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Future Contingencies and the Arrow and Flow of Time in a Non-Deterministic World According to the Temporal-Modal System TM

To Nuel Belnap

Abstract. It is shown how the temporal-modal system of events TM (axiomatized in Appendix) allows for the avoidance of the logical determinism without the rejection of the principle of bivalence. The point is that the temporal and the modal parts of TM are so inter-related that modalities are in-the-real-world-inherent modalities independently of whether they concern actual or only possible events. Though formulated in a tenseless language, whose interpretation does not require the assumption of tense facts at the basic level of reality, TM implies an objective, observer-independent difference between tenses based only on the way in which modalities are distributed along the time continuum. The conclusion is that the arrow of time is an intra-model characteristic of any model of TM that describes the non-deterministic real world up to a certain point of its history, while the flow of time is an inter-model characteristic of the continuous transition between these models.

Keywords: sea-battle puzzle; tenses; in-the-world-inherent modalities; indeterminism; the arrow of time; the flow of time

1. Introduction

Traditionally, time has been represented as an infinite one-dimensional continuum. But this representation is obviously not sufficient. A straight line is also an infinite one-dimensional continuum, but it is not the time continuum.

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Once upon a time, Sir Isaac Newton believed that the main characteristic of time as continuum is that it has an intrinsic direction which it would have, as a part of God's *sensorium*, even if there were no world (Newton 1718, p. 379). Kant was of the same opinion, only that for him the fact that the direction of time is an intrinsic property of time was a consequence of its being a part of our and not of God's *sensorium* (Kant 1956, pp. 63–93). However, in twentieth century Wheeler and Feynman (1949) argued that nothing would be lost in the physicists' description of the world if the relations *earlier than* and *later than* were systematically interchanged, and Grünbaum (1967) concluded that, after all, talk about direction of time must be mind-dependent insofar as it requires a choice between the two alternatives. In this paper, we shall not assume, as Newton and Kant did, that the direction of time—or the *arrow* of time, as it is sometimes called—is something given *a priori*, but we shall argue that we should accept that it is something *objective* and *mind-independent*, at least in a world that is not completely deterministic. This will be shown to be a consequence of the analysis of various problems that concern the relation between *time* and *modality*, on the basis of which we have formulated the system of temporal-modal logic of events **TM**. The whole syntax of **TM** is given in Appendix, where, in addition to the axioms, the most important theorems are listed and briefly commented upon.

The *flow of time* is another characteristic of time whose objectivity we are going to investigate. It should not be confused with the arrow of time, since the flow of time depends on the reality of difference between *tenses*. Russell was one of the first to claim clearly that "in a world in which there was no experience there would be no past, present, or future, but there might well be earlier and later" (1915, p. 212). That is why the majority of physicists believe that the arrow of time is an objective feature of the world worth studying, while the flow of time is only an illusion of creatures with cognitive capacities like ours (Eddington 1920, p. 51; Weyl 1949, p. 116; Einstein 1949, p. 537; Davies 1974, p. 3). Among philosophers, this is the view of *moderate detensers*, who defend the *tenseless theory of time* but admit the objectivity of the direction of time (Smart 1955; Reichenbach 1956; Mellor 1981; Oaklander 1991). However, it will be shown that the system **TM** implies not only the objectivity of the arrow of time but also the objectivity of the flow of time, which means that the very possibility of *indeterminism* requires that the *tense theory of time* is true, according to which the difference between tenses is *real* and *mind-independent*, regardless of whether *tense facts* are postulated at the basic level of reality or not. So, though Einstein did not have to care about it, because he was a hardcore determinist, this result is significant in view of the predominant interpretations of quantum mechanics.

In Section 2, the main problem concerning the relation between time and modality—called by Schlick (1931, p. 202) the *problem of logical determinism*—will be presented in the form in which it was originally formulated and solved by Aristotle through the *restriction* of what was much later called the *principle of bivalence*. Then, it will be shown how Ockham tried to solve the problem when dealing with *predestination* and what his failure in doing it by analyzing modalities consists in. Finally, it will be shown that it was only Łukasiewicz who clearly realized why the problem, as originally formulated, must be understood as directed against the *principle of bivalence*, which led him to *reject* it and to introduce his three-valued logic instead.

In Section 3, it will be suggested how, bearing in mind the results obtained in Section 2, one could try to resolve the main problem concerning the relation between time and modality without restricting or rejecting the principle of bivalence. The point is that this can be done only within a system of *temporal-modal* logic in which the temporal and the modal parts of the system are essentially inter-related so that modalities become i*n-the-real-world-inherent* modalities, while possible worlds do not have their own times but share with actual worlds one and the same time continuum. Following this idea, the main metaphysical and meta-logical assumptions underlying the construction of the intended system **TM** will be discussed.

In Section 4, the *semantics* of **TM** will be informally explained, which will involve the analysis of the specific meanings of *elementary events* and *temporal* and *modal operators*, as well as of the way in which *possible worlds* can be distinguished without reifying them.

In Section 5, we shall reveal the philosophical appeal of **TM** by assessing its assets in dealing with some central questions of the philosophy of time and modality.

And finally, in Section 6, we shall deal with the intended models of **TM**, which will enable us to speak about the *arrow* and the *flow* of time. It will be shown why the *real world up to a certain instant of its history viewed as if it ended at that instant* represents a *model* from a class of isomorphic models of **TM**, which can be therefore called the *privileged model*. And then, while the real world is identical with its own history in any privileged model, the *development* of the real world history represents the *transition*, which will be proved to be continuous, from one given privileged model to other models non-isomorphic to it. Finally, it will be shown that the *arrow* of time is an objective *intra*-model characteristic of time that can be read off from the distribution of modalities in a given model, while the *flow* of time is an objective *inter*-model relation that characterizes the *transition* from one stage of the world history toward the others.

2. Logical determinism

2.1. Aristotle's 'sea battle'

The history of our problem begins with Aristotle's famous *'sea battle' puzzle*, in which the difference between *pastness* and *futurity* plays the central role. If $\mathtt{e}(\mathtt{t}_n)$ is the statement that some event $\mathtt{e}$, say the sea battle, happens at time $\mathtt{t}_n$, the question is if $\mathtt{e}(\mathtt{t}_n)$ *was* true already at some earlier time $\mathtt{t}_m$, given that the sea battle really happens at $\mathtt{t}_n$. Or, similarly, *was* $\neg\mathtt{e}(\mathtt{t}_n)$ true already at $\mathtt{t}_m$ if the battle does not happen at $\mathtt{t}_n$? The problem is that if it is true at $\mathtt{t}_m$ that the sea battle will happen at $\mathtt{t}_n$ or that it will not happen, what is going to happen at $\mathtt{t}_n$ is *predetermined* at $\mathtt{t}_m$, so that it makes no sense to say at $\mathtt{t}_m$ that it is possible both that $\mathtt{e}(\mathtt{t}_n)$ and that $\neg\mathtt{e}(\mathtt{t}_n)$.

