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Game Semantics, Quantifiers and Logical Omniscience

Abstract. Logical omniscience states that the knowledge set of ordinary rational agents is closed for its logical consequences. Although epistemic logicians in general judge this principle unrealistic, there is no consensus on how it should be restrained. The challenge is conceptual: we must find adequate criteria for separating obvious logical consequences (consequences for which epistemic closure certainly holds) from non-obvious ones. Non-classical game-theoretic semantics has been employed in this discussion with relative success. On the one hand, with *urn semantics* [15], an expressive fragment of classical game semantics that weakens the dependence relations between quantifiers occurring in a formula, we can formalize, for a broad array of examples, epistemic scenarios in which an individual ignores the validity of some first-order sentence. On the other hand, urn semantics offers a disproportionate restriction of logical omniscience. Therefore, an improvement of this system is needed to obtain a better solution of the problem. In this paper, I argue that our linguistic competence in using quantifiers requires a sort of basic hypothetical logical knowledge that can be formulated as follows: when inquiring after the truth-value of $\forall x\phi$, an individual might be unaware of all substitutional instances this sentence accepts, but at least she must know that, *if* an element a is given, then $\forall x\phi$ holds only if $\phi(x/a)$ is true. This thesis accepts game-theoretic formalization in terms of a refinement of urn semantics. I maintain that the system so obtained (US^+) affords an improved solution of the logical omniscience problem. To do this, I characterize first-order theoremhood in US^+. As a consequence of this result, we will see that the ideal reasoner depicted by US^+ only knows the validity of first-order formulas whose Herbrand witnesses can be trivially found, a fact that provides strong evidence that our refinement of urn semantics captures a relevant sense of logical obviousness.

Keywords: game-theoretic semantics; logical omniscience; quantifiers; urn semantics; existential import

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1. Introduction

Logical omniscience, a key theorem of normal systems of epistemic logic, states that the knowledge set of ordinary rational agents is closed for its logical consequences. More precisely, in these systems it holds that

$$[\mathsf{K}_s(T) \wedge (T \Rightarrow \phi)] \Rightarrow \mathsf{K}_s(\phi), \tag{$*$}$$

in which K_s is the epistemic modality 's knows that'.

Although epistemic logicians in general judge $(*)$ unrealistic, there is still no clear consensus on how it should be restrained. There are multiple proposals of non-standard systems of epistemic logic imposing different restrictions on the validity of this theorem. For instance, one could restrict $(*)$ by taking into account non-classical epistemic worlds w verifying T and falsifying p, even when T classically entails p. For an overview of the main proposals on this topic [see 8]. Nevertheless, the challenge is most of all conceptual: we must find adequate criteria for separating *obvious* logical consequences (i.e., consequences for which $(*)$ certainly holds) from non-obvious ones.

In the quantificational case of the logical omniscience problem, non-classical *game-theoretic semantics* has been employed with relative success. In particular, based on *urn semantics* [15], an expressive fragment of classical game semantics that weakens the dependence relations between quantifiers occurring in a formula, we can formalize, for a broad array of examples, epistemic scenarios in which an individual ignores the validity of some first-order argument or sentence.

Classical game semantics is a 2-player, zero-sum game in which the participants must logically analyze a formula driven by a specific set of rules (see Definition 1.1 below for the classical rules for propositional connectives and atomic formulas). Usually, we refer to its players as *Eloise* and *Abelard*. The existence of a winning strategy for Eloise in the game determines the satisfiability of the played formula (for more details, see [20, pp. 93–97]). In what follows, symbols α and β play the role of metavariables for players of the game.

By a classic structure we mean a pair $M = \langle D, ^M \rangle$ such that D is a domain of elements and ^{M} is an interpretation function such that:

- For every constant c in our language L, c^M is an element of D;
- For every n-ary function $F \in L$, $F^M : D^n \to D$;
- For every n-ary predicate $P \in L$, $P^M \subseteq D^n$.

The symbol $\models$ means classical satisfiability.

DEFINITION 1.1. Let M be a classic structure. Suppose that, in the i-th round of a match of classical game semantics in M, a player α holds some formula ϕ.

- If ϕ is atomic formula, then α wins the match if, and only if, $M \models \phi$.
- If ϕ is $\neg\psi$, then players α and β switch roles, that is, β holds ψ in the $i+1$-th round.
- If ϕ is $\psi_0 \wedge \cdots \wedge \psi_{n-1}$, then, in the $i+1$-th round, α holds ψ_j, for some $j < n$ chosen by β.
- If ϕ is $\psi_0 \vee \cdots \vee \psi_{n-1}$, then, in the $i+1$-th round, α holds ψ_j, for some $j < n$ of her own choice.

Urn semantics is obtained, on the one hand, by preserving the classical game-theoretic rules for propositional connectives and, on the other hand, by adopting the rules for universal and existential formulas set up in Definition 1.2 below. Consider any family of sets $B = \{B(a_0, \ldots, a_{n-1}) \subseteq \text{dom}(M) : a_i \in B(a_0, \ldots, a_{i-1})$, for all $n \in \mathbb{N}$ and all $i < n\}$. We say that any such set $B(a_0, \ldots, a_{n-1})$ is an *election set* of M and B is a family of election sets of M.[1] In other words, a family of election sets B induces a partial order in $\{\text{dom}(M)^n\}_{n\in\mathbb{N}}$. In what follows, I denote a pair $\langle M, B\rangle$ as M_B.

DEFINITION 1.2. Let M be a classic structure and B a collection of election sets of M. Suppose that, in the i-th round of a match of urn game semantics in M_B, a player α holds some formula ϕ. Further, assume that, if ϕ is a quantified formula, then ϕ is the n-th quantified formula analyzed in the match:

- If ϕ is either atomic or a formula of the form $\neg\psi$, $\psi_0 \wedge \cdots \wedge \psi_{m-1}$ or $\psi_0 \vee \cdots \vee \psi_{m-1}$, then the clauses of Definition 1.1 hold.
- If ϕ is $\exists x\psi$, then α holds $\psi(x/b)$ in the $i+1$-th round of the match, for some $b \in B(a_0, \ldots, a_{n-1})$ of her own choice, in which a_j is the 'witness' of the j-th quantified formula analyzed in the match, for every $j = 0, \ldots, n-1$.
- If ϕ is $\forall x\psi$, then α holds $\psi(x/b)$ in the $i+1$-th round of the match, for some $b \in B(a_0, \ldots, a_{n-1})$ chosen by β.

