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First-Order Logic with Adverbs

Abstract. This paper introduces two languages and associated logics de-
signed to afford perspicuous representations of a range of natural language
arguments involving adverbs and the like: first-order logic with basic ad-
verbs (FOL-BA) and first-order logic with scoped adverbs (FOL-SA). The
guiding logical idea is that an adverb can *come between* a term and the rest
of the statement it is a part of, resulting in a logically stronger statement.
I explain various interesting challenges that arise in the attempt to imple-
ment the guiding idea, and provide solutions for some but not all of them.
I conclude by outlining some directions for further research.

Keywords: adverbs; notation; extensions of first-order logic

*Well now I'd like to raise a question − indeed, a doubt − about Donald's proposal
about what I would call perhaps the semantically significant structure of sentences
reporting actions or events, and what he calls the logical form of those sentences. And
I'm very much inclined to agree [...] that, if we understand logical form in just the
way in which Donald understands it, it would be impossible to improve on his proposal
for construing such sentences as quantifying over particular events or actions. What
I'm not quite so convinced of is the necessity for this proposal. For it seems to me
that such sentences do exhibit a structural − well, a logical structure if you like, or at
least a semantic-syntactic structure, a logical structure − which is sufficient in itself
to validate the inferences which he appeals to logical form, as he understands it, to
show the validity of.*

*And I think that − well, to expand a bit − anybody at all who masters the lan-
guage knows that it's perfectly in order to* add *to any affirmative action or happening
sentence qualifications of time, place, manner, means and so on. And anybody who
understands this knows by the same token that it's equally in order to* drop *those qual-
ifications, to drop those modifications from any affirmative sentence reporting actions
and events without threat to truth-value, if truth is there in the first place.*

P. F. Strawson, in a public discussion with Donald Davidson[1]

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[1] The meeting appears to have taken place in 1997. A video may be found on
YouTube (one version is at `http://www.youtube.com/watch?v=hE71QAOYav4`). It is

1. Introduction

Consider the following natural language argument:

> Socrates is both human and mortal.
> Therefore, Socrates is human.

This can be translated into the language of first-order logic (FOL) as follows:

> $Hs \wedge Ms$
> $\therefore Ms$

The original and the translation differ in some ways. Perhaps the biggest structural difference between the natural language version and the FOL version is that it is natural to regard the 'and' in the natural language version not as a sentential connective, but as a connective at the level of predicates. That is, it is natural to regard 'human and mortal' as a single, complex, predicate. For the purpose of translating natural language arguments and assessing their validity using logical methods, FOL is arguably good enough on this score, at least in many typical contexts. The syntactic difference does not seem very drastic, and it is easy enough to regiment the above natural language argument using a sentential connective. (Still, it is nice to know that we can have a FOL-like logical language which has complex predicates, and to have the details worked out.[2])

Now, consider the following natural language argument:

> Socrates ran slowly.
> Therefore, Socrates ran.

This is, I submit, an instance of a simple and distinctive form of valid argument. And here I think FOL leaves a bit more to be desired than with the previous example. The canonical way of regimenting an argument like this in FOL, due to (Davidson, 1967) (which builds on ideas in (Reichenbach, 1947)), regards the premise and conclusion as quantifying

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well worth watching for Strawson's full comment and Davidson's response. Later in the session (around 41:30 in the linked version), audience member Martin Davies (sitting next to Bryan Magee) re-raises the topic from the audience, leading to further interesting discussion.

[2] See (Stalnaker, 1977) for a logical language designed to give a perspicuous representation of complex predicates, and arguments for its philosophical significance.

over an event, and yields a translation like this (where E is a predicate meaning 'is an event'):

$\exists x(Ex \wedge Rsx \wedge Sx)$
$\therefore \exists x(Ex \wedge Rsx)$

Or, as is customary, we may use a special kind of variable e which ranges only over events, and write:

$\exists e(Rse \wedge Se)$
$\therefore \exists e Rse$

Following Davidson's own strategy for paraphrasing this back into English, it can be read as something like (see Davidson, 1967, p. 92):

> *There is an event which is a running by Socrates and which is slow. Therefore, there is an event which is a running by Socrates.*

The Davidsonian approach shows us a way of doing without adverbs: well and good.[3] But in my view, the original natural language argument cries out for a logical language which can represent it more perspicuously. As follows, for instance:

Rs^{s}
$\therefore Rs$

The goal of this paper is to develop a language and associated logic which looks like this. Such a language promises to be a natural and flexible means of representation in its own right, as well as enabling us to translate a range of natural language arguments into a notation that reflects their linguistic form more closely than is possible in ordinary FOL.

To this end, the basic logical idea that I want to take up and run with (slowly) is that there are symbols, "adverbs", which can *come between* a term and the rest of the statement it occurs in, so that the addition of an adverb to a statement results in a logically stronger statement. The occurrence of such a symbol in a formula represents an extra hurdle that the formula must clear in order to be true. My attempt to implement this one simple idea in the context of standard formal logic has taken me on an unexpectedly interesting journey, involving considerable unforeseen structure, justified by the felt naturalness of the goal.

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[3] If you're worried that the Davidsonian approach might struggle with adverbs that do not have to do with events, see (Parsons, 1990).

The task of adding simple adverbs of the kind envisaged to standard logic should be contrasted with that of modelling the complex behaviour of adverbs and adjectives in natural language. The latter task has been tackled influentially in (Montague, 1970; Thomason and Stalnaker, 1973), providing a basis for subsequent work in that vein. By contrast, I am focused here on the problems that arise in the attempt to add simple adverbs to standard logic in a straightforward way. Such problems as 'What will the semantics of adverbs look like?', 'How should truth in a model be defined?', and 'What about proof-theory?'.

Below I develop a language and associated logic FOL-BA (*first-order logic with basic adverbs*), initially defined model-theoretically, in which adverbs can come between names and the adverbless atomic statements they appear in. Then, noting serious limitations of FOL-BA, I develop a second language and associated logic FOL-SA (*first-order logic with scoped adverbs*) in which adverbs can take scope over one another and can come between terms and more complex statements that they appear in — i.e., not just between names and adverbless atomic statements.

The plan for the paper is as follows. Sections 2 through 4 concern FOL-BA. In Section 2 I informally explain the notation and the inferential desiderata for FOL-BA. In Section 3, I define the language of FOL-BA. In Section 4, I explain the semantics of FOL-BA, define truth in a model, and define the consequence relation. (In an associated appendix I (i) show that the FOL-BA consequence relation meets the inferential desiderata, (ii) extend a tree proof system for FOL to FOL-BA, (iii) show that the tree system is sound and complete for FOL-BA, and (iv) give procedures for translating between FOL-BA and FOL.) Sections 5 through 7 concern FOL-SA. In Section 5 I explain FOL-BA's limitations and introduce the approach I will take in setting up FOL-SA. In Section 6 I define the language of FOL-SA, and in Section 7 I develop its model-theory and define the consequence relation. I conclude the paper briefly in Section 8.

2. Introducing FOL-BA

I translate 'Socrates ran slowly' as follows: Rs^{s}. Unlike in English, adverbs in FOL-BA can be strung together side-by-side without a symbol in between. So if $^{\text{d}}$ is an adverb corresponding to 'doggedly', we may write Rs^{sd} to express what in English we might express by saying 'Socrates ran slowly and doggedly'. Another example which illustrates how this

extension of FOL will depart from ordinary English: 'Socrates is non-essentially ('accidentally') a member of {Socrates}, but {Socrates} is essentially such that Socrates is a member of it' (see Fine, 1994).

On the present approach, these two things can, if we wish, be said in one go:

Socrates^a ∈ {Socrates}^{e}

'The Manor House intrinsically has The Great Beam as a part, but The Great Beam is extrinsically a part of The Manor House' becomes:

$Pb^x h^i$

Note that we need not decide, in general, what if any sub-propositional phrases are being modified upon the addition of an adverb. We need not think of the adverbs as predicate modifiers, or term modifiers, or predicate-place modifiers. They are a category of symbol in their own right and they play a role in truth in a model. (What they definitely are is wff modifiers, so you could think of them as one-place propositional operators, yielding different propositions depending on where inside their operands they are put.)

As we shall see, implementing the basic idea of adverbs results in characteristic inferential behaviour. Things like: 'Socrates runs slowly' implies 'Socrates runs' (*Subtraction*), 'Nothing runs' implies 'Nothing runs fast' (*Addition*), and 'Socrates runs slowly and laboriously' implies 'Socrates runs laboriously and slowly' (*Permutation*).

Borowski (1974)[4] puts the above forward as characteristic features of 'adverb adjunction'. Pörn (1983) takes the capture of these as minimal conditions of adequacy of a logical treatment of adverbs. These authors had a range of linguistic data to account for, leading to constructions quite different from what I present here. By contrast, the project here is to take up and run with a simple idea and implement it in logic. The result has a claim to our interest in virtue of its implementation of a natural guiding idea, and the perspicuous regimentations of natural language arguments it affords. Despite this difference in aim, the two approaches coincide in regarding Subtraction, Addition and Permutation as desirable. However, there is another feature put forward by Borowski

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[4] Borowski departs from Davidson's approach, and uses special sentential operators. 'John kissed Mary at midnight' becomes 'At midnight (John kissed Mary)', and then finally 'A_t (John kissed Mary, midnight)' (Borowski, 1974, p. 491).

and aimed at by Pörn which I will not try to capture — indeed, I will try to capture the opposite. These authors wanted to accommodate statements like 'John left in great haste' (where 'at great haste' is treated as an adverb) and 'John left at a leisurely pace' in such a way that they do not jointly imply 'John left in great haste at a leisurely pace'. By contrast, I want Lj^h and Lj^l jointly to imply Lj^hl. That is simpler and fits just as well with the guiding idea. I call this feature *Compounding*.[5]

A final feature I want to capture is *Repetition*: 'Socrates runs slowly, laboriously, and slowly' is equivalent to 'Socrates runs slowly and laboriously'.