In order to make place for indeterminism Aristotle distinguishes between two sorts of necessity, the conditional and the unconditional one. "What is, necessarily is, when it is; and what is not, necessarily is not, when it is not" (*De interpretatione* 19 a 23). This represents what medieval logicians called *necessitas per accidens*. Once the sea battle happened, it has become necessary that it did. "But not everything that is, necessarily is; and not everything that is not, necessarily is not. For to say that everything that is, is of necessity, when it is, is not the same as saying unconditionally that it is of necessity. Similarly with what is not" (*loc. cit.*). This means that the sea battle could have happened even if it did not happen, and that it could have not happened even if it happened, and, in particular, it means that before $\mathtt{t}_n$ it was possible that it would happen at $\mathtt{t}_n$ as well that it would not happen. "And the same account holds for contradictories: *everything necessarily is or is not, and will be or will not be*" (*loc. cit.; italics added*). This means that Aristotle did not restrict the general validity of the principle of excluded middle: $\Box(p \vee \neg p)$ holds always, and so also when $\mathsf{e}(\mathsf{t}_n)$ is substituted for p and stated at some time earlier than t_n.

"But one cannot divide and say that one or the other is necessary" (*loc. cit.*), namely, that either $\Box\mathsf{e}(\mathsf{t}_n)$ or $\Box\neg\mathsf{e}(\mathsf{t}_n)$ is true. However, a strange thing follows. Not only that at t_m neither $\Box\mathsf{e}(\mathsf{t}_n)$ nor $\Box\neg\mathsf{e}(\mathsf{t}_n)$ is true, but this must hold for $\mathsf{e}(\mathsf{t}_n)$ and $\neg\mathsf{e}(\mathsf{t}_n)$ themselves: "[...] It is necessary that *one part* (ϑάτερονμόριον) of the contradiction (ἀντίφασις) is true or false — not, however, *this one* or *that one* (τόδε ἢ τόδε), but as chance has it; or for one to be true rather than the other, yet not *already true or false* (ἤδη ἀληϑῆἢψ ευδῆ). Clearly, then, it is *not necessary* that *of every* affirmation and opposite negation *one should be true and the other false*" (*loc. cit.; italics added*). So, curiously enough, the father of traditional logic solved the *'sea battle' puzzle* in a non-standard way! First, he restricted the general validity of the *principle of bivalence* by allowing *truth-value gaps*, for he admitted that there are cases in which neither the affirmation of a proposition nor its negation is either true or false. But, at the same time, he did not restrict the general validity of the *principle of excluded middle*, because it is necessary for $p \vee \neg p$ to be true *independently* of whether it is p that will turn out true or whether it is $\neg p$. So, the complex proposition $p \vee \neg p$ can be stated as true even if neither of the component propositions is yet either true of false.

2.2. Ockham on the problem of salvation and predestination

When dealing with the problem concerning the incompatibility between predestination and deserving salvation, which is essentially similar to the *'sea battle' puzzle*, William of Ockham could not apply Aristotle's solution to it. For the existence of truth-value gaps is in "opposition to the Faith", namely, "*to the pronouncements of the Saints*, who say that God *does not know* things that are *becoming* (*fienda*) in a *way different* from that in which [He knows] things that have *already occurred* (*facta*)" (Ockham 1945. q. 1, supp. VI; *italics added*). So, "God knows not only which part of a contradiction is true and which false, but He knows with certainty [regarding] *all future contingents* (*omnia future contingentia*) *which part* (*quae pars*) of the contradiction *will be true* (*erit vera*) and which false" (*loc. cit.; italics added*).

Ockham confessed that "it is impossible to express clearly the way in which God knows future contingents" (*loc. cit.*), but did his best by trying to find conditions under which the truth of p would not be predetermined by God's foreknowledge (*praescientia*) that p. The point is that it is not only p that is contingent, but it is also God's knowledge that p. For example, says Ockham, if this person will be saved, 'God knows that this person will be saved' is true and yet it is possible that He *will never have known* that this person will be saved. Namely, 'God knows that this person will be saved' is true *if and only if* 'this person will be saved' is true, but, according to Ockham, it is not that 'this person will be saved' is true *because* 'God knows that this person will be saved' is true, but *the other way round*.

Ockham's solution is based on the general principle that, if p will be true at $\mathtt{t}_n$, God must be able to know that it will be so at any time $\mathtt{t}_m$ earlier than $\mathtt{t}_n$. But then, since *knowledge* implies *truth* ($Kp \to p$), it must be true at $\mathtt{t}_m$ that p will be true at $\mathtt{t}_n$, which, contrary to the intended solution, predetermines at $\mathtt{t}_m$ the truth of p, precluding the possibility of $\neg p$.

Now, there are contemporary philosophers who call themselves Ockhamist Indeterminists because (without entering the debate about whether divine foreknowledge is possible if indeterminism is true or not) they subscribe to the view that "a genuinely future-tense statement whose embedded clause is contingent" may be true, even though it is about something supposedly presently unknowable and not now-unpreventable (Rosenkranz 2012, p. 618). According to this version of Ockhamism, just as 'One mile to the north from here, p' is true *here*, if p is true *there*, so 'One day hence, p' is true *at present*, if p is true *one day hence*. Without any causal connection between the two respective events which p is about, "some truths are truths by courtesy of other truths" (Rosenkranz 2012, p. 625). So, contrary to those who think that due to the unreality of future all future contingents are false (Tod 2016), those who, like Aristotle, accept the existence of truth-value gaps (see also van Fraassen 1966) so that no future contingent is now either true or false (Thomason 1970, p. 272), and those who claim that due to the time-branching in the non-deterministic universe we need to relativize utterance-truth to a context of assessment and sentence-truth to both a context of utterance and a context of assessment (MacFarlane 2003), Rozenkranz claims that genuine future-tense statements are grounded in future reality and are either true simpliciter or false simpliciter. We disagree. But, as we shall see below, there is something appealing in the Ockhamist view—as well as in some other views just mentioned—which, if properly reinterpreted, can be incorporated into our system **TM**.