[1] Actually, this definition of election set establishes a *perfect information* game semantics. If we assume that $B(a_0, \ldots, a_{n-1}) = B(b_0, \ldots, b_{n-1})$, for any two n-sequences $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$, then we get an alternative system with imperfect information. Although these two systems differ from each other in important aspects, in the present discussion, we may entirely disregard this distinction. For more details, see [13].

In other words, urn semantics relativizes different nested quantifiers occurring in a formula for distinct parts of a model and, in this way, breaks down the classical dependence relations between quantifiers.[2] Consequently, this system is a proper subtheory of classical logic: there are several first-order arguments and formulas which, despite being valid in the latter, are not valid in the former.

Example 1.1. The formula $\neg\forall x\forall y(p(x) \wedge \neg p(y))$ is not valid in urn semantics. To verify this, imagine a structure M with domain $\{a, b\}$ and a collection of election sets B such that $B(\emptyset) = \{a\}$, $B(a) = \{b\}$. Furthermore, assume that $M \models p(a)$ and $M \models \neg p(b)$. The following is a winning strategy for Abelard in a match of urn game semantics in M_B and $\neg\forall x\forall y(p(x) \wedge \neg p(y))$:

1. In the 1st round of the match, Eloise and Abelard switch roles, that is, Abelard holds $\forall x\forall y(p(x) \wedge \neg p(y))$ in the 2nd round.
2. In the 2nd round, Eloise has no other option but to choose $a \in B(\emptyset)$. So, Abelard holds $\forall y(p(a) \wedge \neg p(y))$ in the 3rd round of the match.
3. In the 3rd round, Eloise must choose $b \in B(a)$. So, Abelard holds $p(a) \wedge \neg p(b)$ in the 4th round of the match.

By the definition of atomic satisfiability in M and the classical rule of conjunction, this is a winning position for Abelard. Hence, in urn semantics, $\neg\forall x\forall y(p(x) \wedge \neg p(y))$ is falsifiable and, consequently, is not valid in this logic.

The significance of urn semantics is, in its essence, game-theoretic, though we can also define a Tarskian semantics for this logic.

Definition 1.3. Let M be a classic structure, B a collection of election sets of M and $\langle a_0, \ldots, a_{n-1}\rangle$ a sequence of elements of M such that $B(a_0 \ldots a_i) \in B$, for every $i < n$. For any formula ϕ, M_B *n-satisfies* ϕ with respect to $\langle a_0, \ldots, a_{n-1}\rangle$, in symbols $M_B, a_0, \ldots, a_{n-1} \Vdash_n \phi$, if, and only if, the following holds:

- If ϕ is atomic formula, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi \Leftrightarrow$ M classically satisfies ϕ.

[2] There is a duality between urn semantics and IF-logic [19]. Whilst the latter is a second-order extension of classical logic that introduces expressive resources to control quantificational dependence, in the former we are severely incapable of even expressing this property.

- If ϕ is $\psi \wedge \gamma$, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi$ if, and only if, $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \psi$ and $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \gamma$.
- If ϕ is $\psi \vee \gamma$, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi$ if, and only if, either $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \psi$ or $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \gamma$.
- If ϕ is $\neg\psi$, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi \Leftrightarrow \mathrm{M}_B, a_0, \ldots, a_{n-1} \nVdash_n \psi$
- If ϕ is $\exists x\psi$, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi \Leftrightarrow \mathrm{M}_B, a_0, \ldots, a_{n-1}, b \Vdash_{n+1} \psi(x/b)$, for some $b \in B(a_0 \ldots a_{n-1})$.
- If ϕ is $\forall x\psi$, then $\mathrm{M}_B, a_0, \ldots, a_{n-1} \Vdash_n \phi \Leftrightarrow \mathrm{M}_B, a_0, \ldots, a_{n-1}, b \Vdash_{n+1} \psi(x/b)$, for every $b \in B(a_0, \ldots, a_{n-1})$.

Finally, we say that M_B *urn satisfies* ϕ, in symbols $\mathrm{M}_B \Vdash \phi$, if, and only if, $M_B, \emptyset \Vdash_0 \phi$.

THEOREM 1.1 (Completeness, 13). *For any classical structure M, any collection of election sets B and any formula ϕ, the following are equivalent:*

1. *Eloise has a winning strategy in the urn game semantics in M_B and ϕ.*
2. *$M_B \Vdash \phi$.*

So, by adding to our modal frame urn models as epistemic worlds, we can generate a non-normal epistemic logic with a weaker version of $(*)$. This system preserves all classical validities by defining logical truth as satisfiability in all classical worlds, but does not guarantee full epistemic closure anymore since we can now access non-classical epistemic worlds which do not verify some classical logical consequences.[3]

Nevertheless, urn semantics offers a disproportionate restriction of $(*)$. Roughly stated, this system reduces first-order validity to theoremhood in propositional logic (I furnish more detail on this fact in Section 2.1 below). Conceived as a framework for dealing with the logical omniscience problem, this amounts to say that, according to urn semantics, there are almost no obvious first-order validities.

Now, we have semantic reasons to believe that such a limitation of $(*)$ is inadequate. Some of the main semantic theories available in the

[3] Since we are here dealing with an issue in the epistemology of classical logic, we can conveniently rely on the deduction theorem. Therefore, we can treat the problem of our knowledge of logical consequences as a special case of the problem of our knowledge of logical validities, at least in this context of investigation. To be more precise, we could say that, if someone does not know that a consequence ϕ follows from a set of premises T, this might be described as ignorance of the validity of $\bigwedge T \to \phi$.

philosophical literature relate linguistic competence to some sort of logical knowledge. For instance, according to Brandom's inferentialism [3], to understand a sentence involves to comprehend (and, consequently, at least in part, to know) the network of logical dependencies in which it is involved. Similarly, if we embrace some kind of Davidsonism [5], i.e., the idea that the meaning of a sentence is constituted by its set of truth-conditions, then, since the logical lexicon plays a fundamental role in the development of a theory of truth, we should acknowledge that linguistic understanding presupposes some knowledge about the content of logical expressions.[4] Nonetheless, if the solution of the logical omniscience problem offered by urn semantics is correct, then a reasoner would be capable to comprehend first-order sentences even if she completely ignored the logical relations in which they are embedded: an implausible hypothesis to say the least. Therefore, to achieve a more accurate solution of the problem we need to enhance urn semantics.[5]

To make such an improvement, let us reflect on the semantics of the quantifiers. A traditional idea, earlier stressed by Aristotle in his *Prior Analytics* [1], is that the basic meaning of the universal quantifier is more negative than positive. In Aristotle's words:

> We use the expression 'predicated of every' when none of the subject can be taken of which the other term cannot be said.
>
> [1, A 24b28-29]

So, according to Aristotle, by stating that 'Every x is ϕ' we are in fact saying that there is no x which is not ϕ. This is the 'no-counterexample' interpretation of $\forall$. By this conception, to use a universal formula we do not need to know all its possible instances, we only need to take care

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[4] However, note that Davidson himself is an externalist about meaning. On the contrary, our argument here presupposes some sort of semantic internalism.

