018, 2019, 2020] use. Thus, when we write $\models_{\boldsymbol{L}} \mathit{\Gamma} \Rightarrow \alpha$, for any logic, $\boldsymbol{L}$, we mean $\mathit{\Gamma} \models_{\boldsymbol{L}} \alpha$.

The reason for focusing on **ST** and **CL**, then, is that their comparison raises the question of how to identify a logic or, better, of what the identity conditions of a logic are. This is a really important topic, among other reasons, because it is central to any conception of rivalry between logics. Certainly, we should all agree that if any two pair of logics are identical, they cannot be rivals. So, are **ST** and **CL** identical? Cobreros et. al., seem to think so, claiming that one can classically deal with some paradoxes that affect **CL** (as it is traditionally presented) by means of **ST**, the main reason of its 'classicality' being that they share the same set of valid inferences.[6] This is what Barrio et al. [2020] are going to challenge, namely, the alleged identity of **ST** and **CL**.

In doing so, what they notice is that despite **ST** and **CL** being identical at the inferential level, they do diverge at the level of *metainferences* (inferences between inferences). So, for instance, while Cut,

$$\frac{\varGamma \Rightarrow A, \varDelta \quad \varGamma', A \Rightarrow \varDelta'}{\varGamma, \varGamma' \Rightarrow \varDelta, \varDelta'}$$

is a metainference validated in **CL**, it is not valid in **ST**.[7] One could argue, then, that in order for any two logics to be identical it is a sufficient condition that they agree at the inferential as well as the metainferential levels. But, Barrio et al. [2018, 2019, 2020] argue that there is another logic, namely, **TS/ST**, which is identical to **CL** at the levels of inferences and of metainferences, but which deviates at the metametainferential level (the level of inferences between metainferences). Again, one could try to identify logics claiming that their inferences, metainferences and metametainferences are the same. However, as the authors point out "the phenomenon incarnated by the **ST** approach is pervasive" [Barrio et al., 2020, p. 96], meaning that we can keep on ascending and find another logic that is identical to **CL** at the inferential, metainferential and metametainferential level, but which deviates at the metametametainferential level (the level of inferences between metametainferences).[8] The moral is that there is no finite inferential level, n $(0 \leq n < \omega)$, at which we can stop and claim to have found the logic in the **ST**-family which is identical to **CL**, since there will always be another logic in the **ST**-family which agrees with **CL** up to $n+1$ and deviates thereafter. Hence,

[6] $\forall \varGamma, \alpha$: $\models_{\mathbf{ST}} \varGamma \Rightarrow \alpha$ if and only if $\models_{\mathbf{CL}} \varGamma \Rightarrow \alpha$.

[7] For example, $\frac{p,q \Rightarrow r,s \quad p,q,r \Rightarrow s}{p,q \Rightarrow s}$ is an instance of Cut that is valid in **CL** but not in **ST**.

[8] See [Barrio et al., 2018, 2019, 2020] for the proofs.

this 'ascention in classicality' yields a **ST**-collection, i.e., a hierarchy of increasingly classical logics, which can be recursively defined.[9]

Thus, the authors claim to have found a novel identity criterion for logics: a logic is to be identified with an infinite hierarchy of theorems, inferences, metainferences, metametainferences, and so on. As I said, it is not my purpose here to examine the correctness of the proposal. Surely there are many aspects that could be disputed, but one must acknowledge the plausibility of the proposal, at least due to its formal rigour, so we are going to suppose that the proposal is, *prima facie*, correct.

Now, it is interesting to observe that the authors adopt a semantic (model-theoretic) stance towards validity. We could say, following the distinction at the beginning, that their internal interpretation of the logics in the hierarchy is truth-conditional. One reason for doing so is given by the authors themselves. They explain that "once we grasp the semantic definition of logical consequence for a given system this settles which formulae, inferences and metainferences of any arbitrary level are valid in it" [Barrio et al., 2020, p. 106]. So, for any inferential level n ($0 \leq n < \omega$), if we give the semantic definition of logical consequence for that level, we thereby fix which inferences of levels above and below n are valid.[10]

However, despite their choice for the semantic approach, Barrio et. al., make the following comment in a footnote:

> [. . .] throughout this article we will adopt a semantic stance towards metainferential validity [. . .]. This does not mean that these issues cannot be established and explored by proof-theoretical means, but for matters of space these will be left for another occasion.
>
> [Barrio et al., 2020, p. 98]

[9] Definition 4.6 in [Barrio et al., 2020]. On reflection, we can say that, in a certain sense, there is no increase in classicality. If there are infinite inferential levels, **ST** and **TS/ST**, for instance, are equally non-classical, since they both have ω non-classical inferential levels.