2.3. Łukasiewicz's three-valued logic

Curiously enough, Łukasiewicz was the first in the history of logic to recognize that the *'sea battle' puzzle* was intended to question one of the basic principles of our entire logic. Which one? It cannot be the *principle of contradiction*, because, for Aristotle, this principle is the most general and unrestrictedly valid principle. It cannot be the *principle of excluded middle*, for, as we saw in Section 2.1, its validity is explicitly confirmed by Aristotle in the case of the future sea battle. Expectedly or unexpectedly, it is the *principle of bivalence*, which, though practically operative throughout the history of logic, got its name and explicit formulation only in Łukasiewicz's *Rector's Speech* in 1922 (Łukasiewicz 1922, p. 126), when he contrasted it with the principle of his three-valued logic system, mentioned in Łukasiewicz 1918 and outlined in Łukasiewicz 1920. The reason for such a late recognition of it lies either in its conflation with the *principle of excluded middle* or in its reducibility to the other two principles. For if it holds, for *any* proposition, that the conjunction of it and its negation is always false, whereas the disjunction of it and its negation is always true, then, given the standard way in which $\wedge$, $\vee$ and $\neg$ are defined, it is derivable that *every* proposition must have one and only one of the two truth values, which is exactly what the *principle of bivalence* claims. One way or another, the point of the *'sea battle' puzzle* had been lost for more than 2000 years.

Both Aristotle, implicitly, and Łukasiewicz, explicitly, solved the problem of *logical determinism* by attacking the *principle of bivalence.* Aristotle restricted its general validity through the introduction of truth-value gaps, while Łukasiewicz rejected it in his system of three-valued logic. Łukasiewicz's system contains, in addition to *truth* and *falsity*, a third truth-value, which he calls *indeterminate*, because it is the truth-value of the propositions that are neither true, for they have no real correlate, nor false, for their denials too have no real correlate. Using philosophical terminology, which he admits not to be particularly clear, Łukasiewicz says that what corresponds to these propositions is neither being nor not-being but *possibility.* The problem with Łukasiewicz's solution is that it may sound odd to say that some *statement* is *neither true nor false* and yet *states* something. If it *states* something *possible*, isn't it more natural to say that it states something *true* or *false* about something that is possible?

3. The system of temporal-modal logic of events TM

In what follows, we shall try to do something that, in view of what has been said above, may seem impossible: to construct a consistent system, **TM**, in which *logical determinism* will be avoided and *the principle of bivalence* retained. For the reason given at the end of Section 2.3, to have such a system would be an advantage *ceteris paribus*, but, of course, we have to assess the liabilities of such an enterprise. Do the metaphysical assumptions and consequences of **TM** look intuitively acceptable, in general, and plausible in view of disputes within the philosophy of time and the philosophy of modality, in particular?

3.1. General metaphysical and meta-logical assumptions

There are some inessential assumptions that we shall only mention here. First, we shall assume that there is no temporal beginning of the history of the real world. But the system can easily be adjusted so as to be in accordance with the *Big Bang* cosmology (see Section 6.4 and Appendix). Second, as we shall see, in any model of **TM**, the only *branching points* in the history of the real world are the point at which there are indefinitely many ways in which the world history *can* develop (as in the apex of figure 1), or the points lying within the real part of the time continuum at which the real world history *could have* continued developing otherwise than it actually did, while, according to the Theory of Relativity, there are *real world line branches* with their own times (see Belnap 1992; Rakić 1997; McCabe 2005). But again, the system **TM** could be adjusted so as to make it possible to speak of more real world lines, in accordance with relativity physics. What we have to reject as not in accordance with the leading idea of **TM** is only the plurality of worlds in the sense of David Lewis's modal realism (Lewis 1986 and elsewhere).

We shall now investigate the three main assumptions necessary for the construction of **TM**.

3.1.1. In-the-world-inherent modalities

The reason why Aristotle restricted and Łukasiewicz rejected the *principle of bivalence* consists in the fact that, as Łukasiewicz put it, statements about future contingents as well as their denials have no real correlate. However, at the end of our short historical survey we raised the question whether it is perhaps more natural to say that, though it is true that they have no real correlate because they are about something that is just possible, they can still be true or false. For they can be said to be *true* at least if they say about something possible that it is possible and about something impossible that it is impossible, and *false* at least if they say about something possible that it is impossible and about something impossible that it is possible. It is to be expected that we can easily get what we want by using some appropriate system of modal logic. However, there is a big problem here.

The problem with the proposition about the *future sea battle* has not only to do with the fact that it is contingent and that it concerns something that is just possible. *Time* plays a crucial role in it. Ockham's failure to solve the problem by simply insisting on the fact that the propositions that *p* and that *God knows that p* are both contingent shows that *modal analysis* alone is not sufficient. In order to solve the problem without either restricting or rejecting the *principle of bivalence*,we need a kind of combination of *temporal* and *modal* logic. However, from the first formulation of modal logic systems by Clarence Irving Lewis to Kripke's possible worlds semantics, David Lewis's analysis of counterfactuals and quite recent disputes concerning necessitism and contingentism (Williamson 2013, pp. 1–29), time has not appeared as something that could be decisive for choosing one of the systems or endorsing this or that metaphysical position. Analogously, it is remarkable that in the dispute between those who support the tensed theory of time and those who support the tenseless theory of time modalities are either not mentioned at all or do not play any important role.[1] The disputes in philosophy of modality and philosophy of time have been running in parallel as if the outcome of any of the former could have no essential impact on the outcome of any of the latter, and vice versa. So Timothy Williamson, when considering the opposition between necessitism and contingentism and the opposition between permanentism and temporaryism, though admitting that "most necessitists will be permanentists too" and that "most temporaryists will be contingentists", says that "necessitists are not automatically permanentists, nor are temporaryists automatically contingentists" (2013, p. 4). At best, some authors claim that there is a structural similarity between arguments in the philosophy of modality and in the philosophy of time (see, for instance, Rini and Cresswell 2012, p. 6).

[1] For instance, in the MIT collection *Time, Tense, and Reference* (Jokić and Smith 2003), which contains 14 contributions, there is only one exception.