[5] Here I am thematizing a minimal and quite general notion of logical obviousness; that is, I am assuming that an instance of logical validity is obvious if, and only if, it is an item of basic knowledge shared by all rational agents *qua* competent linguistic users. Of course, one could object that linguistic competence is not a sufficient condition for being rational, and I tend to agree with this objection. In fact, I believe that rationality and, consequently, logical obviousness are context-sensitive notions (for a proposal that explicitly acknowledges this fact, see [8]). Even so, I insist that linguistic competence is a necessary requirement for rationality: given any sense of rationality, if an individual is rational, then she is capable of sharing a language. Hence, although I am not providing a full description of all the different meanings of logical obviousness, I expect to at least contribute an analysis of a hard-core sense of that concept.

whether there are possible counterexamples to it. Consequently, our competence in using universal quantifiers request the following sort of hypothetical logical knowledge:

(R∀) $\forall x\phi$ holds if, and only if, if an element a is given, then $\phi(x/a)$ holds as well.

A competent speaker does not need to be aware of all the possible as which can instantiate $\forall x\phi$ in the above clause. She only needs to know that, *if* such an element is given, then universal instantiation is applicable. On the other hand, if universal instantiation is not applicable (i.e., in case an element a is given such that $\neg\phi(x/a)$), then the use of universal quantification should be prevented.

This account is compatible with the idea that the logical knowledge accompanying our linguistic competence in using quantifiers is sometimes *fallible*. To check this we need to recall *Herbrand's theorem*, a fundamental feature of classical logic according to which a universal formula $\forall x\phi$ is classically unsatisfiable if, and only if, for some set of terms t_0, …, t_{n-1} (usually called *Herbrand witnesses*), $\phi(x/t_0) \vee \cdots \vee \phi(x/t_{n-1})$ is unsatisfiable as well [2, 253–ff.]. Since Herbrandization is available in classical logic, Herbrand's theorem provides theoremhood characterization for this system. Now, if our linguistic competence in using universal quantifiers only demands hypothetical knowledge such as of R∀, it can happen that a reasoner ignores the existence of Herbrand witnesses for a certain first-order formula. Consequently, in this case, she would fail to know that the given formula is valid.

The no-counterexample conception of ∀ has been made precise in game-theoretic terms by means of dialogical logics [14]. We can also get another formalization which makes explicit the fallible aspect of the logical knowledge associated with our use of quantifiers, via the following refinement of urn semantics.

Definition 1.4. For any set $X \subseteq \mathrm{dom}(M)$, a collection of election sets B is *cumulative* for M and X (hereafter, for simplicity, I will say that B is cumulative for $M(X)$) if, for any $B(a_0, \ldots, a_{n-1}) \in B$,

$$X \cup \{a_0, \ldots, a_{n-1}\} \subseteq B(a_0, \ldots, a_{n-1}).$$

If B is a cumulative collection of election sets, then we say that M_B is a cumulative urn structure. Finally, a urn game semantics in M_B and $\phi(t_0, \ldots, t_{m-1})$ is called cumulative if, and only if, B is a cumulative collection of election sets of $M(\{t_0^M, \ldots, t_{m-1}^M\})$.

Let us denote cumulative urn semantics as US^+. Similarly, sometimes I may refer to cumulative urn structures as US^+ models. Regarded as a basis for epistemic logics, US^+ pictures an ideal reasoner who knows that at least every element already considered in the semantic game can be an instance of a universal formula, a minimal epistemic requirement for a person to have competence in understanding quantificational sentences. In the rest of this paper, I argue that US^+ offers a quite accurate weakening of logical omniscience. To show this, I characterize first-order theoremhood in this system. As a consequence of this result, we will see that the ideal reasoner depicted by US^+ only knows the validity of first-order formulas whose Herbrand witnesses can be trivially found, a result that provides strong evidence that our refinement of urn semantics in fact captures a relevant sense of logical obviousness.[6]

2. Theoremhood characterization for US^+

2.1. Preliminary remarks

In what follows, for simplicity, let us focus on first-order languages without function symbols. Moreover, the languages here considered are always finite. The logical signature of our language is $\{\approx, \neg, \vee, \wedge, \forall, \exists\}$.

We can describe theoremhood in urn logics by examining *Hintikka normal forms* (hereafter, HNFs). So, first of all, let us define this class of formulas.

[6] As we will see with more detail in Sect. 3 below, one could argue that any account of the problem of logical omniscience based on variations of urn semantics is just too narrow: it offers a solution to the problem only for a subclass of the set of first-order logical validities. Later on, I present a more complete response to this objection. Nevertheless, from the beginning, it is important to have in mind that the present work has the following motivation. I am convinced that the problem of logical omniscience is multifaceted and the goal of establishing a general solution to it should be pursued with a bottom-up approach; that is, we should start by analyzing smaller sets of examples and then, step by step, we should try to formulate more general hypotheses on the subject. In this sense, in this paper, I explore a game-theoretic strategy that although, on the one hand, presents a still very restricted solution to the problem, on the other hand, highlights aspects of the issue which are completely ignored by other approaches (e.g., the fact that logical obviousness is related to the epistemological grounds of our semantic competence, that this minimal logical requirement can be formalized in terms of some kind of default semantics etc.)