Aiming for Addition and Subtraction means that *non-intersective* or *non-factive* natural language adverbs like 'allegedly' cannot be translated straightforwardly using the sort of adverbs on offer here. They simply fall outside the scope of the present project, in much the same way as non-truth-functional sentential connectives fall outside the scope of truth-functional propositional logic.

Addition and Subtraction are, from the point of view of the basic logical idea I am trying to implement, by far the most important of the inferential desiderata just introduced. It is these that are really inspired by natural language. By contrast, Permutation, Compounding and Repetition embody relatively unimportant design choices that could easily have been made differently. (Compounding essentially just provides a handy way of *conjoining* multiple adverbial claims without having to write the adverbless part multiple times. Repetition and Permutation can be seen as then flowing from that. And once we allow this kind of compounding at the level of adverbs, it becomes natural to consider other kinds. We may for instance envisage a language in which adverbs can be

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[5] This example of John leaving in two different ways needs to be treated differently if we do not want to be able to compound the two statements. Compare: if Socrates is in the habit of running quickly and also in the habit of running slowly, the sentences 'Socrates runs quickly' and 'Socrates runs slowly', and the result of applying compounding to them ('Socrates runs quickly and slowly') *can* be understood in such a way as to all be true. If we instead consider a predicate like 'is running now', we would probably not want to add adverbs for both 'quickly' and 'slowly' to express any truth. But that is because we think 'Socrates is running quickly now' and 'Socrates is running slowly now' can't both be true, for reasons which go beyond the basic logic of adverbs. Such incompatibilities do not mean that we do not want compounding. Similarly, we know things can't be red all over and green all over, but we do not want propositional logic itself to prevent us from inferring the conjunction of 'The ball is red all over' and 'The ball is green all over' from those statements taken separately.

negated, so that something like $Pa^{\sim b}$ can be written and is equivalent to $Pa \wedge \sim Pa^{b}$. I leave this kind of development aside in the present paper.)

3. The Language of FOL-BA

The language of FOL-BA is the language of FOL[6] with the addition of a stock of superscript letters, or *adverbs*.

Vocabulary:

Names: a, b, c, ..., t. If we need more, we use subscripts (i.e. a_2, a_3, ..., b_2, b_3, ...).

Variables: x, y, z, u, v, w. (We use subscripts if we need more.)

Predicates: $=^2$, A^1, B^1, C^1, ..., A^2, B^2, C^2, ... (Superscripts indicate the number of argument places, and may be omitted for convenience. As with names and variables, we use subscripts if we need more.)

Connectives: $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$

Quantifiers: $\forall$, $\exists$

Brackets: (,)

Adverbs: ^{a}, ^{b}, ^{c}, ... If we need more, we add primes (i.e. $^{a'}$, $^{b'}$, $^{c'}$, ..., $^{a''}$, $^{b''}$, $^{c''}$, ...).

Terms: (i) names are terms; (ii) variables are terms; (iii) nothing else is a term.

Place-fillers: (i) terms are place-fillers; if p is a place-filler and $\underline{a}$ is an adverb, then $p^{\underline{a}}$ is a place-filler[7]; (iii) nothing else is a place-filler.

Wffs of FOL-BA:

(i) Where $\underline{P}^n$ is an n-place predicate and $\underline{p}_1$, ..., $\underline{p}_n$ are place-fillers, the following is a wff: $\underline{P}^n\ \underline{p}_1 \ldots \underline{p}_n$. A wff of this kind is called an *atomic* wff.

(ii) Where α and β are wffs and $\underline{x}$ is a variable, the following are wffs: $\sim\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \supset \beta)$, $(\alpha \equiv \beta)$, $\forall\underline{x}\alpha$, $\exists\underline{x}\alpha$.

(iii) Nothing else is a wff.

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[6] I leave term-forming functions out of the picture for simplicity's sake, but they pose no difficulties. The presentation of FOL that I adapt and extend here is from (Smith, 2012, p. 280) and preceding. In Smith's presentation, there are no variable assignments and truth is defined directly (not via satisfaction) but what follows could easily be adapted to a satisfaction-based presentation of quantification in FOL.

[7] I.e., place-fillers followed by adverbs are place-fillers

4. The Model-Theory of FOL-BA

4.1. What is a FOL-BA Model?

A model of a fragment of FOL-BA may be thought of as an ordered quadruple $\langle D, N, P, A \rangle$, where:

(i) D is a non-empty set of objects (*domain*).

(ii) N is a function which maps each name to its *referent*, a member of D.

(iii) P is a function which maps each n-place predicate to its *extension*, a (possibly empty) set of n-tuples containing members of D.[8] In the special case of the two-place predicate for identity, $=$, the extension must be the set containing all and only the ordered pairs containing some member of D followed by itself, i.e., the set of repetitive ordered pairs involving members of D.

(iv) A is a function which maps each adverb to its *extension*, a (possibly empty) set of triples $\langle Pr, i, num \rangle$, where Pr is an n-place predicate, i is an instance of Pr (i.e., a member of Pr's extension), and *num* is a number from 1 to n.

Here, (i)–(iii) give the familiar ingredients of a (certain style of) FOL model and (iv) introduces extensions for adverbs: the extension of an adverb is the set of triples of predicates, predicate-instances and predicate-places that the adverb applies to.

Intuitively, if in the instance i of Pr, the nth place is occupied a'ly (i.e. in the manner signified by some adverb [a]), the extension of [a] will include $\langle Pr, i, n \rangle$. For example, if Socrates is running quickly to Athens, the extension of [q] will include $\langle R, \langle \text{Socrates, Athens} \rangle, 1 \rangle$.

Before proceeding to the definition of truth in a FOL-BA model, a couple of remarks on the way things are being done here.

First, it may be wondered why we put *numbers* tracking predicate-places in the third places of members of adverbs' extensions, rather than objects from the relevant predicate-instances. Why do not we represent that Socrates is running quickly to Athens by putting $\langle R, \langle \text{Socrates,} \text{Athens} \rangle, \text{Socrates} \rangle$ in the extension of [q]? Well, if we did it that way, then

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[8] In the case of one-place predicates, it is convenient to think of their extensions simply as subsets of the domain, but for the purposes of the definition of truth in a model below, the extensions of one-place predicates are sets of 1-tuples, as defined above.

the formulas $Fa^{b}aa$, $Faa^{b}a$ and $Faaa^{b}$, for example, would be logically equivalent to each other. But this is not part of what we want, and does not flow naturally from our guiding idea. We should allow for the case where a given object appears more than once in an instance of a predicate, and in different manners in the different appearances, so we keep track of places using numbers in the members of the extensions of adverbs.

Second, what might become of the idea of a zero-place predicate, which one sometimes sees in presentations of FOL, in the present context? The presentation of FOL I am building on does not include zero-place predicates, and for simplicity I have not attempted to accommodate them formally. But I would like to indicate briefly how this could be done, for zero-place predicates may appear to pose a difficulty: if a sentence like 'It is raining' is best regimented as a zero-place predicate, and if the adverbs of FOL-BA occupy (along with terms) places in predicates, it may seem as though we are left with no good way of regimenting 'It is raining hard' using one of our adverbs. This difficulty is, I think, more apparent than real. Now that we have adverbs, places in predicates play a dual role; they may harbour terms, and they may harbour adverbs. And with 'It is raining hard' we find ourselves wanting to use an adverb but no term. One solution is to replace the category of zero-place predicates with a special category of one-place predicates — special because the one place may be left empty, or filled with a special kind of place-filler that is only allowed to contain adverbs. Then, we could put these special predicates in the extensions of adverbs — e.g., the one for 'It is raining' could be put into the extension of the adverb for 'hard'. Or we could move away from the idea that a particular place in a predicate can harbour both terms and adverbs. On this scheme, $n{>}0$-place predicates in FOL (and FOL-BA as developed here) become predicates with n places for terms and n places for adverbs. 'It is raining' could then be regimented using a predicate with zero term-places and one adverb-place.)

4.2. Truth in a FOL-BA Model

We take an existing definition of truth in a model for FOL from (Smith, 2012, p. 280) and modify the clause for atomic wffs by adding a subclause. Intuitively speaking, the added clause asks us to go from left to right checking, for each adverb-occurrence, whether that occurrence

comes between the term it shares a place with and the rest of the atomic wff. If we find a certain ordered triple in the adverb's extension, all is well and the wff could be true as far as that adverb-occurrence is concerned. (We may think of an adverb's extension as a kind of record held by the adverb concerning the occasions on which Truth is permitted to get past an occurrence of that adverb and into the proposition that contains it.) Given that we want Compounding, this one-by-one checking suits our aims.

Before defining truth on a model, two pieces of terminology.

The term $\underline{t}$ at the beginning of a place-filler $\underline{p}$ may be denoted $term(\underline{p})$. For example, $term(a) = a$, $term(b^{\mathrm{t}}) = b$ and $term(x^{\mathrm{t}}) = x$.

We use $\alpha(\underline{x})$ to stand for an arbitrary wff with no free occurrence of any variable other than x, and $\alpha(\underline{a}/\underline{x})$ to stand for the wff resulting from $\alpha(\underline{x})$ by replacing all free occurrences of x in $\alpha(\underline{x})$ with the name $\underline{a}$.