Due to such a separation between the philosophy of modality and the philosophy of time, any *temporal modal logic* has turned out to be a *combination* of a temporal logic system and a modal logic system, where both are *chosen independently* one from another for any reasons whatsoever. In view of our intention to avoid *logical determinism* by retaining the *principle of bivalence*, such a way of proceeding is hopeless. For what we need is a *temporal-modal* system [intentionally written with a dash] in which the temporal and the modal part are essentially inter-related, so that possible worlds are not considered as having their own times but, being anchored in the real world, share with it the same time continuum. There would be no problem of the *possible sea battle* if it were not supposed that it is the *future* sea battle, or better and more precisely, the *sea battle that is said to happen tomorrow.* The problem arises because *tomorrow* is an interval of the same time continuum within which real world is situated, i.e., within which the real world history has been developing. The proposition that states that the sea battle will happen tomorrow is about a *real possibility.* That is why Nuel Belnap says that "if a certain possibility is real, [...], it must be part and parcel of *Our World*", and that's why "the brilliantly conceived doctrine of Lewis 1986 (and elsewhere) ought to be rejected" (Belnap 2007, p. 87, n. 2).

So, our first metaphysical assumption will be that there is something that we shall call *in-the-world-inherent modalities* that our system of temporal-modal logic of events **TM** should enable us to speak of.

3.1.2. Quine's slogan revised

We shall take it that the set of all elementary events (to be defined shortly) that happen in sub-intervals of a time interval $\mathtt{t}_n$ makes up a *world actualized on* $\mathtt{t}_n$.[2] For the sake of simplicity, a special assumption will be that there is no sub-interval of $\mathtt{t}_n$ on which nothing happened, since we may suppose that the world never comes into complete standstill. Then, the *real world* can be said to consist of all actualized worlds. For a *time interval* on which a world is actualized we can say that it is *actual* as well. This seems intuitively plausible independently of how the ontological status of actual time intervals is further analyzed (as supervening on the real world history or in some other way). The problem is,

[2] We use "$\mathtt{e}$ happens *on* $\mathtt{t}_n$" in order to imply clearly that an event $\mathtt{e}$ *stretches over whole* $\mathtt{t}_n$.

however, what to say about *tomorrow*, which we need in order to speak about the sea battle that is to happen or not to happen tomorrow.

Given the way in which we speak of *actual* intervals, it is natural to say that the time interval to which *tomorrow* refers is *non-actual.* At the same time, however, in the interval-based continuum system that we shall use to speak of time intervals, individual variables t_1, t_2, ..., t_n, ... should range over the set of all time intervals, including those to which we refer by *tomorrow*, *the day after tomorrow*, etc. But then, according to Quine's famous slogan "To be assumed as an entity is to be reckoned as the value of a variable" (Quine 1961, p. 13), the time interval to which we refer by *tomorrow* already *exists* independently of the fact that it is *non-actual.*

Existence and *actuality*, though not identical, are two so closely related notions that common philosophical usage requires to say that *whenever* there is something that exists there is also something that is actual, and vice versa. True, this does not mean necessarily that *whatever* exists should be said to be actual, and that *whatever* is actual should be said to exist. The problem with *tomorrow* is that we have a situation in which there is something that allegedly exists while there is nothing actual. Moreover, the only thing that exists is a non-actual time interval! What to do? Fortunately, there are other, independent reasons given by more philosophers why Quine's requirement ought to be rejected (see, for instance, Fine 2009), and in order to resolve our problem, we shall simply do this.

However, there is something sound in the idea of Quine's, which becomes acceptable for us if we alter his slogan so as to be not about the *existence* but only about the *individuation* of elements of the universe of discourse. We shall start building **TM** by introducing a set of ten axioms (see Appendix) that defines implicitly the structure of the one-dimensional continuum, so that each interval is well-individuated by this set of axioms. But it would be bizarre to say that, due to the fact that by defining a one-dimensional structure we let individual variables range over the elements of the set that is one-dimensional and continuous, we are automatically committed to the existence of elements of all imaginable one-dimensional continua, be they straight line segments, time intervals or whatever else. We should rather say that *if* there are entities whose structure is one-dimensional and continuous, their elements are *well-individuated* by the given set of ten axioms (Bernays 1922, p. 95). According to Hilbert, all models of a consistent set of axioms exist, but in a trivial sense, just as possible structures. This meaning of 'existence' is intra-mathematical (Hallett 1995, pp. 37–52), and it is not to be confused with the existence we are dealing with in the given context.

With the suggested revision of Quine's slogan, we have got what we need. We may differentiate between *actual* and *non-actual* intervals without committing ourselves to the existence of something not actual. More importantly, we can speak of the actual and the non-actual part of one and the same time continuum, where the latter is an imaginary continuation of the former. So, the specificity of the *time* continuum consists in the fact that in any model of **TM** it should be represented as consisting of two parts, the *real* and the *imaginary* one, with the *apex* as the boundary between the two.

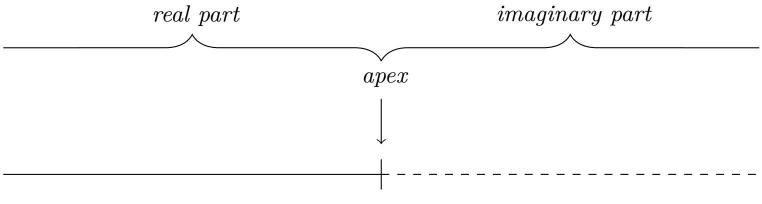

Figure 1.

As the continuation of the real part of the continuum, the imaginary part even gets the metric, without which it would be metrically amorphous. *Tomorrow* is supposedly an interval as long as *yesterday*, and for this it is not required that what was the measure yesterday must exist tomorrow. Let us remember that, analogously, physicists speak of our universe as 13.798 ± 0.037 billion years old, though the years they are speaking about are solar years extrapolated backwards in time when the Solar System did not exist.

Events that happened on intervals of the actual part of the time continuum are events of actualized worlds, while those said to happen on intervals of the non-actual part of the time continuum are events of possible worlds that only may become actual.

3.1.3. Non-locality of truths and archives of factual and modal truths

Events in the history of the real world are considered as things that happen *locally*. It is raining here but it is not raining there. It is raining today but it was not raining yesterday. However, if true, the propositions that express these local facts do not hold only locally. If it is true that it is not raining there, it is also true here that it is not raining there, and if it was not raining yesterday, it was not true only yesterday but it is also true today that it was not raining yesterday. So, given that '*proposition*' is used as a semantic term concerning an interpretation of a sentence under which it becomes either true or false, while '*truth*' is a shorthand for '*true proposition*', we can endorse the *principle of the non-locality of factual truths* in view of both *spatial* and *temporal* aspects of the real world. We may call this principle *Leibniz's principle of truth mirroring* (Leibniz 1976, §§ 56–62), but where the mirrors are not the monads of his metaphysics but segments of the spatio-temporal world.