For a finite language L, let $At(x_0, \ldots, x_{n-1})$ be the set of atomic formulas of L with variables within $\{x_0, \ldots, x_{n-1}\}$. For any set of formulas Φ, let $\overline{\Phi} = \{\neg\phi : \phi \in \Phi\}$ and $\mathrm{E}\Phi = \{\exists x\phi : \phi \in \Phi\}$. $\bigwedge \Phi$ and $\bigvee \Phi$ denote, respectively, the conjunction and disjunction of the elements in Φ. For any sets of formulas Φ and Ψ, $\Psi \setminus \Phi$ means the result of substracting the elements of Φ from Ψ. Finally, by the *rank* of a formula we mean a function r such that:

- $r(\neg\phi) = r(\phi)$;
- $r(\phi \wedge \psi) = r(\phi \vee \psi) = max\{r(\phi), r(\psi)\}$;
- $r(\exists x\phi) = r(\forall x\phi) = r(\phi) + 1$.

DEFINITION 2.1. A formula $\theta(x_0, \ldots, x_{n-1})$ of L is a *state description* if, and only if, the following holds:

- If $r(\theta) = 0$, then, for some $\Phi \subseteq At(x_0, \ldots, x_{n-1})$, θ is
$$\bigwedge \Phi \wedge \bigwedge \overline{At(x_0, \ldots, x_{n-1}) \setminus \Phi}.$$
- If $r(\theta) = m + 1$, then, for some set Γ of state descriptions with rank m, θ is
$$\bigwedge \mathrm{E}\Gamma \wedge \forall y \bigvee \Gamma.$$

A Hintikka normal form of rank m is a disjunction of state descriptions with same rank.

For any state description, we always consider alphabetic variations in which different quantifiers bind distinct variables.

State descriptions sententially present, in a systematic way, all existential types realized in a model. In this way, these sentences define equivalence classes of models of a language: models satisfying a same state description of rank m realize the same existential types for sequences of length m. In classical logic, urn semantics and US$^+$ we have *HNF translatability*, i.e., every formula with rank m is equivalent to a chain of HNFs of rank k, for every $k \geqslant m$. Each disjunctive in an HNF of a formula ϕ identifies an equivalence class (modulo existential types) of models satisfying ϕ [10, p. 84].

For each one of these logical systems, we can characterize logical theoremhood by surveying the syntactic properties of HNFs. In the rest of this section, we will need to resort to a notion of *component* defined by the clauses below.

- ϕ and ψ are components of $\phi \wedge \psi$ and $\phi \vee \psi$;

- $\neg\phi$ and $\neg\psi$ are components of $\neg(\phi \wedge \psi)$ and $\neg(\phi \vee \psi)$;
- ϕ is a component of $\neg\neg\phi$;
- For any term t, $\phi(x/t)$ is a component of $\exists x\phi$ and $\forall x\phi$;
- Transitivity: If γ is a component of ψ that is a component of ϕ, then γ is a component of ϕ.

A *component chain* is a sequence of formulas $\langle \phi_0, \ldots, \phi_{n-1} \rangle$ such that, ϕ_{i+1} is a component of ϕ_i, for every $i < n-1$.

Hintikka [9] proved that, in classical logic, a formula ϕ is unsatisfiable if, and only if, for an HNF ψ of ϕ, each state description θ disjunctive of ψ lacks at least one of the following syntactic properties:

1. If there is a component chain of the form

$$\langle \theta, \ldots, \exists x\theta_1, \ldots, \exists y\theta_2, \ldots, \theta_0(x/t, y/s) \rangle,$$

then there is also a component chain of the form

$$\langle \theta, \ldots, \exists y\theta_1', \ldots, \exists x\theta_2', \ldots, \theta_0(x/t, y/s) \rangle;$$

2. If there is a component chain of the form

$$\langle \theta, \ldots, \exists x\theta_1, \ldots, \forall y\psi_1, \ldots, \theta_0(x/t, y/s) \rangle,$$

then there is also a component chain of the form

$$\langle \theta, \ldots, \forall y\psi_2, \ldots, \exists x\theta_2, \ldots, \theta_0(x/t, y/s) \rangle;$$

3. For any formula of the form $\exists x\theta_1$ component of θ such that $r(\exists x\theta_1) \geqslant 2$, there is a component chain

$$\langle \theta, \ldots, \exists x\theta_1, \theta_1(x/t), \exists y\theta_2, \ldots, \theta_0(x/t, y/t) \rangle$$

such that $\theta_0(x/t, y/t)$ has rank 0 and is not contradictory (that is, neither there are literals $P(t_0, \ldots, t_{n-1})$ and $\neg P(s_0, \ldots, s_{n-1})$ occurring in θ_0 such that, for every $i < n$, $t_i \approx s_i$ is a literal of θ_0, nor there is a literal $t \not\approx t$ ocurring in θ_0).

These syntactic properties form a complete set of consistency conditions for state descriptions (and, consequently, for HNFs) in classical logic. Drawing upon these conditions, we can define a proof procedure for this logical system: to determine whether some formula is classically valid, we only need to verify if there is an HNF translation of its negation that does not fulfill at least one of those consistency conditions. In virtue of the semi-decidability of classical logic, if a formula is logically valid, such a procedure will end at some point (not specifiable a priori).

Urn semantics is much more flexible regarding the consistency of state descriptions. Mendonça [13] proved the following result:

Theorem 2.1. *Let θ be a state description such that different quantifiers occurring in it bind distinct variables and consider $\gamma_0, \dots, \gamma_{n-1}$ all the state descriptions of rank 0 occurring in it. Then, θ is satisfiable in urn semantics if, and only if, $\gamma_0 \wedge \dots \wedge \gamma_{n-1}$ is satisfiable in classical logic.*

This theorem reveals that, as mentioned in the previous section, urn semantics reduces first-order validity to propositional theoremhood, an extreme result. In the following I will show that US^+ affords a milder weakening of classical validity.