[10] This is because of their definition of local validity, which they endorse, and because we can consider formulae as inferences with empty premises. The details should not worry us here, though. It is sufficient to understand that the semantic definition of logical consequence can be given for any inferential level, in the same way as we did for **ST** at the level of regular inferences, and that we can thereby fix which inferences of any arbitrary level are valid. In particular, in the **ST**-collection, the inferences belonging to the level for which the semantic definition is given and those below it will behave classically, while the levels above will be the non-classical logic, **LP**, developed by Priest.

Well, I claim that one cannot give a good enough justification for the choice and the correctness of a logic, understood under this novel identity criterion, from a purely proof-theoretic standpoint.

Recall what was said about pure and applied logics. It is possible to define the **ST**-collection of logics proof-theoretically and, so, to internally interpret its logics. But the authors aimed at an identity criterion that could be used for problems related to rivalry between logics. Yet, rivalry only makes sense when we apply the pure logics to a domain with some ends. Thus, it is in the context of applications that we must justify the correctness of a logic and so, we need to externally interpret a logic in order to decide among different rival candidates, i.e. pure logics, that aim at becoming a theory of what follows from what in that domain of application.

But, what justification can we get, from proof-theoretic standards, for the validity of an inference of level 999? Or of level 1000? Or of any level that we have never conceived nor, *a fortiori, used*? According to Prawitz [2015], one of the leading figures in proof-theoretic semantics, one should adopt a first person perspective (because we are involved in the practice we are trying to codify) when pondering over the validity of some inference and, so, over whether one should include an inference in one's deductive practice. This is because,

> Unlike physics where one may think that one is studying given phenomena, logic is a field where the object of study, the correctness of inferences, is essentially created by our decisions about what the logical constants are to mean. [Prawitz, 2015, pp. 24, 25]

But if the meaning of the logical constants is justified by our use of them and we can have no clue of how we use them in, say, level 2021, then, there is no possible justification for the choice of our logic. The fundamental problem of proof-theoretic semantics is that there are inferences that are crucial, because they are necessary identity conditions of a logical system, but that cannot be justified by their (inferentialist) standards, because there is not (and cannot be) a human inferential practice justifying each possible basic set of inferences of any level.[11]

[11] By 'basic set of inferences' meaning those from which every other inference is deducible. So, I am not claiming that we need to justify every possible inference, otherwise there would be unjustifiable inferences even in the first level, for instance, those having a sufficient amount of premises so that no human inferential practice could justify them.

That is, if it is physically impossible to decide over every inferential level, and if there cannot be an inferential practice backing up every basic (meta...meta) inference, then we run out of inferentially acceptable grounds for justification.

Someone on the side of Prawitz could reply that we do not need to go through every possible inference of each level to decide whether they are correct or not, because they are just instances of the same schematic rule, in such a way that if someone has reasons for accepting Explosion as a valid inference, she already has reasons for accepting metaExplosion, metametaExplosion, and so on and so forth. But such a homogeneity between inferential levels is something that should be justified, and just appealing to some sort of formality of logic[12] seems quite a dogmatic move, even more by inferentialist standards.[13]

However, notice that if such a justification for homogeneity could be obtained, that would immediately rule out logics like **ST**, which seem to rely on the fact that different norms of reasoning may apply for different inferential levels. This is semantically captured by defining logical consequence as a mixed notion. Thus, the inferentialist who defends **ST** finds herself in a dilemma. She can either opt for homogeneity, which is dubious and hard to justify, and rule out mixed logics like **ST**, or she can reject homogeneity in order to preserve mixed accounts of logical consequence, which would leave her without a proper justification for the choice of logic. Both horns seem rather unattractive.

Hence, why do I claim that model-theory stands in a better situation for giving an account of validity and for justifying our choice of logic? First, because I believe, contrary to Prawitz, that logic and physics are not so unlike, in the sense that a logic should aim, when canonically applied, at being a theory of correct reasoning. Thus, just as we cannot *decide* the laws of physics, we cannot decide either the laws of logic. This means that the ontology of the domain in which we are reasoning constrains the inferences that we are allowed to make. And, just as within the other sciences, the selection of the best logical theory will

[12] See [MacFarlane, 2000] for an analysis of different notions of formality.

[13] In [Cobreros et al., 2021] there is an attempt to justify a possible intuition about higher-level inferences in terms of acceptance and rejection of propositions, but the attempt shows the difficulty of making sense even of metainferences at level 2. In any case, this would count as an internal interpretation and would not be of much help in a situation in which we had to decide among rival logical systems of the hierarchy.

have to proceed by abduction, taking into account the adequacy with data, strength, simplicity, and the like.