It may be suspect whether the truth that it was raining yesterday should imply that it was true the day before yesterday that it would be raining the day after (see theorem $\mathrm{Th}_{\mathbf{TM}}15$ in Appendix). For we wish that the day before yesterday it was only possible that it would be raining the day after. However, the very use of *yesterday* means that the *day after* refers to the interval that is *already included* into the real world, so that the possibility of non-raining is *eo ipso* precluded. This possibility is a real possibility only if the *apex*, as the boundary between the actual and the imaginary part of the time continuum, precedes the time interval of which it is said that it is raining on it.

The last example is of great importance because it shows that all truths may be said to hold indistinguishably on any time segment of the real world only if by 'truths' we mean *factual* and *logical* truths. If we define the *archive of truths* as the set of all the propositions that are true on a given interval, then *only modal truths* about the possible or necessary occurrence or non-occurrence of events, which vary from one actual interval to another, can supply the criterion for discriminating actual intervals by using their archives. For instance, the truth that the day before yesterday it was *possible* that it would not be raining the day after remains preserved in the *yesterday's archive of truths*, while the truth that it is *necessary* that it was raining yesterday, because it actually was, does not and will never appear in the *day before yesterday's archive of truths*.[3]

[3] Our notion of archive of truths should not be confused with the notion of *chronicle* (or *history*, or *route* in Prior 1967). Adopting a suggestion of Kripke, in formulating his Ockhamist system Prior (1967, pp. 122–137) used branching time to represent different possible developments of the world history after some instant t (cf. Øhrstrøm and Hasle 2020): in such a representation, chronicles are maximal linearly ordered subsets of the branching (i.e. partially ordered) time structure. Hence, chron-

4. The semantics of TM

In any model of the standard axiomatization of the infinite one-dimensional continuum the elements over which individual variables range are null-dimensional entities. This means that in the case of the time-continuum, the elements are to be understood as *durationless instants* analogous to *non-extended points*. However, since **TM** is to be a logical system of *events*, we should rather take the elements of the time-continuum to be *time intervals*, because events normally happen in time intervals and only derivatively at instants of time (as in the case of the instantaneous collision of two bodies).

In Oberwolfach 1969, Hamblin gave a sketch of a logic of intervals "within the lower predicate calculus without mentioning or otherwise depending on the concept of an instant" (Hamblin 1972, p. 327). However, his axioms give a logic of linear order for intervals which is only dense (Hamblin 1972, p. 328) and not continuous. Now, contrary to Cantor's first condition for continuity, which in the interval-based system can be formulated in the standard first-order language (see axiom $\mathrm{A}_{\mathbf{T}}8$ in Appendix), for the formulation of the second condition an infinite number of elements, both in the instant-based and in the interval-based system of the continuum, must be explicitly mentioned (cf. Cantor 1962, p. 195). In order to be able to do this, we have chosen the infinitary language $L_{\omega_1\omega_1}$, as the weakest possible extension of the first-order language, which allows the formation of infinite conjunctions and disjunctions prefixed by an infinite number of quantifiers (see axioms $\mathrm{A}_{\mathbf{T}}9$ and $\mathrm{A}_{\mathbf{T}}10$ in Appendix). The use of $L_{\omega_1\omega_1}$ makes it possible to have just a single sort of individual variable interpreted directly as ranging over the basic set of time inter-

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icles concern the order between elements of the time continuum, whereas archives of truth register what has happened or could have happened on them. Accordingly, in the Ockhamist system an instant can belong to multiple chronicles, but in the system **TM** to every interval corresponds a single archive of truths. We did not wish to represent the rejection of logical determinism and allowing for different continuations of the world history after a certain interval by introducing branching in the strucutre of time: in **TM**, the fact that an event can happen but also can fail to happen on a future interval is represented by means of diffrent possible worlds accessible from the actual world corresponding to an interval that ends at the present moment. Not only is it more natural to think of possibilities in terms of possible worlds than in terms of some non-standard structure of time, but we would also like to make room to speak about different (non-actualized) possibilities even in situations which would require branching time for their description (e.g., in Einstein's thought experiments with twins).

vals, without any further commitments of the second-order language (see Arsenijević et al. 2008b and Arsenijević et al. 2014).

So, the temporal part of **TM** will be formulated as an interval-based system of the time continuum (see Appendix) without mentioning instants. But sometimes, as in the case when we want to speak of an *apex* (see Section 3.1.2), we shall have to speak about instants as well. Fortunately, this will represent no problem, since, in accordance with Arsenijević's generalized definition of the syntactically and semantically only trivial differences between two formal systems (see Arsenijević 2003a), the interval-based and the instant-based system of time continuum can be shown to have the same expressive power, meaning that, due to the two sets of truth preserving translation rules between the formulae of the two systems, any truth of one of the two systems can be expressed in the other one (see Arsenijević et al. 2008a). So, we shall be able to speak both about time intervals and time instants within any of the two systems at will.

In addition to individual constants t_1, t_2, ..., t_i, ... that denote particular time intervals, and individual variables t_1, t_2, ..., t_i, ... that range over the set of all the time intervals independently of whether they are actual or non-actual, the language of **TM** contains, besides logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ and quantifiers $\exists$, $\forall$, binary predicate constants $=, \prec, \succ, \lhd, \cap, \subset$ and unary predicate constants ε, e, e_2, ..., e_i, ...

The binary constants denote the identity, the precedence, the succession, the abutment, the overlapping and the inclusion relations, respectively, in which two time intervals can stand. Their meaning is supposed to be understandable intuitively. It is only to be noted that $\succ$, $\lhd$, $\cap$ and $\subset$ can be defined via $=$ and $\prec$ (see Appendix).

The unary constants denote elementary events ε, e, e_2, ..., e_i, ..., which can be understood as predicates of individual constants and variables. Finally, **TM** contains temporal operators $\{\mathsf{t}_1\}$, $\{\mathsf{t}_2\}$, ..., $\{\mathsf{t}_i\}$, ..., and $\{t_1\}$, $\{t_2\}$, ..., $\{t_i\}$, ..., and modal operators $\Diamond$ and $\Box$. Let us explain in detail the intended meaning of all these symbols.

4.1. Elementary events

In the interval-based system as such there are no atomic intervals. So, in order to introduce elementary events we must find some characteristic of events which makes them elementary independently of metrics.

In his 1969 talk in Oberwolfach, Prior said that "time enters physical science through intervals by which one event may be earlier or later than another" (Prior 1972, p. 323), and then, in a number of discussions with Hamblin (Hamblin 1972, p. 328) suggested an axiom (axiom 10 in Hamblin 1972) which we may use to define elementary events implicitly. The axiom says that if e is an elementary event, then, if it is true that it happens on interval t_n, it is true that it happens on every of its subintervals (see $A_{\mathbf{TM}}6$ in Appendix).