2.2. Main result

As before, we start by setting out consistency conditions for state descriptions in US^+. Let us expand our language L with a set of Henkin constants in the following way. First, let us enumerate both the set of formulas of L with a free variable and a set of new constants D_0 with the same cardinality. For every formula $\phi_i(x)$ of L in which x is a free-variable, let $d_i \in D$ be the Henkin constant of $\exists x \phi_i(x)$. Consider $L = L_0$ and $L_1 = L_0 \cup D_0$. Now, assume L_n is already defined. Let us enumerate both the set of formulas of L_n with one free-variable and a set of new constants D_n with the same cardinality. For every formula $\phi_j(x)$ of L_n in which x is a free-variable, let $d_j \in D_n$ be the Henkin constant of $\exists x \phi_j(x)$. $L_{n+1} = L_n \cup D_n$. Therefore, $L^* = \bigcup_{n \in \mathbb{N}} L_n$ is the desired expansion of L with Henkin constants.[7]

Furthermore, suppose that, for any component chain σ,

$$\mathrm{Q}x_0\phi_{k0}, \dots, \mathrm{Q}x_{n-1}\phi_{k(n-1)}$$

are all quantified formulas such that $\langle \mathrm{Q}x_i\phi_{ki}, \phi_{ki}(x_i/s_i)\rangle$ is a partial segment of σ, for some term s_i of the language. Then, we say that $s_0, \dots, s_{n-1}$ are the *witnesses* of σ.

[7] Although we are always assuming that L is a finite language, L^* is infinite. This is not a problem, provided we keep in mind that we are characterizing theoremhood for validities of L, i.e., L^* plays just an auxiliary role in the proof of our main result.

DEFINITION 2.2. Let θ be a state description with free-terms $t_0, \dots, t_{n-1}$ and rank k. For every $0 < q \leqslant k$, let

$$\psi := \bigwedge \mathrm{E}\Gamma \wedge \forall y \bigvee \Gamma(s_q, \dots, s_{q+k-1})$$

be any state description component of θ with rank q and such that $s_q, \dots, s_{q+k-1}$ are the witnesses of $\langle \theta, \dots, \psi \rangle$. θ is *+-consistent* if, and only if, for every $t \in \{t_0, \dots, t_{n-1}, s_q, \dots, s_{q+k-1}\}$, there is some component chain $\langle \theta, \dots, \psi, \dots, \theta'(y/t) \rangle$ such that $\theta'(y/t)$ has rank 0 and is not contradictory.

To describe theoremhood in US^+, our strategy is to show that, taking into consideration +-consistent state descriptions, we can generate a special set of formulas for which there is a US^+ model. In what follows, we say that a set of formulas Γ is *closed for molecular components* if the following holds:

- If $\phi \wedge \psi \in \Gamma$, then $\phi, \psi \in \Gamma$;
- If $\neg(\phi \wedge \psi) \in \Gamma$, then either $\neg\phi$ or $\neg\psi$ are in Γ;
- If $\phi \vee \psi \in \Gamma$, then either ϕ or ψ are in Γ;
- If $\neg(\phi \wedge \psi) \in \Gamma$, then $\neg\phi, \neg\psi \in \Gamma$;
- If $\neg\neg\phi \in \Gamma$, then $\phi \in \Gamma$.

DEFINITION 2.3. A *+-set* Δ is a union of sets of formulas $\bigcup_{n \in \mathbb{N}} \Delta_n$ such that:

1. For every term t, $\neg(t \approx t) \notin \Delta_n$;
2. For any two literals $P(t_0, \dots, t_{m-1})$ and $\neg P(s_0, \dots, s_{m-1})$, $\{P(t_0, \dots, t_{m-1}), \neg P(s_0, \dots, s_{m-1})\} \cup \{t_i \approx s_i : i < m\} \not\subseteq \Delta_n$;
3. For every atomic formula p, if $p \in \Delta_n$, then $p \in \Delta_{n+1}$; If $\neg p \in \Delta_n$, then $\neg p \in \Delta_{n+1}$;
4. Δ_n is closed for molecular components;
5. If $\exists x \phi_i \in \Delta_n$, then $\phi(x/d_i) \in \Delta_{n+1}$;
6. If $\forall x \phi \in \Delta_n$, then, for any formula $\theta \in \Delta_0$ with free-terms $t_0, \dots, t_{n-1}$, for every component chain $\sigma := \langle \theta, \dots, \forall x \phi \rangle$ defined in $\bigcup_{j \leqslant n} \Delta_j$ with witnesses $s_0, \dots, s_{m-1}$ and for any $t \in \{t_0, \dots, t_{n-1}, s_0, \dots, s_{m-1}\}$, $\phi(x/t) \in \Delta_{n+1}$; Now, assume there are two component chains $\sigma := \langle \theta, \dots, \forall x \phi \rangle$ and $\sigma' := \langle \theta, \dots, \exists y \phi_i \rangle$ both defined in $\bigcup_{j \leqslant n} \Delta_j$ and with same witnesses: then, $\phi(x/d_i) \in \Delta_{n+1}$;
7. If $\neg\exists x \phi \in \Delta_n$, then, for any formula $\theta \in \Delta_0$ with free-terms $t_0, \dots, t_{n-1}$, for every component chain $\sigma := \langle \theta, \dots, \neg\exists x \phi \rangle$ defined in

$\bigcup_{j \leqslant n} \Delta_j$ with witnesses $s_0, \ldots, s_{m-1}$ and for any $t \in \{t_0, \ldots, t_{n-1}, s_0, \ldots, s_{m-1}\}$, $\neg\phi(x/t) \in \Delta_{n+1}$; Now, assume there are two component chains $\sigma := \langle\theta, \ldots, \neg\exists x\phi\rangle$ and $\sigma' := \langle\theta, \ldots, \exists y\phi_i\rangle$ both defined in $\bigcup_{j \leqslant n} \Delta_j$ and with same witnesses: then, $\neg\phi(x/d_i) \in \Delta_{n+1}$;

8. If $\neg\forall x\phi_i \in \Delta_n$, then $\neg\phi(x/d_i) \in \Delta_{n+1}$.

Definition 2.4. A +-set Δ is *hereditary* if, and only if, for every $n \in \mathbb{N}$, for any formula $\phi \in \Delta_n$, there is some formula $\theta \in \Delta_0$ and a component chain $\langle\theta, \ldots, \phi\rangle$ defined in $\bigcup_{j \leqslant n} \Delta_n$.

Lemma 2.1. *For any +-consistent state description θ with rank $k > 0$, there is a hereditary +-set Δ such that $\theta \in \Delta_0$.*

Proof. There is no difficulty in defining, based on θ, a family of sets $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$ satisfying clauses 3–8 of Definition 2.3. We only need to show that Δ so defined meets conditions 1 and 2.