We now define truth in a FOL-BA model $\mathfrak{M}$ as follows:

1. An atomic wff $\underline{P}^n\ \underline{p}_1 \ldots \underline{p}_n$ without free variables is true in $\mathfrak{M}$ iff
   (i) $\langle term(\underline{p}_1)$'s referent, $\ldots$, $term(\underline{p}_n)$'s referent$\rangle$ is in the extension of $\underline{P}^n$, and
   (ii) each k such that $0 < k \leqslant n$ and each adverb $\underline{a}$ in $\underline{p}_k$, $\langle \underline{P}^n$, $\langle term(\underline{p}_1)$'s referent, $\ldots$, $term(\underline{p}_n)$'s referent$\rangle, k\rangle$ is in the extension of $\underline{a}$.
2. $\sim\alpha$ is true in $\mathfrak{M}$ iff α is false in $\mathfrak{M}$.
3. $(\alpha \wedge \beta)$ is true in $\mathfrak{M}$ iff α and β are both true in $\mathfrak{M}$.
4. $(\alpha \vee \beta)$ is true in $\mathfrak{M}$ iff one or both of α and β is true in $\mathfrak{M}$.
5. $(\alpha \supset \beta)$ is true in $\mathfrak{M}$ iff α is false in $\mathfrak{M}$ or β is true in $\mathfrak{M}$ (or both).
6. $(\alpha \equiv \beta)$ is true in $\mathfrak{M}$ iff α and β are both true in $\mathfrak{M}$ or both false in $\mathfrak{M}$.
7. $\forall \underline{x}\alpha(\underline{x})$ is true in $\mathfrak{M}$ iff for every object o in the domain D of $\mathfrak{M}$, $\alpha(\underline{a}/\underline{x})$ is true in $\mathrm{M}^{\underline{a}}{}_o$, where $\underline{a}$ is some name not assigned a referent in $\mathfrak{M}$, and $\mathfrak{M}^{\underline{a}}{}_o$ is a model just like $\mathfrak{M}$ except that in it the name $\underline{a}$ is assigned the referent o.
8. $\exists \underline{x}\alpha(\underline{x})$ is true in $\mathfrak{M}$ iff there is at least one object o in the domain D of $\mathfrak{M}$ such that $\alpha(\underline{a}/\underline{x})$ is true in $\mathrm{M}^{\underline{a}}{}_o$, where $\underline{a}$ is some name not assigned a referent in $\mathfrak{M}$, and $\mathfrak{M}^{\underline{a}}{}_o$ is a model just like $\mathfrak{M}$ except that in it the name $\underline{a}$ is assigned the referent o.

Note that the truth-conditions for atomic wffs involving the identity predicate are covered by clause 1 above. Hence, a true identity statement

is just as capable of becoming false upon the addition of adverbs as any other atomic wff.

4.3. The FOL-BA Consequence Relation

We define consequence in the standard way:

> $\Gamma \models_{\text{FOL-BA}} \alpha$ iff there is no model $\mathfrak{M}$ in which all members of Γ are true and α is false.

Note that FOL-BA is an intensional, i.e. non-extensional, logic, in that co-extensive predicates cannot always be substituted *salva veritate*. Whether an adverb-containing atomic wff is true in a model depends not only on the extensions of the predicates and adverbs involved, but on *which* predicate figures in the wff.[9] And the resulting intensionality is as it should be: we want to be able to translate 'John dances gracefully' and 'All and only dancers are tuba players' as Dj^{g} and $\forall x (Dx \equiv Tx)$ without these formulas implying Tj^{g} (i.e. 'John plays the tuba gracefully').

See the appendix for further development of FOL-BA (and see the end of B, there for a worked example of translating a natural language argument and giving a tree proof of its validity). We now turn to a second, richer logic.

5. Introducing FOL-SA

There are serious limitations of FOL-BA having to do with scope: adverbs cannot take scope over one another and can only come between terms and the atomic statements they are part of. But there is reason to want adverbs which can take scope over one another and which can come between a term and a more complex statement.

Consider:

John reluctantly danced gracefully.

Read naturally, this is not equivalent to:

John danced both reluctantly and gracefully.

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[9] Note that if one were to associate predicates with properties and relations, so that two predicates may be mapped to different properties while being alike in extension, then the properties could figure in the extensions of adverbs in place of the predicates that express them. Then, different predicates standing for the same property could be substituted *salva veritate*.

John might have been quite happy to dance, and only reluctant to do so gracefully. The best we can do in FOL-BA is something like:

Dj^{r}

where 'D' is simply stipulated to mean 'danced gracefully'. That is a failure of articulation, preventing us from capturing the fact that 'John danced' follows logically from 'John reluctantly danced gracefully'.

A related difficulty arises with the following pair of sentences:[10]

Peter slowly checked all the lightbulbs.
Peter checked all the lightbulbs slowly.

Read naturally, these do not mean the same thing. With the first, the 'slowly' is about the checking-of-all-the-lightbulbs: the sentence could be true even if Peter checked some non-slowly. Not so with the second, and it is only the second which can be given an articulate translation into FOL-BA.

A good way to handle these phenomena in a perspicuous logical notation, I submit, is to use brackets tagged with particular adverbs, indicating that occurrences of those adverbs within the brackets may come between the terms they appear alongside and the material in the brackets. Making the natural choice to attaching the tags to the left brackets, we can then represent:

John reluctantly danced gracefully.

using

$(^{\mathrm{r}}(^{\mathrm{g}}Dj^{\mathrm{gr}}))$

and

Peter slowly checked all the lightbulbs.

using

$(^{\mathrm{s}}\forall x(Lx \supset Cp^{\mathrm{s}}x))$

which is compatible with Peter not having been slow with each lightbulb:

$\sim\forall x(Lx \supset (^{\mathrm{s}}Cp^{\mathrm{s}}x))$

We will also want a way of pairing the adverbs tagging the brackets with some but not all occurrences of the same adverb within. For consider a formula like:

$(^{\mathrm{s}}\forall x(^{\mathrm{s}}Ra^{\mathrm{s}}xb^{\mathrm{s}}))$

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[10] This type of example is discussed in (Thomason and Stalnaker, 1973, p. 200).

This does not tell us which of the bracket-tagging occurrences corresponds to which of the term-flanking occurrences. But we still want all four occurrences to be occurrences of the same adverb.

One solution would be to draw lines pairing up bracket-tagging occurrences with term-flanking occurrences. That can be a nice way of doing things when writing by hand, but typesetting such formulas and implementing their syntax in detail is unnecessarily complicated: with such a device in place, formulas are no longer straightfowardly treatable as linear strings of symbols.

Instead, we can append indices — Hindu-Arabic numerals — to adverbs when necessary. Taking the ambiguous formula above: if we want the first bracket-tagging adverb in the formula to go with the occurrence flanking the b and the second bracket-tagging adverb to go with the occurrence flanking the a, we can write:

$$(^{\text{s}1}\forall x(^{\text{s}}Ra^{\text{s}}xb^{\text{s}1}))$$

or equivalently

$$(^{\text{s}}\forall x(^{\text{s}1}Ra^{\text{s}1}xb^{\text{s}}))$$

or we could use indices on all occurrences, as in

$$(^{\text{s}2}\forall x(^{\text{s}1}Ra^{\text{s}1}xb^{\text{s}2}))$$

In FOL-BA, adverbs just figure in some atomic wffs, and more complex wffs are built up exactly as in FOL. In this richer setting it becomes natural to have a special category: adverb wffs.

Turning to semantics, we find a further interesting challenge. It is natural to want pairs of formulas like:

$$(^{\text{s}}\forall x(Lx \supset Cp^{\text{s}}x))$$

and

$$(^{\text{s}}\sim\exists x(Lx \wedge \sim Cp^{\text{s}}x))$$

to be logically equivalent.

No such challenge arose with FOL-BA, since in FOL no atomic wff is equivalent to any other. But now we can state more complex conditions that things may satisfy in certain manners, and such conditions can be stated in different ways.

A natural first thought is to appeal to the notion of logical equivalence in the truth-rule for adverb wffs. We could group together, in the members of the extensions of adverbs, equivalent formulations of the came condition.

One minor issue is that we really want a model-relative notion of
"equivalence-modulo-identity": we want the notion, not of having the
same truth-value on all models whatsoever, but of having the same truth-
value on all models where the same pairs of names co-refer as on the
model in question. We want pairs like $(^sPa^s)$ and $(^sPb^s)$ to have the
same truth-value whenever a and b have the same referent.

A trickier issue is circularity: the notion of logical equivalence — and
in turn, the model-relative notion just sketched — involves the notion of
truth on a model. We can't simply help ourselves to this when giving
a truth-rule for adverb wffs, since that rule is part of the definition of
truth on a model.

Accordingly, we break the language up into levels: sets of formu-
las where the number of nestings of adverb-wffs within adverb-wffs is
bounded. Then, instead of simply having a notion of a model of the
language, we define the notion of a *level n* model, i.e., a model for the
level n fragment of FOL-SA, i.e., the set of wffs of level n or less. Starting
with the adverbless level 0 fragment and building up from there, we avoid
circularity. At any level n above zero, adverbs will have extensions, and
these will involve sets of conditions which are required to be "equivalent-
modulo-identity" *but where we only consider models of lower levels.*

When specifying a model, the user can pick a level as high as they
need to make the specification, and when specifying a member of the
extension of a particular adverb, they can formulate the desired con-
dition however they wish. Then, what goes into the extension of the
adverb is the set of conditions containing that formulation along with
all the formulations which are "equivalent-modulo-identity" considering
only models of lower levels. From there, one can move to higher levels
as needed (i.e., if one wants to work with wffs of higher level than any
used in the specification of the model).

The next two sections make precise the syntactic and semantic ideas
sketched above.

6. The Language of FOL-SA

Vocabulary:

Names, Variables, Predicates, Connectives, Quantifiers, Brackets: The
same as in FOL-BA.

Adverbs: The same as in FOL-BA but we shall also call these 'Plain
adverbs' below.

Indices: 1, 2, 3, ... I.e., Hindu-Arabic numerals for natural numbers.

Terms: (i) names are terms; (ii) variables are terms; (iii) nothing else is a term.