There is, however, a problem with the requirement stated in Prior's axiom, which is discussed in Arsenijević (2002, pp. 126–128). Elementary events denoted by the unary constants e, e_2, …, e_i, … can be of any kind whatsoever. A unary constant can denote a happening at the atomic level as well as some everyday event like raining. The only condition for an event to be elementary is that it happens uninterruptedly. According to the axiom, it should mean that if (in some specified space area) it was uninterruptedly raining yesterday, there was no subinterval, however short, on which it was not raining. Does this mean that it was raining during some interval that lasted a billionth part of a second? It seems that there are just two possible answers: to say that it makes no sense to say that it was raining during such an interval, or to say that it was not raining on it. But neither answer will do. The first would mean that the axiom is not applicable in such cases as uninterrupted raining, which would exclude a great number of normal situations in which we speak of events that happen uninterruptedly. As for the second answer, it would mean that in all cases like uninterrupted raining the statement of the axiom is false, and moreover, if we make a partition of *yesterday* into abutting subintervals that lasted a billionth of a second each, we get that, yesterday, it was not raining at all! For the solution to this seeming paradox we have to realize that, if we accept that it makes no sense to say that on some very short interval *taken per se* it was raining, we should accept also that it makes no sense to say that it was not raining on it *taken per se*. This suggests that we may stipulate that the decision about whether it was or it was not raining on such an interval depends on whether it is *included* in an interval of which it does make sense to say that it was raining on it and on which it was raining, or it is included in an interval of which it does make sense to say that it was raining on it but on which it was not raining.

It is important to realize why *simple predicates* secured by *Prior's axiom* have a special treatment in **TM**. If complex predicates were introduced into the system, uniform substitution would not hold for them. For instance, if an event e failed to happen on an interval t_n but nevertheless occurred on its subinterval t_m, replacing the simple predicate e with the negative predicate '$\neg\mathsf{e}$' in Prior's Axiom would not save the truth value. Hence, the system **TM** is committed to a kind of an *event atomism*, which is not analogous to physical atomism, since there is no minimal duration of an elementary event. As in the standard predicate calculus, $\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n)$ will be always true, only that either of the disjuncts may happen to be true if there is a world fully actualized on t_n, while it is $\neg\mathsf{e}(\mathsf{t}_n)$ that will be true in any other case. Namely, $\neg\mathsf{e}(\mathsf{t}_n)$ is true both if there is no world actualized on t_n at all as well as if there is a subinterval of t_n on which no world is actualized. Of course, the spatial localization of e must also be provided by its description, for $\mathsf{e}(\mathsf{t}_n)$ is true only if e's happening on t_n occurs in a specified area.

Finally, since in every actual world any particular event may fail to happen, it would follow that there could be an actual world at which nothing happens. However, divorcing actuality from happenings in a system of events such as **TM** is counterintuitive, and, in any case, it is something we do not want to allow. So, in order to ban this possibility we require that something happens in any actual world. Since our system does not provide for quantification over events, among them there should be one designated event—denoted by ε—such that, as we have already suggested, it occurs in every actual world and accompanies an occurrence of any other event (see axiom $\mathrm{A}_{\mathbf{TM}}7$ and theorem $\mathrm{Th}_{\mathbf{TM}}2$ in Appendix). The notion of such a universal happening is quite natural and, in the context of modern physics, can be understood as the *motion of matter* (which is the basic happening in the ancient Ionian physics as well).

4.2. Temporal and modal operators and their iterations and combinations

The temporal operator $\{\mathsf{t}_n\}$ (where any other individual constant or variable can be substituted for t_n) can be prefixed to any formula of the system by assigning to t_n a consistent set of propositions which we called in Section 3.1.3 *the archive of truths*. So, for any formula A and any interval t_n, $\{\mathsf{t}_n\}A$ is a formula whose truth value depends solely on whether A belongs to t_n's archive of truths or not. If an interval t_n is non-actual, no actual world has "dawned" on it yet to be described and, therefore, its archive is empty. In such a case, $\{\mathsf{t}_n\}A$ is false for every formula A (see theorem $\mathrm{Th}_{\mathbf{TM}}3$ in Appendix). Not even logical truths, which are true *of* any interval, be it actual or not, are true *at* a non-actual interval — surely not because the formulae expressing them are false, but because they are not stored in the archive of $\mathtt{t}_n$, which is empty. This simple example demonstrates that prefixing a temporal operator is generally not a trivial matter, since it affects the truth value of the ensuing complex formula.

If we consider an archive of truths associated with an actual interval, we find a non-empty set of formulae concerning a world actualized on it as well as all the true formulae from the worlds actualized on each of its subintervals. Accordingly, an interval is said to be *only partly actual* if and only if there is a subinterval of the given interval whose archive is not empty as well as a subinterval whose archive is empty.

Let us summarize what the archive of an actual interval contains. First, it contains all logical truths and the truths of **TM**. As for factual truths, for an event $\mathtt{e}$ occurring in the description of a possible world and an interval $\mathtt{t}_n$, $\mathtt{e}$ either did or did not occur on $\mathtt{t}_n$, so that the truth about its occurrence, or about its non-occurrence, should be supposed to be recorded in every archive (see axiom $\mathrm{A}_{\mathbf{TM}}1$ in Appendix). An archive belonging to an actual interval, therefore, contains all the truths about what happened, or failed to happen, on that interval as well as all the truths about what happened on intervals that precede it (which are, hence, actual themselves). The archive also contains all the truths about what happened on all the actual intervals that end only later. However odd the last fact may seem, it is in accordance with *Leibniz's principle of truth mirroring*, which we have adopted (see Section 3.1.3). Finally, we take, in accordance with the axioms of **TM**, that any actual interval's archive contains the truths that state that nothing has occurred on non-actual intervals (see theorem $\mathrm{Th}_{\mathbf{TM}}16$ in Appendix). That is why non-empty archives *cannot be discriminated* between themselves with regard to the *factual* truths about events they contain: *all such truths can be read off from any one of them.*

The non-empty archives register truths about *archiving* on other actual intervals due to the possibility that formulae describing the history of the process of registering facts have *iterated* temporal operators. For example, $\{\mathtt{t}_n\}\{\mathtt{t}_m\}A$ means that the archive of an interval $\mathtt{t}_n$ contains the formula $\{\mathtt{t}_m\}A$, which says that A is recorded in $\mathtt{t}_m$'s archive. But again, no such formulae are sufficient for differentiating various non-empty archives (see axiom $\mathrm{A}_{\mathbf{TM}}2$ in Appendix). As we shall see now, by turning to modal operators, *only modal truths* about possible or necessary occurrence or non-occurrence of events, which vary from one actual interval to another, can supply the criterion for discriminating actual intervals by using their archives.