Assume either clause 1 or clause 2 does not hold. Due to the syntactic structure of θ, this can only happen if, for some formula $\forall y\psi$, there is no component chain $\langle\theta, \ldots, \forall y\psi, \psi(y/t'), \ldots, \theta_0\rangle$ such that θ_0 has rank 0 and is not contradictory, for $t' \in \{t : t \text{ is a free term in } \theta\} \cup \{s : s \text{ is a witness of } \langle\theta, \ldots, \forall y\psi\rangle\}$. However, this is in conflict with the definition of +-consistency. Therefore, by reductio ad absurdum, Δ satisfies conditions 1 and 2. □

Lemma 2.2. *For any hereditary +-set Δ there is a cumulative urn structure M_B such that, for any $\phi \in \Delta_0$, $M_B \Vdash \phi$.*

Proof. To prove this result, we proceed by constructing a syntactic US^+ model for Δ_0.

Let C be the set of closed terms of our first-order language (assume that C includes all Henkin constants necessary for defining Δ). For every $t \in C$, let $[t] = \{s : t \approx s \in \Delta\}$. $C^{\approx} = \{[t] : t \in C\}$ will be the domain of our structure.

Let ^{M} be an interpretation function such that:

- For any constant c, c^M is $[c]$;
- For any n-ary predicate P, $P^M = \{\langle t_0^M, \ldots, t_{n-1}^M\rangle : P(s_0, \ldots, s_{n-1}) \in \Delta$, for any $s_i \in [t_i]\}$.

(Remember that, for simplicity, we are examining only languages without function symbols).

The definition of ^{M} entails immediately the following fact:

Fact 2.1. *For any atomic formula p, $p \in \Delta$ if, and only if, $M \models p$.*

Now, let B be a collection of election sets such that, for any $n \in \mathbb{N}$, $B(s_0^M, \ldots, s_{n-1}^M)$ is the sum of two sets of elements:

- First, $B(s_0^M, \ldots, s_{n-1}^M)$ includes all elements t^M such that, for any component chain σ defined in $\bigcup_{j \leqslant n} \Delta_n$ with its first element in Δ_0 such that its witnesses are congruent to $s_0, \ldots, s_{n-1}$ and $t_0, \ldots, t_{m-1}$ are all the free-terms occurring in the first formula of σ,

$$t^M \in \{[s_i] : i < n\} \cup \{[t_l] : l < m\};$$

- Secondly, for any component chain σ defined in $\bigcup_{j \leqslant n} \Delta_j$ with its first element in Δ_0 such that its witnesses are congruent to $s_0, \ldots, s_{n-1}$ and $\exists x \phi_l$ is its last component, $d_l^M \in B(s_0^M, \ldots, s_{n-1}^M)$.

We will prove the following fact:

FACT 2.2. *For every formula ϕ, if $\phi \in \Delta_n$, then, for any $s_0^M, \ldots, s_{n-1}^M$ such that there is a component chain σ defined in $\bigcup_{j \leqslant n} \Delta_j$ whose first element is in Δ_0, ϕ is its last element and whose sequence of witnesses are congruent to $s_0, \ldots, s_{n-1}$, it holds that $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \phi$.*

If $\neg\phi \in \Delta_n$, then for any $s_0^M, \ldots, s_{n-1}^M$ such that there is a component chain σ defined in $\bigcup_{j \leqslant n} \Delta_j$ whose first element is in Δ_0, $\neg\phi$ is its last element and whose sequence of witnesses are congruent to $s_0, \ldots, s_{n-1}$, it holds that $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \neg\phi$.

We prove this fact by induction on ϕ.

Let ϕ be a conjunction $\psi \wedge \gamma$. If $\psi \wedge \gamma \in \Delta_n$, since Δ_n is closed for molecular components, $\psi, \gamma \in \Delta_n$. By the induction hypothesis, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \psi, \gamma$. Then, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \psi \wedge \gamma$.

If $\neg(\psi \wedge \gamma) \in \Delta_n$, then either $\neg\psi$ or $\neg\gamma$ are in Δ_n. By the induction hypothesis, either $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \neg\psi$ or $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \neg\gamma$. So, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \neg(\psi \wedge \gamma)$.

A similar reasoning holds for disjunction.

Let ϕ be $\neg\psi$. If $\neg\psi \in \Delta_n$, by induction, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \neg\psi$. If $\neg\neg\psi \in \Delta_n$, then $\psi \in \Delta_n$ and, by induction, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \psi$.

Let ϕ be $\exists x \psi_{ki}$. If $\exists x \psi_{ki} \in \Delta_n$, then $\psi_{ki}(x/d_{ki}) \in \Delta_{n+1}$. By the inductive hypothesis, $M_B, s_0^M, \ldots, s_{n-1}^M, d_{ki}^M \Vdash_{n+1} \psi_{ki}(x/d_{ki})$. Now, note that $d_{ki}^M \in B(s_0^M, \ldots, s_{n-1}^M)$. So, $M_B, s_0^M, \ldots, s_{n-1}^M \Vdash_n \exists x \psi$.

Let ϕ be $\forall x \psi$. Suppose $M_B, s_0^M, \ldots, s_{n-1}^M \nVdash_n \forall x \psi$. So, for some $t^M \in B(s_0^M, \ldots, s_{n-1}^M)$, $M_B, s_0^M, \ldots, s_{n-1}^M, t^M \nVdash_{n+1} \psi(x/t^M)$. By the inductive hypothesis, $\psi(x/t) \notin \Delta_{n+1}$. Consequently, $\forall x \psi \notin \Delta_n$.

Similar reasonings hold for the cases of $\neg\forall x \psi$ and $\neg\exists x \psi$. □

Finally, we can describe first-order theoremhood in US$^+$.

THEOREM 2.2. *For a formula ϕ, let $\bigvee \Gamma$ be an HNF translation of it. Then, ϕ is satisfiable in* US$^+$ *if, and only if, every $\gamma \in \Gamma$ is $+$-consistent.*

PROOF. Sufficiency follows immediately from Lemmas 2.1 and 2.2.

To prove necessity, assume that no $\gamma \in \Gamma$ is $+$-consistent. Without loss of generality, let us fix one such γ. Then, there is a component chain $\sigma := \langle \gamma, \ldots, \theta_0 \rangle$ such that all quantified formulas occurring in it are existential and θ_0 is contradictory. I will show that Abelard has a winning strategy for any urn game semantics in γ.