Second, precisely because there are ontological constraints, model-theory is a better tool to be externally interpreted and so, to provide justification for the inferences we include as well as to be an informative semantics (in the sense of Priest) of the domain. To be clear, I do not agree with Popper [1979], for instance, who sees in Tarski's work the achievement of a scientifically respectable definition of *correspondence* truth. As I see it, model-theoretic semantics provides a way of rigorously interpreting a logic into its domain of application. The reason is that model-theoretic accounts can, and should, be seen as reductive theories of validity, i.e., theories about arguments whose conclusion follows from the premises, *in virtue of* their preserving *truth in a model*. Thus, the reduction of validity to truth enables a natural association between meaning, truth and world. In order to systematically account for validity, we introduce the ontological features of the domain we are reasoning about by properly constructing the model-theoretic semantics, which means, defining the possible semantic values (true, false, both, undecided, etc.), deciding which of them are designated, what does it take for a formula to be satisfied, and so on. But, as I warned above, this does not force that the relation (meaning, truth, world) is a realist one, by means of correspondence truth, as Popper claims.

For instance, a possible way of externally interpreting some model-theoretic semantics, is to philosophically justify that the domain of application for that logic is an epistemically constrained one, because it involves reasoning about objects which, although objective, are not mind-independent. One could reasonably argue in those terms when formalising our discourse about, say, moral facts or mathematical objects. In such cases, one would, then, construct the model-theory in order for it to implement an appropriate conception of truth for that ontology (e.g., Wright's [1992] superassertibility). This is usually done by adopting intuitionism, introducing more truth-values, going for Kripke models, etc.

Notice, then, that this way of proceeding, within the model-theoretic approach, would already give us reasons for rejecting, or at least doubting, the vast majority of the logics belonging to the **ST**-collection.[14] For

[14] As applied logics. They are, however, perfectly well-defined mathematical objects. Nevertheless a reviewer criticises this way of semantically 'rulling out' a logic, by appealing to cases studied in [Cozzo, 1994], where it is shown how even

what reason would we have for a logic which treats the first 8849 levels as behaving 'classically' but takes the 8850th one, and all succeeding ones, to behave non-classically? What crucial ontological, alethic or semantic differences could there possibly be between those levels? In fact, I would say that there probably are non-arbitrary philosophical reasons to justify either full **CL** or **ST**. **CL** because one would argue that there should be no difference between the way we reason about something and our reasoning about reasoning itself; i.e., that theory and metatheory (of any level) should agree. Alternatively, one could argue that logic and metalogic are applied to radically different objects and domains, and, therefore, that there are some inferences which appear to be valid in one case but not valid in the other. If that were the case, **ST** would have some philosophical justification because it behaves classically up to inferences but deviates from metainferences upwards. So, for instance, if the inferences are between propositions referring to medium-sized objects, they could behave classically, since the mind-independence of those objects is an ontological property that might allow us to justify and externally interpret the bivalence, the meaning of negation, etc. But once the inferences are between other inferences, that is, between mathematical entities, whose ontological status is a matter of philosophical dispute, those inferences could behave non-classically.

However, one could still doubt whether these semantics can be informative at, say, level 2021, but I would argue that they are. The reason is that the target inferential levels in which satisfaction is defined are the level of formulas, for **CL** and the level of inferences, in order to distinguish strict/tolerant-satisfaction, in the **ST** case. These, together with the kind of philosophical interpretations that we have provided, should suffice to justify the semantics of higher-levels recursively built, because the ontological constraints do not vary, just in the same way that we can justify the semantics for a conjunction with 2021 conjuncts. The problem with inferentialist approaches with realist scruples is that it is difficult to see what they can hold on to once we loosen the intuitions about a practice that we cannot even conceive for very high levels.[15]

We see that if one is eager to conserve the variety of technical tools

trivial inconsistent languages can have interesting semantic properties. But I do not 'rule out' those logics in absolute terms, but relative to their 'correctness' or adequacy for modelling a given inferential practice.

[15] I thank an anonymous reviewer for pressing on this point.

that different logical systems offer, including mixed ones like **ST**, while being able to externally interpret them for their justification, one should, then, embrace the model-theoretic approach. This should not be taken to mean, however, that there is nothing useful in the proof-theoretic approach. In fact, it could be possible to have a proof-theoretic internal interpretation of a logic, $\mathfrak{L}_i$, justifying the $\mathfrak{L}_i$-*validity* of its inferences, while we approach that very same logic model-theoretically in order to externally interpret it and argue for its correctness as a theory of what follows from what in a given domain of reasoning.