In the semantics of modal logic systems, formula $\Box A$ is said to be true if and only if A is true in all accessible possible worlds, and $\Diamond A$ is taken to be true if and only if there is an accessible possible world in which A is true. In the standard possible world semantics, the truth of a formula prefixed by a modal operator is assessed from a single world, and, therefore, it is not necessary to point to the world from which the accessible worlds are accessible. However, in **TM**, there is *one real* world but *an infinite number of actual worlds*, so that some possible worlds are accessible from some actual worlds but not from others, depending on what happened in between on the intervals on which these actual worlds have been actualized. If, for instance, an event e happened on an interval t_n, then on an earlier interval t_m it was possible for e not to occur on t_n (granted by the axiom $\mathrm{A}_{\mathbf{TM}}16$), while on t_n itself this possibility is precluded (as a result of the axiom $\mathrm{A}_{\mathbf{TM}}14$). Thus there is a merely possible world, in which e does not happen on t_n, which is accessible from the world actualized on t_m but not from the world actualized on t_n.

The fact that the archives of truths on actual intervals differ only in view of modal truths means that the difference consists entirely in the fact that *different sets of accessible possible worlds* are associated with them. Consequently, the formulae with a modal operator outside the scope of a temporal one lack a determinate truth value, as in such cases it is not specified which actual world's set of accessible possible worlds is to be taken into account. In **TM** we can meaningfully talk about possibilities only by bearing in mind what has, up to a certain time, already been actualized. So the status of formulae such as, for example, $\Box\mathsf{e}(\mathsf{t}_n)$ and $\Diamond\mathsf{e}(\mathsf{t}_n)$, should be understood by analogy to the well-formed but open formulae in predicate logic, which become definitely true or false only after some further qualification. Formulae with iterated modalities can, accordingly, be true or false only if the sequence of modal operators is, as a whole, subjected to a temporal operator. Hence, we can also speak of merely possible worlds being accessible from other merely possible worlds but only provided that the first merely possible world in the chain is accessible from some actual world. In other words, the talk of possible possibilities, possible necessities, etc., has to be *anchored* in the real world.[4]

[4] In **TM**, temporal versions of both Barcan and the converse Barcan formula

4.3. Discriminating possible worlds without reifying them

Since the language of **TM** is not the second-order language in which we could use either the notion of the set of all actual and non-actual events or the notion of the maximally consistent set of propositions—which would imply, contrary to the leading idea of **TM**, the *reification* of possible worlds—we must find some other way to individuate possible worlds without using these notions.

For describing possible worlds ascribable to a time interval, actual or non-actual, let us form, recursively, the *equivalence classes* differing in view of all and only elementary events that they contain. Then, each equivalence class should be associated with just one possible world actualizable on various time intervals, and can be, therefore, understood as a *universal* predicable to various time intervals as *particulars*. In such a way we obtain the equivalence classes $\omega_1, \omega_2, \ldots, \omega_i, \ldots$, where $\omega_1 = \langle \varepsilon \rangle$, $\omega_2 = \langle \varepsilon, \mathsf{e}_1 \rangle$, $\omega_3 = \langle \varepsilon, \mathsf{e}_2 \rangle$, $\omega_4 = \langle \varepsilon, \mathsf{e}_1, \mathsf{e}_2 \rangle$, $\omega_5 = \langle \varepsilon, \mathsf{e}_3 \rangle$, $\omega_6 = \langle \varepsilon, \mathsf{e}_1, \mathsf{e}_3 \rangle$, $\omega_7 = \langle \varepsilon, \mathsf{e}_2, \mathsf{e}_3 \rangle$, $\omega_8 = \langle \varepsilon, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3 \rangle$, and so on and so forth.

Let us illustrate the function of the equivalence classes of possible worlds with the use of the following two diagrams. In figure 2 the possible world represented through the equivalence class ω_2 (which supposedly consists of ε and e_1) is predicated to t_n as well as to t_k and t_m, but not to t_l, because of the occurrence of e_2, due to which it is only ω_4 (which consists of ε, e_1 and e_2) that is predicated to t_l.

Similarly in view of what is represented in figure 3. The possible world represented through the same equivalence class ω_2 is predicated to t_n, t_q and t_s, but not to t_p and t_r because, due to the occurrence of e_2 on t_p and e_3 on t_r, ω_4 (consisting of ε, e_1 and e_2) is predicated to t_p, and ω_6 (consisting of ε, e_1 and e_3) to t_r.

It is important to notice that with regard to the distribution of possible worlds along a time interval there is nothing analogous to *Prior's axiom*, which supposedly holds for elementary events. Namely, although

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can be proved (see $\text{Th}_{\textbf{TM}}21$ and $\text{Th}_{\textbf{TM}}22$ in Appendix). These two theorems together establish that the same time continuum runs through all the possible worlds (since the time intervals are being quantified over in the theorems). This puts our system in sharp contrast to what has become known in the literature as Leibniz's system (cf. Øhrstrøm and Hasle 1995, pp. 270–282; 2020), in which chronicles (see footnote 2) are envisaged not as belonging to a tree-structure, but as parallel lines: every chronicle has its own time with appropriate moments; instants are only later defined as equivalence classes of contemporaneous moments that belong to different time lines.

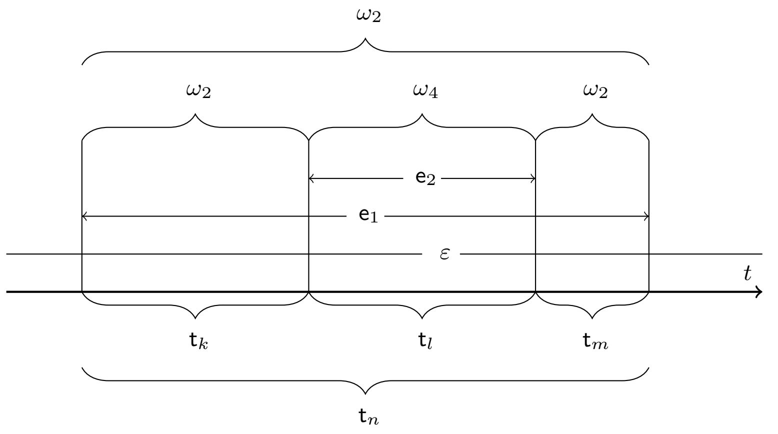

Figure 2.