Indexed adverbs: (i) plain adverbs are indexed adverbs; (ii) a plain adverb followed by an index is an indexed adverb; (iii) nothing else is an indexed adverb.

Place-fillers: (i) terms are place-fillers; (ii) if p is a place-filler and $\underline{a}$ is an indexed adverb, then $p^{\underline{a}}$ is a place-filler[11]; (iii) nothing else is a place-filler.

Wffs of FOL-SA:

(i) Where $\underline{P}^n$ is an n-place predicate and $\underline{p}_1$, ..., $\underline{p}_n$ are place-fillers, the following is a wff: $\underline{P}^n \, \underline{p}_1 \ldots \underline{p}_n$ Wffs of the above kind are called *atomic* wffs.

(ii) Where α and β are wffs and $\underline{x}$ is a variable, the following are wffs: $\sim\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \supset \beta)$, $(\alpha \equiv \beta)$, $\forall\underline{x}\alpha$, $\exists\underline{x}\alpha$.

(iii) Where $\alpha(\underline{p})$ is a wff containing an occurrence o of a place-filler $\underline{p}$ which is not and does not contain a bound variable, the result $(\underline{a1}\cdots\underline{an}\alpha \, (p^{\underline{a1}\cdots\underline{an}}))$ of putting n indexed adverbs $\underline{a1}\cdots\underline{an}$ that do not appear in $\alpha(\underline{p})$ to the immediate right of the occurrence o of $\underline{p}$, putting a right-bracket to the immediate right of $\alpha(\underline{p})$, and putting a left-bracket followed by $\underline{a1}\cdots\underline{an}$ to the immediate left of $\alpha(\underline{p})$, is a wff. Wffs of the above are called *adverb* wffs.[12]

(iv) Nothing else is a wff.

An *adverb opening* is a left-bracket followed by an indexed adverb, and its *companion* is the right-bracket that came with this opening in the construction of the wff that it is the beginning of. The *level* of a wff is the number of consecutive adverb openings you can encounter in a row before hitting the companion of the first adverb opening you encountered.

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[11] I.e., place-fillers followed by indexed adverbs are place-fillers)

[12] It should be noted that this definition of adverb wffs forgoes the possibility of putting adverbs next to *different terms* in one go, the way we can in FOL-BA (recall Section 2). For example, while we can write $Pb^x h^i$ in FOL-BA, we cannot write $(^{xi}Pb^x h^i)$ in FOL-SA as set up above. There is no deep difficulty in modifying the syntax and semantics of FOL-SA to allow this, but the modification as I formulated it makes the syntax and semantics a bit harder to comprehend (as presented here; I suspect and hope that it is possible to give a more streamlined presentation than I have managed). Alternatively, a string like $(^{xi}Pb^x h^i)$ may be regarded as an abbreviation of $(^x Pb^x h) \wedge (^i Pbh^i)$.

7. The Model Theory of FOL-SA

We define the notion of a *level n* model, i.e., a model for the *level n* fragment of FOL-SA, i.e., the set of wffs of level n or less.

We begin with the notion of a level 0 model. Then we define truth in such a model. We then, in effect, use these notions to define the notion of a level 1 model and truth in such a model. We then use *these* notions to define the notion of a level 2 model, and so on. More precisely, having defined the level 0 notions, we show how, given the notion of a level n model and the notion of truth in such a model, to define the notion of a level $n+1$ model and the notion of truth in such a model.

A model of the level 0 fragment of FOL-SA is really just a model of the language of FOL. It may be thought of as an ordered triple $\langle D, N, P \rangle$, where:

(i) D is a non-empty set of objects (the *domain*).

(ii) N is a function which maps each name to its *referent*, a member of D.

(iii) P is a function which maps each n-place predicate to its *extension*, a (possibly empty) set of n-tuples containing members of D. In the special case of the two-place predicate for identity, $=$, the extension must be the set containing all and only the ordered pairs containing some member of D followed by itself, i.e., the set of repetitive ordered pairs involving members of D.

We again use the $\alpha(\underline{x})$, $\alpha(\underline{a})/\underline{x})$ terminology explained when setting up FOL-BA.

We define truth in a level 0 model $\mathfrak{M}$ as follows:

1. An atomic wff $\underline{P}^n\underline{p}_1 \dots \underline{p}_n$ without free variables is true in $\mathfrak{M}$ iff $\langle \underline{p}_1\text{'s referent}, \dots, \underline{p}_n\text{'s referent} \rangle$ is in the extension of $\underline{P}^n$.

2. $\sim\alpha$ is true in $\mathfrak{M}$ iff α is false in $\mathfrak{M}$.

3. $(\alpha \wedge \beta)$ is true in $\mathfrak{M}$ iff α and β are both true in $\mathfrak{M}$.

4. $(\alpha \vee \beta)$ is true in $\mathfrak{M}$ iff one or both of α and β is true in $\mathfrak{M}$.

5. $(\alpha \supset \beta)$ is true in $\mathfrak{M}$ iff α is false in $\mathfrak{M}$ or β is true in $\mathfrak{M}$ (or both).

6. $(\alpha \equiv \beta)$ is true in $\mathfrak{M}$ iff α and β are both true in $\mathfrak{M}$ or both false in $\mathfrak{M}$.

7. $\forall\underline{x}\alpha(\underline{x})$ is true in $\mathfrak{M}$ iff for every object o in the domain D of $\mathfrak{M}$, $\alpha(\underline{a}/\underline{x})$ is true in $\mathrm{M}^{\underline{a}}{}_o$, where $\underline{a}$ is some name not assigned a referent in $\mathfrak{M}$, and $\mathfrak{M}^{\underline{a}}{}_o$ is a model just like $\mathfrak{M}$ except that in it the name $\underline{a}$ is assigned the referent o.

8. $\exists \underline{x} \alpha(\underline{x})$ is true in $\mathfrak{M}$ iff there is at least one object o in the domain D of $\mathfrak{M}$ such that $\alpha(\underline{a}/\underline{x})$ is true in $\mathrm{M}^{\underline{a}}{}_o$, where $\underline{a}$ is some name not assigned a referent in $\mathfrak{M}$, and $\mathfrak{M}^{\underline{a}}{}_o$ is a model just like $\mathfrak{M}$ except that in it the name $\underline{a}$ is assigned the referent o.

Before defining the notion of an $n+1$ level model for $n \geqslant 0$, some terminology.

Call an open wff which contains exactly one occurrence of a free variable a *condition*.

Two conditions α and β of level n or less are *i-equivalent* w.r.t. an $n+1$-level model $\mathfrak{M}$ iff, on all models of level 0, ..., n which assign the same objects to the same names as $\mathfrak{M}$, the set of objects satisfying $\alpha =$ the set of objects satisfying β. (The notion of satisfaction is the obvious one: An object o in the domain of a model $\mathfrak{M}$ *satisfies* a condition c on a model $\mathfrak{M}$ iff the result of replacing c's free variable with a name for o is true on $\mathfrak{M}$.)

A set S of conditions of level n or less is *maximal i-equivalent w.r.t. an $n+1$-level model $\mathfrak{M}$* iff (i) no member α of S is i-equivalent w.r.t. $\mathfrak{M}$ to an open wff β of level n or less that is not a member of S, and (ii) for any two members of S α and β, α and β are i-equivalent w.r.t. $\mathfrak{M}$.

For $n \geqslant 0$, an $n+1$-level model may be thought of as an ordered quadruple $\langle D, N, P, A \rangle$, where:

  (i) A non-empty set D of objects (the *domain*).
 (ii) N is a function which maps each name to its *referent*, a member of D.
(iii) P is a function which maps each n-place predicate to its *extension*, a (possibly empty) set of n-tuples containing members of D. In the special case of the two-place predicate for identity, $=$, the extension must be the set containing all and only the ordered pairs containing some member of D followed by itself, i.e., the set of repetitive ordered pairs involving members of D.
 (iv) A is a function which maps each adverb to its *extension*, a (possibly empty) set of pairs $\langle o, S \rangle$, where o is an object in D, and S is a set of conditions of level n or less that is maximal i-equivalent w.r.t. $\mathfrak{M}$.

We will also use the term *extension* in connection with indexed adverbs, meaning the extension of the plain adverb preceding the index.

Before defining truth in an $n+1$ level model for $n \geqslant 0$, some more terminology.

The term $\underline{t}$ at the beginning of a place-filler $\underline{p}$ may be denoted *term($\underline{p}$)*, as explained when setting up FOL-BA.

Given a closed adverb wff α we may speak of the *focus of* α, meaning the name occupying the place-filler $\underline{p}$ which featured in α's construction, and the *condition of* α, meaning the open wff that you get by taking α, replacing its focus with a free variable $\underline{x}$, and removing the outer brackets and the indexed adverbs that were added during α's construction.

We now define truth in an $n+1$ level model for $n \geqslant 0$:

1. An atomic wff $\underline{P}^n\underline{p}_1...\underline{p}_n$ without free variables is true in $\mathfrak{M}$ iff $\langle \text{term}(\underline{p}_1)\text{'s referent}, \ldots, \text{term}(\underline{p}_n)\text{'s referent} \rangle$ is in the extension of $\underline{P}^n$.

2–8. The same as above in the definition of truth in a 0 level model.

9. An adverb wff $(\underline{a1} \cdots \underline{an}\alpha(\underline{p}^{\underline{a1}\cdots\underline{an}}))$ is true in $\mathfrak{M}$ iff (i) $\alpha(\underline{p})$ is true in $\mathfrak{M}$ and (ii) for each indexed adverb $\underline{a}$ in $\underline{a1}\cdots\underline{an}$, $\langle o, S \rangle$ is in the extension of $\underline{a}$, where o is the referent of the focus of $(\underline{a1}\cdots\underline{an}\alpha(\underline{p}^{\underline{a1}\cdots\underline{an}}))$ and the condition of $(\underline{a1}\cdots\underline{an}\alpha(\underline{p}^{\underline{a1}\cdots\underline{an}}))$ is a member of S.