Assume that, in the i-th round of a match of urn game semantics in γ, Eloise holds a formula of the form

$$\psi := \bigwedge \mathrm{E}\Gamma' \wedge \forall y \bigvee \Gamma'.$$

- Then, in the $i+1$-th round, Abelard requires Eloise to hold $\bigwedge \mathrm{E}\Gamma'$;
- In the $i+2$-th round, Abelard requires Eloise to hold an existential formula component of σ.

By iterating this procedure $r(\gamma)$ times, Abelard forces Eloise to hold all components of σ, including θ_0. □

3. US$^+$ and logical omniscience

Just like urn semantics, US$^+$ affords a solution for the logical omniscience problem by providing a platform for generating non-classical epistemic worlds. Let $\mathcal{F} = \langle W, W^*, <, I, I^* \rangle$ be a epistemic modal frame such that W is the collection of all classical worlds, W^* is the collection of all US$^+$ worlds (i.e., worlds behaving as US$^+$ models), $<$ is an accessibility relation between worlds in $W \cup W^*$, I is a valuation for worlds in W and I^* is a valuation for worlds in W^*. Consider that $\mathcal{F}$ is a normal frame, i.e., the modal axiom K and the necessitation rule hold in $\mathcal{F}$. Let an inference be logically valid if it holds in all classical worlds. In this way, we immediately have the failure of logical omniscience. For some logical validity ϕ (namely, those validities identified in Theorem 2.2), even though ϕ is satisfied in all classical worlds, it is not satisfied in some US$^+$ world w^*. If, for some world w, $w < w^*$, then it holds that

$$w \models \phi \wedge \neg \mathsf{K}_s(\phi),$$

where K_s is the operator of epistemic necessity.

Now, we need to ask: does US^+ offer a reasonable solution for the logical omniscience problem? To answer this question, we need to evaluate whether cumulative urn models draw a plausible epistemological picture. Bear in mind that, in the present context of investigation, I am not assuming any sort of ontological realism concerning epistemic possible worlds. Here, epistemic worlds serve just as theoretical tools for explaining the epistemic stance of a regular individual regarding logic. This being so, it is enough to add to our modal frame syntactic models such as the one defined in the proof of Lemma 2.2. In this sense, our question might be restated as follows: does syntactic cumulative urn models deliver a plausible account of the epistemic situation of ordinary reasoners?

An important feature of US^+ is its *finite model property*: for instance, syntactic models such as the one built in the proof of Lemma 2.2 are finite. Hence, any formula satisfiable in US^+ is satisfied by a finite structure. In the context of this semantic system, the finite model property is a consequence of the *decidability* of the calculus. Roughly stated, to verify if a certain formula is satisfiable, we only need to build its syntactic structure which is, in the case of US^+, a finitary procedure.

This is exactly what we would expect of a framework describing ordinary reasoners' epistemic attitudes towards logic. On the one hand, in the process of analyzing the truth-conditions of a formula that happens to express a classical logical truth, one might be unable to pinpoint its set of Herbrand witnesses. In such cases, all the reasoner can do is to remark that the elements so far given in the analysis do not validate the formula. This finite set of elements constitutes a *surface* countermodel to the formula. In other words, our epistemic situation informs us that, *by default*,[8] the formula is not valid, a piece of information that is encapsulated in finite US^+ countermodels. On the other hand, if US^+ preserves the validity of a classical logical truth, there is an obvious sense in which the search for its Herbrand witnesses is computationally triv-

[8] This description of the epistemological requirements of our semantic competence in using quantifiers exhibits some similarities with *default logic* [16]. In both cases, the logical lexicon is depicted in terms of a set of rules whose application is by default and defeasible. Nonetheless, there is an important difference between default logic and the present proposal: whereas proponents of the former have used it to advance an analysis of the meaning of logical connectives and quantifiers, the present paper introduces US^+ only as a framework to explain how ordinary reasoners epistemically represent the semantics of our language.

ial: the Herbrand witnesses of the formula are within the set of elements composing its syntactic and finitarily generated US^{+} model.[9]

Nevertheless, one might wonder whether this default account of logical obviousness does not lose too many classical validities by allowing a sort of vacuous satisfiability of universal formulas. If no element previously occurred in the semantic analysis of a universal formula, is it vacuously satisfiable in US^{+}? The answer is no. Even though Definition 1.4 only demands from the domain of universal quantifiers that they include all elements already examined (if there are any), here we have one of those cases where the framework we are working with (namely, state descriptions and HNF translations) imposes additional constraints that cannot be withdrawn.

Hodges [10] calls state descriptions 'game normal forms' in virtue of the fact that these formulas identify Ehrenfeucht-Fraïssé games between models. A different reason to adopt this nomenclature is the fact that state descriptions define canonical versions of game semantics. That is, a semantic game in an HNF translation of a formula ϕ might be seen as a canonical version of a game semantics in ϕ itself. Moreover, observe that each component chain of a state description determines a different match of a game semantics in this formula. Now, suppose that, at the i-th and the $i+1$-th rounds of a canonical game semantics, Eloise holds, respectively, the formulas $\bigwedge \mathrm{E}\Gamma \wedge \forall y \bigvee \Gamma$ and $\forall y \bigvee \Gamma$. Owing to the proof of Lemma 2.2, we know that, in the $i+2$-th round, Abelard can demand that Eloise hold $\bigvee \Gamma(y/t)$, in which t might be not just all the elements considered in previous rounds of the match, but also the witnesses of each existential formula in $\mathrm{E}\Gamma$. In other words, due to a syntactic feature of state descriptions and HNFs, the domain of the universal quantifier must include the elements hitherto surveyed in the match as well as the

[9] We need to be very careful here to not misidentify the perspective of the *modeler* (an *external* perspective) with the standpoint of the rational agent whose epistemic situation is being analyzed (an *internal* perspective). US^{+} models are available to the modeler, not to the rational agent. In other words, I am not claiming that ordinary reasoners have US^{+} models in their heads. However, there is a possible translation between such external and internal viewpoints. The fact that the construction of syntactic US^{+} models is computationally low-cost gives body, in computational terms, to a reasonable conception of the information available to the reasoner on the satisfiability of a formula. So as an US^{+} model only checks witnesses which have occurred before in a given semantic game, to evaluate whether a formula is satisfied, a rational agent can only compare it with the information offered so far by previously considered elements.

witnesses which would be considered had the game evolved in a different way (i.e., had, in the $i+1$-th round, Abelard demanded that Eloise hold some formula in $\mathrm{E}\Gamma$ instead of $\forall y \bigvee \Gamma$).