5. From model-theory to localism

Let us briefly address, before concluding, a point that was hinted at above. As I said, I defend a form of pluralism that I call 'localism' as opposed to 'globalism', borrowing the notions from [Haack, 1978] and [Priest, 2005]. As such, it is not a novel notion[16], but I guess that what makes the view distinctive, somehow, is how I understand localism in relation to pluralism and the conceptual improvement that this yields to the debate. My understanding of pluralism/monism and localism/globalism is such that localism (globalism) is not a form of pluralism (monism). In fact, I define them as orthogonal to each other, in the sense of being theoretically independent, driven by reasons concerned with applications of logic.

Thus, the localist thesis states that there is a multiplicity of types of objects that configure various domains of discourse, in such a way that reasoning about these different kinds of objects requires adopting different logics. The logic **ST**, for instance, is in itself a nice example of localism since, as we noted above, different norms of reasoning seem to be governing different inferential levels. Globalism, on the other hand, is the position defending that the application of logic is global, in the sense that logical laws and valid arguments must be applicable regardless of the content (the different objects we might reason about). Under the assumption that there is a canonical application of logic (assumption that we have made following [Priest, 2005]), this means that localism implies, contrary to globalism, that there are *sub-canonical applications*, i.e., different domains of reasoning that require different logical theories.

[16] Zardini [2018] uses 'regionalism' in the same spirit as I use 'localism'.

Pluralism and monism, however, are understood *à la* Beall and Restall [2006], which is probably the most influential conception. In a nutshell, Beall and Restall defend the view that the application of logic is not relative to domains of reasoning. That is, they claim, like Priest, that the application of logic is *all-purpose*.[17] Their pluralism consists, then, in the thesis that there are equally legitimate logics, i.e., correct logics, for the canonical application.

Let us present, now, the enriched theoretical framework that our definitions yield. Hopefully, this will also further clarify how their differences rest upon the issue of applications.

- *Globalism-Monism*: there is just one correct logic and it is neutral with respect to the domain to which it is applied.
- *Globalism-Pluralism*: there are a variety of logics that are equally correct and their application is global, i.e., independent of the objects of reasoning.
- *Localism-Monism*: Different domains of discourse require different logics, but there is only one correct logic for each domain.
- *Localism-Pluralism*: Different domains of discourse require different logics and there might be various equally correct logics for a given domain.

Notice that the model-theoretic approach that we have argued for supports a localist stand towards logic, but excludes, in principle, neither monism nor pluralism. We have motivated the model-theoretic approach, partly by appealing to its suitability for being externally interpreted and, therefore, for being well justified. This is crucial, precisely, when disputing the correctness of a logic against others for a given application. But the applications that fall under the heading of 'analysis of reasoning' and, so, that are canonical, are many and diverse, such as reasoning in mathematics[18], reasoning about semantic paradoxes, about vague phenomena, quantum phenomena and so on and so forth.[19]

To my mind, while the universality and topic neutrality of logic would be theoretical virtues, they are just desiderata that have to be given up, or restricted, in the face of empirical evidence. And what the evidence

[17] This notion is from [Field, 2009].

[18] Following [Shapiro, 2014], this could already imply localism if some parts of mathematics are intuitionistic and others classical.

[19] I am aware of the globalist responses in the literature and of the problems raised against localism, e.g., mixed inferences, but a full-fledged defence of localism will have to wait for another occasion.

shows is different logical systems arising continuously and competing for becoming the accepted theory that systematizes reasoning in a given domain. If the day when we discover a logic that succeeds at being the best candidate for every and each canonical application comes, then we will certainly see this as a virtue. But the evidence available right now falls far short of supporting such a situation and I would not be surprised if, as appears to be the case, reasoning about different phenomena, with diverse ontologies, requires different rules of reasoning.

6. Conclusion

I started by analysing different conceptions of what it means to interpret a logic and how those conceptions relate to the pure/applied distinction in logic. That framework has served the purpose of introducing a long-standing dispute between proof-theoretic and model-theoretic semantics, which might be seen as two opposing views on accounting for validity and on how to justify such accounts in the light of canonical applications. I have argued for the model-theoretic approach by trying to reveal a fundamental flaw of proof-theoretic semantics, namely, an intrinsic deficiency in providing justification for the inferences that are to be considered as valid (i.e., correct, not just $\mathfrak{L}_i$-*valid*).

Thus, I have aimed at proposing a new argument that could stress that shortcoming for justification, coming from a novel identity criterion for logical systems. The case of the **ST**-collection of logics shows the *prima facie* possibility of there being inferential levels systematising what follows from what in a given domain that cannot be grounded in any human inferential practice, while at the same time being constitutive of the identity of a logic. More generally, I expect that the argument reinforces the idea that there must be something more to validity than the use of some connectives represented in some abstract rules and that a logic cannot be externally interpreted nor, therefore, justified, if its domain of application is completely overlooked.

Finally, I believe that the model-theoretic approach favours a localist conception of logic, which I see as a further virtue of model-theory, given the current empirical evidence.

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