Finally, consequence may be defined as follows:

$\Gamma \models_{\text{FOL-SA}} \alpha$ iff there is no model $\mathfrak{M}$ (of any level) in which all members of Γ are true and α is false.

8. Conclusion

FOL-BA and FOL-SA provide a perspicuous framework for translating and assessing the validity of a range of natural language arguments. The framework may also be used more directly for representing information about, and reasoning about, various domains. We can use adverbs to formulate theses about particular properties and relations of interest, for instance:

$\forall x \forall y (x \in y \supset x \in y^{\text{e}})$
Set membership is essential w.r.t. its 2nd place. (Sets have their members essentially.)

There is also further work to be done in developing the framework itself, especially with regard to FOL-SA. That FOL-SA meets the inferential desiderata laid out in Section 2 when introducing FOL-BA can be shown easily along the same lines as the proof given in the appendix for FOL-BA, using the notions of positive and negative occurrences. On the proof-theory side, the question of whether a sound and complete

proof theory can be given — and if so, what forms it might take — remains open. It is also not clear to me whether anything corresponding to the schemes for translating back and forth between FOL and FOL-BA given in the appendix can be carried over to FOL and FOL-SA. More generally speaking, I would like to see some light shed on the metalogical properties of FOL-SA, including its expressive power.

Finally, there is scope for extending the framework in various ways; recall for instance the remark, at the end of Section 2, about logical complexity at the level of adverbs themselves. Here are some further ideas for how the framework might be extended. The possibilities of quantifying into adverb position may be explored (this may be of philosophical interest in connection with recent work on higher-order metaphysics (for an overview, see Fritz and Jones, forthcoming), e.g., to formulate theses about *ways*). Adverbs in a modal setting may be investigated, and not just by adding modal sentential operators; we could study modal adverbs whose semantics involve other possible worlds. A many-valued investigation may be pursued, with adverbs providing an additional degree of freedom for truth-value variation, whose interactions with other factors may then be studied. Distinctive anti-extensions for adverbs may be introduced so that, e.g., the full semantic value of an adverb translating 'happily' contains information allowing us to discriminate between someone doing something unhappily, or merely neutrally (i.e. neither happily nor unhappily). We could also set up postulates or rules for adverbs of philosophical interest. For instance, entailments involving the adverbs 'essentially' and 'accidentally' could be captured, giving us a "logic of essence" different from that of (Fine, 1995).[13] Finally, provision may be made for non-factive adverbs like 'allegedly'. Addition and Subtraction will not hold of non-factive adverbs, but Permutation, Compounding and Repetition may.

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[13] 'Essentially' and 'accidentally', as well as 'necessarily' and 'contingently', seem to behave in accord with FOL-BA (whereas 'possibly', being non-factive, does not), but note that 'necessarily necessarily' in FOL-BA just boils down to 'necessarily', since FOL-BA adverbs cannot have others in their scope. Further entailments specific to these adverbs, which we might want to try to capture by building on FOL-BA, include: 'Socrates is essentially human' entails 'Socrates is not accidentally human', and 'Socrates is essentially human' entails 'Socrates is necessarily human'.

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Appendix

A. FOL-BA Meets its Inferential Desiderata

We now show that FOL-BA meets the inferential desiderata outlined in Section 2: Subtraction, Addition, Permutation, Compounding and Repetition. For this purpose, we take $\sim$ and $\wedge$ as primitive. Likewise, we take $\exists$ as primitive and regard $\forall$ as $\sim\exists\sim$.

First, let us state the desiderata precisely. For Subtraction and Addition, we will use the syntactic notions of a *positive* and a *negative* occurrence of an atomic wff.[14] When an atomic wff occurs in the scope of an even number of negations, it is a positive occurrence, and is negative otherwise. By extension, we will speak of the positive and negative occurrences of place-fillers.

Let $\alpha(\underline{q}//\underline{p})^{+}$ be the result of replacing zero or more positive, non-quantifier-possessed[15] occurrences of a place-filler $\underline{p}$ with $\underline{q}$ in a wff α. Let $\alpha(\underline{q}//\underline{p})^{-}$ be the result of replacing zero or more negative, non-quantifier-possessed occurrences of a place-filler $\underline{p}$ with $\underline{q}$ in a wff α. We use $^{\alpha}$ and $^{\beta}$ for arbitrary sequences of adverbs, and we restrict the wff variable α to closed wffs.

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[14] Thanks to an anonymous referee for suggesting the use of the notions of positive and negative occurrences for the purpose of stating and proving these laws.

[15] That is, we rule out occurrences of variables to the immediate right of quantifier symbols.

We can now state Subtraction, Addition, Permutation, Compounding and Repetition as follows:

$\alpha \models \alpha(\underline{p}'//\underline{p})^+$, where $\underline{p}'$ is the result of removing zero or more occurrences of adverbs from $\underline{p}$.

$\alpha \models \alpha(\underline{p}'//\underline{p})^-$, where $\underline{p}'$ is the result of adding zero or more occurrences of adverbs to $\underline{p}$.

$\alpha(\underline{p}^{\alpha\beta}) \models \alpha(\underline{p}^{\beta\alpha})$

$\alpha(\underline{p}^{\alpha}), \alpha(\underline{p}^{\beta}) \models \alpha(\underline{p}^{\alpha\beta})$

$\alpha(\underline{p}^{\alpha}) \models \alpha(\underline{p}^{\alpha\alpha})$

That Permutation, Compounding, and Repetition hold is obvious from the semantics. We now prove that Subtraction holds (the proof for Addition is similar).

We prove the law for the case of removing exactly one adverb occurrence. The case of zero is vouchsafed by the fact that $\alpha \models \alpha$ for all α, and the case of more than one is vouchsafed by iterating the case of exactly one.

Call an occurrence of a place-filler $\underline{p}$ in a wff α *open to truth-preserving subtraction with respect to* α iff, in an arbitrary model $\mathfrak{M}$ where α is true, the result of replacing that occurrence of $\underline{p}$ with $\underline{p}'$ is also true in $\mathfrak{M}$. Similarly, we will speak of an occurrence of a place-filler's openness to *falsity*-preserving subtraction with respect to a wff, defined as above but with 'false' in place of 'true'. When it is understood from context that α contains an occurrence of $\underline{p}$, we will write α' to mean the result of replacing that occurrence of $\underline{p}$ with $\underline{p}'$.

We want to show that an arbitrary positive occurrence of a place-filler $\underline{p}$ in a wff α is open to truth-preserving subtraction with respect to α. We show a stronger claim, namely that *both* the following properties hold of an arbitrary wff α:

(Pos-T-Pres) If a place-filler $\underline{p}$ occurs positively in α, then that occurrence of $\underline{p}$ is open to truth-preserving subtraction with respect to α.

(Neg-F-Pres) If a place-filler $\underline{p}$ occurs negatively in α, then that occurrence of $\underline{p}$ is open to falsity-preserving subtraction with respect to α.

We do this by induction on complexity of formulas.[16]

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[16] The proof is informed by ideas from (van Benthem, 2008, p. 21) and (Sánchez Valencia, 1991, p. 96) about monotonic reasoning (in the context of the programme of "Natural Logic"). Indeed, adverb subtraction and addition may be seen as forms of monotonic reasoning; for instance, removing an adverb from a wff is similar to replac-

Base case. All place-fillers occur positively in an atomic wff, so (Neg-F-Pres) holds vacuously. (Pos-T-Pres) holds immediately given subclause (ii) in the truth-rule for atomic wffs; subtracting an adverb just means we have one less thing to check.

Induction step. Our inductive hypothesis IH is that (Pos-T-Pres) and (Neg-F-Pres) hold for all wffs of complexity $0, \ldots, n$. We consider an arbitrary wff α of complexity $n + 1$.

Case 1. α is of the form $\sim\beta$.

We establish (Pos-T-Pres). Suppose there is a positive occurrence of a place-filler $\underline{p}$ in α. In that case, $\underline{p}$ occurs negatively in β. So by (Neg-F-Pres) for β, that occurrence of $\underline{p}$ is open to falsity-preserving subtraction with respect to β. That is, in all models $\mathfrak{M}$ where β is false, β' is also false. But (since α is β's negation), that means that in all models $\mathfrak{M}$ where α is true, α' is also.

(The proof of (Neg-F-Pres) for α is similar; swap 'positively' with 'negatively', 'truth'/'true' with 'falsity'/'false', and '(Pos-T-Pres)' with '(Neg-F-Pres)'. Likewise, we omit proving (Neg-F-Pres) for the two remaining cases below.)

Case 2. α is of the form $(\beta \wedge \gamma)$.

Suppose there is a positive occurrence of a place-filler $\underline{p}$ in α. In that case, that occurrence of $\underline{p}$ is either in β or it is in γ. Suppose, without loss of generality, that it is in β that $\underline{p}$ occurs. By IH ((Pos-T-Pres) for β), in all models $\mathfrak{M}$ where β is true, β' is also. But (since α is $(\beta \wedge \gamma)$), that means that in all models $\mathfrak{M}$ where α is true, α' is also.

Case 3. α is of the form $\exists \underline{x}\beta(\underline{x})$.