Therefore, in US$^+$, universal quantifiers have a kind of *existential import*: by holding that $\forall y\phi$, Eloise must commit to $\phi(y/t)$ as well, in which t is a witness of an existential formula occurring in an alternative development of the game semantics. The contemporary debate on existential import turns around the question of whether certain kinds of quantified expressions — most notably, the universal quantifier — have existential import and, in the affirmative case, whether this existential import is for semantic or for pragmatic reasons (see [7] for a critical assessment of this discussion). Although we cannot contribute a response to this inquire solely based on US$^+$, conceived as a formalization of our epistemic situation with respect to logic US$^+$ implies the idea that at least our epistemic representation of the semantics of universal quantifiers ascribes an existential import to these expressions.[10]

Now, even if it is correct to say that US$^+$ provides an appropriate epistemological description, one might still ask: does US$^+$ catch a bold sense of logical obviousness? Maybe there are some obvious first-order validities which are not captured by this system but can still be collected by some extension of it. We can get potentially relevant extensions of US$^+$ by adding to this logic bounded versions of the consistency conditions for state descriptions in classical logic (see conditions 1–3 in Section 2.1 above). Yet, note that we cannot define weaker versions of condition 1: any attempt in this direction collapses with condition 1 itself. Moreover, the definition of +-consistency (Definition 2.2) already captures condition 3. Therefore, what is left for us is to try to workout condition 2.

For a component chain $\sigma := \langle \ldots, \phi, \ldots, \psi, \ldots \rangle$, let the *distance* between ϕ and ψ in σ be the number of quantified formulas occurring

[10] Since we are offering a purportedly more realistic, less idealized, solution to the logical omniscience problem, one could ask for empirical evidence in favour of this proposal. Even though this is the natural next step of the investigation, the literature on applications of urn semantics (and variations) to problems in pragmatics and linguistics is still incipient. French [6] speculates about the applicability of urn semantics to model conversational situations where an utterer overlooks relations of dependence between quantifiers. Analogously, at this point we can only conjecture that, through US$^+$, we might not just model concrete situations where the epistemic condition of a person disregards relations of quantificational dependence, but also determine the semantic features of quantifiers that this individual can't ignore.

between ϕ and ψ in this component chain. The following item identifies a scheme of weaker versions of consistency condition 2.

- If there is a component chain of the form

  $$\langle \theta, \ldots, \underbrace{\exists x\theta_1, \ldots, \forall y\psi_1}_{\text{distance } m}, \ldots, \theta_0(x/t, y/s)\rangle,$$

  then there is also a component chain of the form

  $$\langle \theta, \ldots, \underbrace{\forall y\psi_2, \ldots, \exists x\theta_2}_{\text{distance } m}, \ldots, \theta_0(x/t, y/s)\rangle.$$

It is not clear from the outset which subclasses of cumulative urn models validate instances of this scheme. In any case, from an epistemological point of view, for sufficiently small m, this principle might be of interest for a philosopher who claims that ordinary reasoners are capable not just of universally instantiating elements formerly given in a semantic analysis but can also anticipate instances that have not occurred yet in the process. Does such an extension of US^+ provide a reasonable description of our epistemic condition? To answer this question we need to precisely define this family of logical systems and to survey some of their formal features just as we have done here for US^+. However, this is an issue for another paper.

Finally, one could object that the scope of the present proposal is too narrow in an important way: my aim here is only to analyze the quantificational case of the logical omniscience problem, but I do not consider whether this solution is generalizable for the propositional case as well. Of course, this could be seen as a drawback. In this sense, [17] and [11] among others argued that an inherent difficulty of attempts, such as the present one, based on Hintikka normal forms and urn semantics, is that they seem to be restricted to the analysis of quantifiers. Furthermore, we can find in the literature alternative solutions which are from the start applicable both to the quantificational as well as to the propositional cases of the logical omniscience problem. Paradigmatic examples are Jago [12] and D'Agostino [4]. So, the narrowness of scope of the present work seems to lack justification.

Now, against this objection, we can say two things. First, at this point it is still not settled whether the present proposal is generalizable for the propositional case of the problem. However, if we take into account the well-known similarities between the game-theoretic semantic features of the conjunction and universal quantification, then a possible generalization suggests itself: perhaps we could argue that, so as our use of universal quantification depends on the hypothetical knowledge stated in R∀, our use of the conjunction is also based in a similar kind of knowledge. Therefore, we would have an account of the epistemological conditions constraining our linguistic competence in using this logical connective analogous to what we have done here for the case of universal quantification.

Secondly, note that Jago and D'Agostino favour proof-theoretic readings of logical obviousness. In this sense, Jago affirms that

> whether a given deduction is informative [...] is a matter of how difficult the inference is [...] More precisely: it is a factor of the shortest number of inference steps required to move from premises to conclusion, relative to some fixed set of inference rules. [12, p. 13]

D'Agostino [4] makes a similar suggestion when he claims that a validity is obvious if, and only if, its logical proof does not require the use of 'virtual information'. Notwithstanding, in the present paper I am pursuing a different philosophical project: I do not characterize logical obviousness in proof-theoretic terms, but rather in game-theoretic terms. I am exploring the idea that a logical validity is obvious if it can be recognized as such in the context of a semantic game. Therefore, although there are alternative solutions which work pretty well both for the quantificational and the propositional cases of the problem, some of their fundamental premises are not ours, and, consequently, those works are not easily comparable.

4. Conclusion

Exploring the idea that our competence in using universal quantifiers demands, as a basic epistemic requirement, the capacity to control the occurrence of counterexamples — a kind of hypothetical logical knowledge here formulated in terms of R∀ –, in this paper we have seen how we can improve urn game semantics to obtain a better solution of the logical omniscience problem. We achieved this improvement by means of the extension US^+, a system of game semantics defined by a subclass of urn models (namely, cumulative urn models). Supported by a characterization of first-order theoremhood in this logic, we showed that

US$^+$ offers a finitary and decidable picture of the epistemic situation of ordinary reasoners with respect to logic. Moreover, according to this theoretical framework, we verified that at least our epistemic representation of the semantics of universal quantifiers associates an existential import with these expressions. Finally, we have seen that, perhaps, this US$^+$-based epistemological account might also be further improved by the addition of bounded versions of a classical principle of consistency of state descriptions — though we leave this possibility open for further research.

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