Suppose there is a positive occurrence of a place-filler $\underline{p}$ in α (i.e. in $\exists \underline{x}\beta(\underline{x})$). Now we need to establish on this basis that in all models $\mathfrak{M}$ where $\exists \underline{x}\beta(\underline{x})$ is true, $\exists \underline{x}\beta(\underline{x})'$ is true also. Suppose for reductio that there is a model $\mathfrak{M}$ where $\exists \underline{x}\beta(\underline{x})$ is true but $\exists \underline{x}\beta(\underline{x})'$ is false. By the truth-rule for existential formulas, there is a model $\mathfrak{M}^{\underline{a}}{}_o$, which is just like $\mathfrak{M}$ except that it assigns a referent o to a new name $\underline{a}$, such that $\beta(\underline{a}/\underline{x})$ is true in $\mathfrak{M}^{\underline{a}}{}_o$. By IH ((Pos-T-Pres) for $\beta(\underline{a}/\underline{x})$), $\beta(\underline{a}/\underline{x})'$ must be true in $\mathfrak{M}^{\underline{a}}{}_o$. But then, by the truth-rule for existential formulas, $\exists \underline{x}\beta(\underline{x})'$ is true — not false — on $\mathfrak{M}$. Contradiction.

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ing a predicate with one with wider extension. In these works it is stated (Sánchez Valencia mentions that it can be shown by induction) that, when a predicate P occurs positively in a wff of FOL, that wff is (upward) monotonic in P (i.e., P may be replaced with a predicate whose extension is a superset of P's while preserving truth).

We have now established that FOL-BA meets our inferential desiderata.

B. FOL-BA Trees

We can get an FOL-BA tree system by adding three rules to a standard FOL tree system: one for growing the tree, one for closing paths, and one governing the saturation of open paths. Let us first summarise the underlying tree system for FOL. (Full, reader-friendly explanations may be found in (Smith, 2012, p. 315) and (Jeffrey, 1967).)

We continue to use $\alpha(\underline{x})$ and $\alpha(\underline{a}/\underline{x})$ as explained when giving the definition of truth in a model. Similarly, to state the rule of Substitution of Identicals, we use $\alpha(\underline{a})$ to stand for an arbitrary wff in which the name a occurs one or more times, and use $\alpha(\underline{b}//\underline{a})$ to stand for any wff resulting from $\alpha(\underline{a})$ by replacing some (but not necessarily all) occurrences of $\underline{a}$ in $\alpha(\underline{a})$ with the name $\underline{b}$. (See Figure 1 for a table of tree rules.)

A tree is finished when all its paths are either *closed* or *saturated*. A path is closed when some formula α and its negation $\sim\alpha$ both appear on the path, or when a negated identity statement involving two occurrences of the same name occurs on the path (as indicated above under 'Identity'). A path is saturated when (i) all possible applications of one-time rules have been made (one-time rules being all those above except for the rule for unnegated universal quantifier formulas and the SI rule, which can be applied multiple times on a single input), (ii) the unnegated universal quantifier rule has been applied to all formulas to which it applies using all the names appearing on the path and (iii) all possible applications of the SI rule which result in atomic[17] formulas new to the path have been made.

We now turn to explaining the three new rules.

Adverb Compounding. If α and β are atomic wffs which are the same except for adverbs, we may write down the wff γ that results from adding to α, at the end of each of its place-fillers, all adverbs that appear in the corresponding place-filler of β (in the order that they appear in β).

Using Greek letter superscripts so that α^α and α^β stand for arbitrary adverbings of an adverbless atomic wff α, we use $\alpha^{\alpha\beta}$ to represent the wff that results from adding to α^α, at the end of each of its place-fillers,

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[17] Smith (2012) states (but does not always adhere to) a stronger requirement, in effect dropping the word 'atomic' — but this weaker requirement suffices.

	Disjunction	
$(α ∨ β)$ ✓ ／　＼ α　　β		$\sim(α ∨ β)$ ✓ $\sim α$ $\sim β$
	Conjunction	
$(α ∧ β)$ ✓ α β		$\sim(α ∧ β)$ ✓ ／　＼ $\sim α$　　$\sim β$
	Conditional	
$(α ⊃ β)$ ✓ ／　＼ $\sim α$　　β		$\sim(α ⊃ β)$ ✓ α $\sim β$
	Biconditional	
$(α ≡ β)$ ✓ ／　＼ α　　$\sim α$ β　　$\sim β$		$\sim(α ≡ β)$ ✓ ／　＼ α　　$\sim α$ $\sim β$　　β
	Negation	
	$\sim\sim α$ ✓ α	
	Existential quantifier	
$∃\underline{x}α(\underline{x})$ ✓$\underline{a}$ (new $\underline{a}$) $α(\underline{a}/\underline{x})$		$\sim∃\underline{x}α(\underline{x})$ ✓$\underline{a}$ $∀\underline{x}\sim α(\underline{x})$
	Universal quantifier	
$∀\underline{x}α(\underline{x}) \setminus \underline{a}$ (any $\underline{a}$) $α(\underline{a}/\underline{x})$		$\sim∀\underline{x}α(\underline{x})$ ✓$\underline{a}$ $∃\underline{x}\sim α(\underline{x})$
	Identity	
Closure rule: $\sim\underline{a} = \underline{a}$ ✗	Substitution of Identicals (SI):	$α(\underline{a})$ $\underline{a} = \underline{b}$ (or $\underline{b} = \underline{a}$) $α(\underline{b}//\underline{a})$

Figure 1.

all adverbs that appear in the corresponding place-filler of $α^β$. We can now summarise Adverb Compounding:

$$\frac{α^α}{α^β}{α^{αβ}}$$

For example, if we have $Pa^{\text{rssr}}b^{\text{r}}$ and $Pa^{\text{tu}}b^{\text{tt}}$, we may write: $Pa^{\text{rssrtu}}b^{\text{rtt}}$, or, taking the two inputs in the opposite order: $Pa^{\text{turssr}}b^{\text{ttr}}$. But note that we may not write: $Pa^{\text{rssrtu}}b^{\text{r}}$ since then we have not added *all* the adverbs from the second formula to the first — in particular, the $^{\text{t}}$'s contained in the second term of the second formula have been lost. While we could relax this, the present rule makes it easier to check trees for finishedness and correctness.

Note also that we may not write: $Pa^{\text{rstusr}}b^{\text{trt}}$ since then, even though no adverbs were lost, they have been mixed together in their order. We could instead have a rule which permits this, but again, the present rule facilitates checking.

Note that Adverb Compounding is not a one-time rule and that we do not tick off its inputs. (The reason for this is explained below when we discuss saturating open paths.)

Closing Paths. Close a path whenever it contains two wffs α and $\sim\beta$ such that (i) α and β are atomic and are the same except for adverbs, and (ii) every adverb that occurs in β's nth place (for any n) also occurs in α's nth place.

For example, a path closes if the following both appear on it:

$Pa^{\text{rstusr}}b^{\text{rtt}}$ and $\sim Pa^{\text{rsr}}b^{\text{trt}}$

Saturating Open Paths. With the addition of Adverb Compounding comes the need for a condition which a path must satisfy in order to be saturated, since Adverb Compounding cannot be a one-time rule. The reason we cannot simply tick off and forget about the inputs to an application of Adverb Compounding has to do with branching. Consider this valid argument:

Fa^{q}
$Fa^{\text{r}} \vee Fa^{\text{s}}$
$\therefore Fa^{\text{qr}} \vee Fa^{\text{qs}}$

If we permitted the ticking off of Adverb Compounding's inputs, a tree for this argument would play out as in Figure 2.

The other path needs to close, but the move we made to close the left path has prevented us from making the analogous move on the right: the needed $^{\text{q}}$ in Fa^{q} has been lost.

Since we need to allow inputs to be reused, we need a new rule for the saturation of open paths. We cannot use exactly the strategy used in the case of Substitution of Identicals — namely, that all applications

$$\text{F}a^\text{q}\checkmark$$
$$\text{F}a^\text{r} \vee \text{F}a^\text{s}\ \checkmark$$
$$\sim(\text{F}a^\text{qr} \vee \text{F}a^\text{qs})\ \checkmark$$
$$\sim\text{F}a^\text{qr}$$
$$\sim\text{F}a^\text{qs}$$
$$/ \qquad \backslash$$
$$\text{F}a^\text{r}\checkmark \qquad \text{F}a^\text{s}$$
$$\text{F}a^\text{qr} \qquad ...$$
$$\times$$

Figure 2.

which would lead to atomic formulas *new* to the path in question need to be made before the path is saturated — since one could just keep getting wffs with more and more adverb occurrences. We need only be interested in formulas that are new *in a certain respect*: namely, in respect of which adverbs occur at all in which places.

Hence: if the only remaining possible applications of Adverb Compounding result in wffs that involve the same underlying adverbless wff and have the same adverbs occurring in the same places as wffs already on the path, then all the necessary applications have been made.[18]

That concludes the exposition of the three new rules governing the construction of trees. Next, we extend the procedure for reading off a model, give two example proofs of validity, and show the soundness and completeness of the system.

Reading Off a Model

We extend the following procedure for reading off a model of an FOL fragment from a saturated open path:

1. Count how many different names occur on the path and put that many things in the domain.
2. Assign a unique referent from the domain to each name occurring on the path.
3. Perform any necessary "trimming": for each atomic identity wff on the path involving two different names, make those two names refer to the same object and remove the now unreferred-to object from the domain.

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[18] Terminological note: to say that $\text{F}a^\text{qs}$ and $\text{F}a^\text{sq}$ have the same adverbs occurring in the same places may sound false given an intuitive understanding of 'places', but we mean 'places' in the sense in which n-place atomic wffs have n places.

1.	$\sim(Fa^i \supset Gb)$	
2.	$\sim\sim(Fa \supset Gb^j)$	
3.	Fa^i	(1, neg. $\supset$ elim.)
4.	$\sim Gb$	(1, neg. $\supset$ elim.)
5.	$Fa \supset Gb^j$	(2, $\sim\sim$ elim.)

$$\diagup \qquad \diagdown$$

6.	$\sim Fa \qquad Gb^j$	(5, $\supset$ elim.)
	X $\qquad$ X	

Figure 3.

4. Populate the extensions of the predicates: for each atomic wff on the path (apart from identity wffs), add the corresponding tuple to the extension of the wff's predicate. (The tuple corresponding to *Pabc*, for example, will be $\langle a\text{'s referent}, b\text{'s referent}, c\text{'s referent}\rangle$.) If no atomic wff involving some predicate requiring an extension appears on the path, leave that predicate's extension empty.

The above remains unchanged, except now adverbs will appear among some atomic wffs — these can simply be ignored when doing the above. We add the following step:

5. Populate the extensions of the adverbs: for each adverb-containing atomic wff on the path, go through its adverb occurrences one by one and add the *corresponding triple* to the adverb's extension: the predicate involved, followed by that atomic wff's corresponding tuple, followed by the adverb's place (a number). (For example, the corresponding triple of the second occurrence of ^{r} in $Ra^r b^s c^{rs}$ is $\langle R, \langle a\text{'s}$ referent, $b\text{'s referent}, c\text{'s referent}\rangle, 3\rangle$.) If no atomic wff involving some adverb requiring an extension appears on the path, leave that adverb's extension empty.

Two Examples. In our tree system, a proof that $\alpha_1, \ldots, \alpha_n \models \beta$ takes the form of a tree, beginning with $\alpha_1, \ldots, \alpha_n$ and the negation of β, on which all paths close. Let us look at a couple of examples.

Firstly, Figure 3 is a tree proof showing that

$$\sim(Fa^i \supset Gb) \models \sim(Fa \supset Gb^j)$$

(which is not immediately obvious, insofar as it is not immediately obvious where adverbs can be added while preserving truth and where they can be subtracted).

1.	$\forall x(Rx \supset Rx^{\text{f}})$	
2.	$\sim\forall x(Rx \supset Rx^{\text{w}})$	
3.	$\sim\exists x Rx^{\text{f}}$	
4.	$\exists x \sim(Rx \supset Rx^{\text{w}})$	(2, neg. $\forall$ elim.)
5.	$\forall x \sim Rx^{\text{f}}$	(3, neg. $\exists$ elim.)
6.	$\sim(Ra \supset Ra^{\text{w}})$	(4, $\exists$ elim.)
7.	Ra	(6, neg. $\supset$ elim.)
8.	$\sim Ra^{\text{w}}$	(6, neg. $\supset$ elim.)
9.	$Ra \supset Ra^{\text{f}}$	(1, $\forall$ elim.)

$$/ \qquad \backslash$$

10.	$\sim Ra \qquad Ra^{\text{f}}$	(9, $\supset$ elim.)
11.	$\mathsf{X} \qquad \sim Ra^{\text{f}}$	(5, $\forall$ elim.)
	X	

Figure 4.

Secondly, let us translate the following natural language argument and use a tree to show its validity:

Everything which runs runs fast.
Not everything which runs runs well.
Therefore, there are fast runners.

We translate as follows:

$\forall x(Rx \supset Rx^{\text{f}})$
$\sim\forall x(Rx \supset Rx^{\text{w}})$
$\therefore \exists x Rx^{\text{f}}$

And Figure 4 is a proof of its validity.

C. Soundness and Completeness

We sketch proofs of soundness and completeness with respect to unsatisfiability. (Soundness and completeness with respect to validity follows easily.)

Soundness. The claim is:

(S) If there is a tree beginning with wffs α_1, ..., α_n on which all paths close, then there is no model in which α_1, ..., α_n are all true.

Taking and manipulating the contrapositive gives us:

(SC) If there is a model in which α_1, ..., α_n are all true, then every tree beginning with α_1, ..., α_n has at least one open path.

We then show by induction something which implies (SC), namely:

(SC′) If there is a model in which α_1, ..., α_n are all true, then every tree beginning with α_1, ..., α_n has at least one *satisfiable* path.

(A path is *satisfiable* iff the set of wffs on it is satisfiable.) (SC′) implies (SC) since a satisfiable path must be open; a path can only be open or closed, and no closed path is satisfiable: we only close a path when it contains some wff which cannot be true, or some pair of wffs which cannot be true together.

Assume for conditional proof that there is a model in which α_1, ..., α_n are all true. On this basis, we show by induction on the number of (tree-growing) rule applications made on a tree that every tree beginning with α_1, ..., α_n has at least one satisfiable path.

Base case. When we have made 0 rule applications, our tree has one path containing just α_1, ..., α_n, and we have assumed that there is a model on which α_1, ..., α_n are all true.

Induction step. If all trees beginning with α_1, ..., α_n on which $0...n$ rule applications have been made have a satisfiable path, then all trees beginning with α_1, ..., α_n on which $n+1$ rule applications have been made have a satisfiable path.

We consider an arbitrary tree on which $n+1$ rule applications have been made, and consider each rule, showing that if it was the last-applied rule, then our tree must have a satisfiable path. In other words, we show that applying a rule to a tree with a satisfiable path always results in a tree with a satisfiable path.

That this holds for the FOL rules is established in (Smith, 2012, p. 363) and the reasoning is straightforward. Here we consider Adverb Compounding, the one extra tree-growing rule we have. The reasoning is straightforward in this case too.

By our inductive hypothesis, our tree had a satisfiable path p before the application of Adverb Compounding. Adverb Compounding takes two wffs α^α and α^β on a path as input and gives as output a third wff, $\alpha^{\alpha\beta}$, added to any open paths on which the two inputs lie. If the inputs were not on p, then p remains unchanged and so our tree still has a satisfiable path. If the inputs were on p, then we extended p by adding $\alpha^{\alpha\beta}$ to it. But then our extended path must be satisfiable, since the set of wffs on p is satisfiable and $\alpha^{\alpha\beta}$ must be true in any model $\mathfrak{M}$ in

which α^α and α^β are both true. (Recall, α^α and α^β are atomic wffs which are the same except for adverbs, and so this is just a special case of Compounding.)

Completeness. The claim is:

(C) If there is no model in which the wffs α_1, ..., α_n are all true, then there is a tree beginning with α_1, ..., α_n in which all paths close.

(We note that a tree in which all paths close must be finite. This, which is important since a tree must be finite in order to count as a proof, is ensured by two facts: (i) all FOL-BA trees containing infinitely many wffs, since they only ever split into finitely many branches at any point, must contain an infinitely long path (an application of König's lemma (see König, 1927; Kleene, 1967/2002, p. 302) (ii) all closed paths must be finite, since at any stage of constructing a tree there can only be finitely many wffs on a path,[19] and whenever we add to a path upon applying a rule, we only add finitely many wffs and we must check for closure before making any further rule applications.)

Taking and manipulating the contrapositive gives us:

(CC) If all trees beginning with α_1, ..., α_n have at least one open path, then there is a model in which α_1, ..., α_n are all true.

We assume for conditional proof that all trees beginning with α_1, ..., α_n have at least one open path. This implies that all such *finished* trees have at least one open path. Take such a tree[20] and call one of its open

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[19] Abstractly, there do exist "trees" with what might be called 'dense paths' — paths containing two wffs that are separated by an infinite number of wffs — but we will just stipulate that these do not count as trees in our proof system.

[20] There must be such a tree, since any tree can be finished, i.e., for any initial finite list of wffs, there *is* a finished tree which begins with that list. This is guaranteed by the existence of tree building procedures which fully mechanize the process of constructing a tree in such a way that all paths must either close or be saturated by following the process (which may take infinitely many steps). (For two such procedures see (Smith, 2012, p. 236) and (Smullyan, 1968, p. 59).) Such procedures are easily adapted to our system: the extra rule Adverb Compounding never leads to new names appearing on a path and may for instance be prioritised lower than propositional and negated quantifier rules but higher than unnegated quantifier rules when cycling through the wffs on a path looking for rules to apply. Importantly, the tree which results from the application of such a procedure must count as a tree, i.e., must not have 'dense paths' in the sense of the previous footnote, since there are only enumerably many names in the language, they are gone through in a specified order, and each time a new name is introduced only finitely many wffs need to be added before the path closes or we move on to the next name. The result is that only enumerably many wffs need to be added to a path in order to saturate it, and each

paths — which must be saturated, since the tree is finished — p. Now, we show by induction on complexity of formulas that the model $\mathfrak{M}$ we read off p[21] makes every wff on p true, including those with which the tree begins, and hence there is indeed a model in which α_1, ..., α_n are all true.

Base case. Wffs of complexity 0 that may appear on p fall into three categories: (i) adverbless wffs of the form $\underline{P}^n\underline{p}_1\ldots\underline{p}_n$, (ii) adverbless wffs of the form $\underline{a} = \underline{b}$, and (iii) adverb-containing atomic wffs (which may involve an ordinary predicate or the identity predicate). All are made true on $\mathfrak{M}$ by construction, i.e., by the rules for reading off a model. (Wffs of category (i) are made true because we put the relevant n-tuple in the extension of the relevant predicate, wffs of category (ii) are made true because we ensure that the names (if they are distinct) have the same referent, and wffs of category (iii) are made true because we make their adverbless base wffs true in one of the above ways, and put the relevant triples in the extensions of the relevant adverbs.)

Induction step. If all wffs of complexity 0, . . . , n that appear on p are true on $\mathfrak{M}$, then all wffs of complexity $n+1$ that appear on p are true on $\mathfrak{M}$.

We need to consider a case for each operator. The reasoning for $\forall$, $\exists$, $\vee$, $\wedge$, $\supset$ and $\equiv$ is standard and may be found in (Smith, 2012, pp. 364–367). Likewise for $\sim$, which is broken up into subcases for different forms of negand, but we need to consider the new case where the negand is an adverb-containing atomic wff.

We want to show that if $\sim\alpha$ appears on p, and α is an adverb-containing atomic wff, then $\sim\alpha$ is true on $\mathfrak{M}$. Now, α itself cannot appear on p, since then p would close (by the ordinary "α and $\sim\alpha$" closure rule). Nor could a wff just like α but with some more adverbs added appear on p, since in that case p would also close (by our new closure rule). But then α must be false on $\mathfrak{M}$, since, when reading a model off a saturated open path, we only put the necessary n-tuple into the extension of α's predicate, and the triples necessary to make α true into the extensions

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one appears some finite point along the path. I only sketch this reasoning because these somewhat subtle issues arise already with FOL tree systems, which are known to be complete.

   [21] This talk of 'the model $\mathfrak{M}$ that we read off a saturated open path p' should be taken with a grain of salt, since in the case of an infinite open path we may be unable to complete the reading off. We really just mean the model $\mathfrak{M}$ that corresponds to a finished open path via the reading-off procedure, whether or not that procedure can actually be carried out completely.

of the adverbs occurring in α, if α or some wff just like α but with some more adverbs added appears on that path. (The compounding rule, together with the new saturation rule, ensures that the necessary triples will never merely come piecemeal, some from one formula and some from another.[22]) Since α is false on $\mathfrak{M}$, $\sim\alpha$ is true on $\mathfrak{M}$.

We have now seen how a standard tree system for FOL and associated soundness and completeness results can be extended to FOL-BA.

D. Translating Between FOL-BA and FOL

In this section we show that FOL-BA wffs can be translated into FOL and back again. We can define a mapping from FOL-BA wffs $\alpha^{\text{FOL-BA}}$ to FOL wffs α^{FOL}, and a mapping from FOL-BA models $\text{M}^{\text{FOL-BA}}$ to FOL models M^{FOL}, such that α^{FOL} is true on M^{FOL} iff $\alpha^{\text{FOL-BA}}$ is true on $\text{M}^{\text{FOL-BA}}$. Furthermore, we can show that α^{FOL} will be true on all FOL models iff $\alpha^{\text{FOL-BA}}$ is true on all FOL-BA models. This enables us to extend many metatheoretic results established for FOL to the case of FOL-BA.

First let us see how to translate a FOL-BA wff and how to specify a FOL model that corresponds to an arbitrary FOL-BA model in the above-described way.

Let us take the following atomic wff of FOL-BA as an example:

$Ra^{\text{pqqr}}b^{\text{rpq}}$

First, we write the wff without the adverbs:

Rab

Call this the *base conjunct.* Then, for every occurrence of an adverb, we conjoin another atomic wff — an *adverb-occurrence conjunct* — whose predicate is written as the predicate in the original FOL-BA wff followed by a numeral (not super- or sub-scripted) corresponding to the place in which the adverb occurrence appears, and whose arity is one higher than the original FOL-BA predicate. We copy the terms over from the base conjunct and add one more: a non-italicized name which looks like the

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[22] I say 'merely' because, depending on the order of formulas on the path and the order in which they are considered, the necessary triples may come piecemeal in a particular reading off, but then there will always be a single formula whose contribution alone would have made α true if it had been examined first.

adverb occurrence except not a superscript — call it an *adverb name.* So after adding our first adverb-occurrence conjunct, we get:

$(Rab \wedge R_1ab\text{p})$

(Note that we can distinguish predicates used for adverb-occurrence conjuncts by the fact that they contain full-size numerals. Similarly, we can distinguish adverb names from other names by the fact that they are not written in italics.)

Completing the translation, we get:

$(Rab \wedge R_1ab\text{p} \wedge R_1ab\text{q} \wedge R_1ab\text{q} \wedge R_2ab\text{r} \wedge R_2ab\text{p} \wedge R_1ab\text{q})$

An atomic wff of FOL-BA involving the identity predicate gets translated in exactly the same way. A non-atomic wff of FOL-BA gets translated by replacing each atomic wff component with that component's translation into FOL. (In the case of an adverbless atomic wff of FOL-BA the translation is homophonic, i.e., is just the same wff.)

It is obvious from the foregoing procedure that the translation is reversible; from a proper FOL translation of an FOL-BA wff we can recover the FOL-BA wff it is a translation of.

Note that the first-order language we are using here contains the adverb-free fragment of FOL-BA. Added to the alphabet needed to generate this fragment, we have, for each n-place predicate $\underline{P}$, n further $n{+}1$-place predicates $\underline{P}_1, \ldots, \underline{P}_n$ (for use in adverb-occurrence conjuncts), and a series of non-italicized names corresponding to adverbs. It is easily verified that this extended language remains an enumerable, first-order language.

Now we turn to specifying the FOL model $\mathfrak{M}^{\text{FOL}}$ that corresponds to an arbitrary FOL-BA model $\mathfrak{M}^{\text{FOL-BA}}$. We begin by taking over everything from $\mathfrak{M}^{\text{FOL-BA}}$ except for adverbs' extensions. It just remains to describe how we determine referents for our new names and extensions for our new predicates. We add a second domain (or 'sort') A to our model containing our new names, and assign each new name to itself as referent. (Since we won't need to quantify over members of A, we won't need sort symbols or new variables: any variables we write down continue to range over the original domain D only. For the sake of agreement with standard presentations of many-sorted logics, we may regard such symbols as available but unused. In any case, our translation of FOL-BA is a translation into many-sorted FOL — in particular, two-sorted FOL. Using standard techniques (for explanation, see Various, 2017, §11.2).

for translating many-sorted FOL into ordinary FOL, we can if we wish take that further step.)

We determine the extensions of our new predicates as follows. (Recall that for each new $n+1$-place predicate $\underline{P\#}$, there is a corresponding old predicate $\underline{P}$, and that for each new name $\underline{n}$ there is a corresponding adverb which has an extension in our original model $\mathfrak{M}^{\text{FOL-BA}}$ — call it $\underline{n}$'s adverb.)

We put an $n+1$-tuple, call it *Candidate*, in the extension of an arbitrary new $n+1$-place predicate $\underline{P\#}$ iff:

i. The n-tuple obtained by removing the last element of *Candidate* — call it *the base tuple* — is in the extension of $\underline{P}$.

ii. The last ($n+1$th) element of *Candidate* is a new name $\underline{n}$ whose adverb's extension in $\mathfrak{M}^{\text{FOL-BA}}$ contains the triple $\langle \underline{P}$, the base tuple, $\#\rangle$ (where $\#$ is the number designated by the numeral in $\underline{P\#}$).

It is clear from this construction that the translation α^{FOL} of a closed atomic wff $\alpha^{\text{FOL-BA}}$ is true on the constructed model $\mathfrak{M}^{\text{FOL}}$ iff $\alpha^{\text{FOL-BA}}$ is true on the model $\mathfrak{M}^{\text{FOL-BA}}$ used for the construction. A straightforward induction on complexity of formulas shows this is the case for the translation α^{FOL} of any closed wff $\alpha^{\text{FOL-BA}}$.

This does not automatically mean that a wff $\alpha^{\text{FOL-BA}}$ is true on all models iff its translation α^{FOL} is true on all models, since the translation α^{FOL} of a FOL-BA logical truth could in principle be true in all FOL models that can arise from translation, and yet false in some other FOL models. Indeed, this would happen in the case of a simpler scheme in which each adverb-containing atomic wff of FOL-BA gets translated into a single atomic FOL wff using a special predicate devised just for that FOL-BA predicate plus some particular addition of adverbs. Atomic wffs of FOL-BA that are not logically independent from one another, e.g., Fa^{b} and Fa^{bc}, turn into logically independent FOL wffs on this simpler scheme, so the translation of a FOL-BA logical truth like (Fa^{bc} $\supset Fa^{\text{b}}$) would be false on some models. Our scheme, however, is up to the job. To verify this, we make a backward road from FOL to FOL-BA models — a procedure which, given a wff $\alpha^{\text{FOL-BA}}$, its translation α^{FOL}, and an arbitrary two-sorted model $\mathfrak{M}^{\text{FOL}}$,[23] gives us a model $\mathfrak{M}^{\text{FOL-BA}}$ such that $\alpha^{\text{FOL-BA}}$ is true on $\mathfrak{M}^{\text{FOL-BA}}$ iff α^{FOL} is true on $\mathfrak{M}^{\text{FOL}}$.

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[23] I.e., an arbitrary two-sorted model $\mathfrak{M}^{\text{FOL}}$ where the old, italicized names and variables refer to and range over members of one sort D, and where the new, unitalicized names refer to members of another sort A.

We begin by taking over everything from $\mathfrak{M}^{\text{FOL}}$ except for its second domain A, any adverb names, and any special predicates used in adverb-occurrence conjuncts. Then, for every adverb name $\underline{a}$ that appears in α^{FOL}, we give an extension in $\mathfrak{M}^{\text{FOL-BA}}$ to $\underline{a}$'s adverb $\underline{\underline{a}}$, determined as follows. We put a triple $\langle \underline{P}, \langle 1, \ldots, n \rangle, \# \rangle$ consisting of a predicate $\underline{P}$, followed by an n-tuple, followed by a number — call such a triple *Candidate* — in the extension of $\underline{\underline{a}}$ iff:

i. $\langle 1, \ldots, n \rangle$, $\underline{a}$'s referent in $\mathfrak{M}^{\text{FOL}} \rangle$ is in the extension of $\underline{P}\#$ in $\mathfrak{M}^{\text{FOL}}$;
ii. $\langle 1, \ldots, n \rangle$ is in the extension of $\underline{P}$ in $\mathfrak{M}^{\text{FOL}}$.

It is clear from the construction that a closed atomic wff $\alpha^{\text{FOL-BA}}$ is true on $\mathfrak{M}^{\text{FOL-BA}}$ iff its translation α^{FOL} is true on $\mathfrak{M}^{\text{FOL}}$. A straightforward induction on complexity of formulas shows that this is also true of any closed wff $\alpha^{\text{FOL-BA}}$.

Putting the results of these two procedures together, we can see that a wff $\alpha^{\text{FOL-BA}}$ is true in all FOL-BA models iff its translation α^{FOL} is true in all two-sorted FOL models.

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