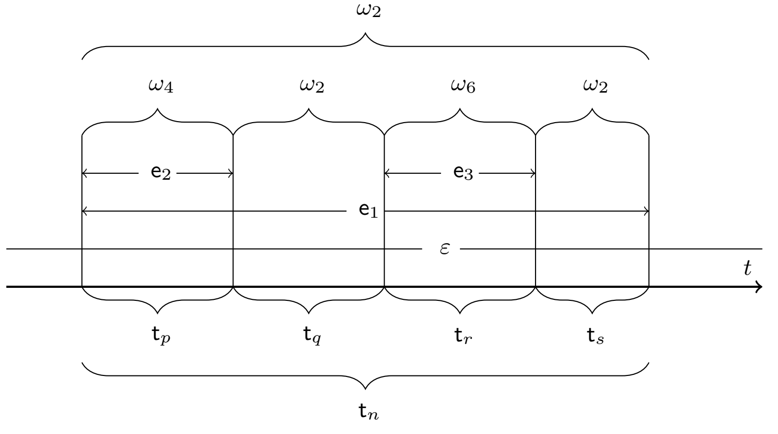

Figure 3.

according to both figures ω_2 is truly predicable to t_n, it is not truly predicable to all of its subintervals.

It is also important to note that although only one distribution of descriptions is allowed if we speak of actual worlds, this does not mean that a variety of distributions is prohibited if we speak of merely possible worlds. And even if some world represented through ω_2 is actualized on t_n so that e_2 does not happen either on t_l or t_p and e_3 does not happen on t_r, it remains that on intervals before t_n it was possible that e_2 happens on t_l or t_p, or that e_3 happens on t_r.

Finally, it is also important to realize that the list of equivalence classes serves only to individuate possible worlds after ascribing their members to particular intervals, which *does not* mean that the only truths about an interval are those that concern the occurrence of the events mentioned in the equivalence class associated with that interval. So, although e_2 and e_3 do not figure in the description whose extension is ω_2, after the distribution of descriptions according to *FIGURE 2* not only $\varepsilon(\mathsf{t}_n)$ and $\mathsf{e}_1(\mathsf{t}_n)$ but also $\neg\mathsf{e}_2(\mathsf{t}_n)$, $\neg\mathsf{e}_3(\mathsf{t}_n)$ and indefinitely many other negative propositions become true *of* the world associated with t_n through ω_2. If that world were also actual, various *negative* propositions would also be true *at* it because the interval t_n would then have a non-empty archive. The same holds for various *modal* truths, which will hold *of* and *at* the world from the equivalence class ω_2 associated with t_n, depending on whether the world is merely possible or actual as well as on how other possible worlds are actualized and distributed over time intervals. The trick of the proposed manner of discriminating possible worlds sufficiently consists in doing so by taking into account only those events occurring in the elementary formulae that are true of them, avoiding all other formulae true *of* or *at* them.

5. The philosophical appeal of TM

In this section we shall deal with the philosophical consequences of **TM** that relate to some central problems in the philosophy of time and modality. Let us start again with the 'sea battle' puzzle, which has generated nearly all other questions concerning the relation between time and modality.

5.1. The solution to the 'sea battle' problem

If $\mathsf{e}(\mathsf{t}_n)$ states that e occurs on t_n that is non-actual, then, if it is stated on some actual interval t_m, $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$ is *false*, but not because e specifically failed to happen on t_n but because nothing happened on t_n since there is no actual world on it. It is straightforwardly wrong to say *today* that *there is* such a thing as the *sea battle happening tomorrow*. That is why both $\neg\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_n)$ are true (see theorem $\text{Th}_{\mathbf{TM}}16$), so that, in relation to the given case, both $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n) \vee \neg\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$, and $\{\mathsf{t}_m\}(\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$ hold.

However, by saying that the sea battle will happen tomorrow we actually mean something *else*, which according to **TM** should be expressed as $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$, where t_m refers to *today* and t_n to *tomorrow*, namely that it is now true that it will be true tomorrow that the sea battle happens then. This is *false*, which is exactly what we want to be the case, since we want to avoid *logical determinism* (see theorem $\mathrm{Th}_{\mathbf{TM}}17$ in Appendix). What $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ says is that $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ belongs to t_m's archive of truth, which is not the case, since what $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ says is that $\mathsf{e}(\mathsf{t}_n)$ belongs to t_n's archive of truths, and this is not the case as t_n's archive of truths is empty.

But now, it is not only $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ that is false, but also $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$, since on t_n, which is non-actual, not only $\mathsf{e}(\mathsf{t}_n)$ but also $\neg\mathsf{e}(\mathsf{t}_n)$ does not belong to t_n's archive of truths, because it is empty (see theorem $\mathrm{Th}_{\mathbf{TM}}18$ in Appendix). When prefixed by a temporal operator that denotes a non-actual interval, even logical truths, like the principle of excluded middle, do not render a true complex proposition, so that in the given case $\{\mathsf{t}_n\}(\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$ is false.[5]

Both being false, $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and$\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ are not contradictory but only contrary, and that is why both $\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ may be true (see theorem $\mathrm{Th}_{\mathbf{TM}}10$ in Appendix). Today, it is both possible that the sea battle will happen tomorrow as well as that it will not happen, namely, $\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_n) \wedge \{\mathsf{t}_m\}\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ is *true*. But this does not mean that at t_m it is true on t_n that it is both possible that the sea battle happens and that it does not happen, since $\{\mathsf{t}_m\}\{\mathsf{t}_n\}(\Diamond\mathsf{e}(\mathsf{t}_n) \wedge \Diamond\neg\mathsf{e}(\mathsf{t}_n))$ is *false* (see theorem $\mathrm{Th}_{\mathbf{TM}}11$ in Appendix).

Let us now compare the answer in the system **TM** with different proposals concerning the truth of future contingents mentioned in Section 2.2: Are future contingents all false (Todd), true or false simpliciter (Rosenkranz), neither true nor false (Thomason), or dependent on the context of the assessment (MacFarlane)?

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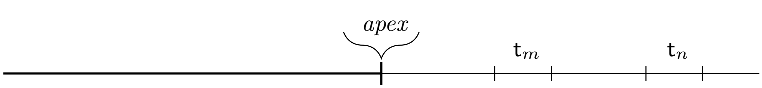
[5] If t_n is a non-actual interval, then both $\{\mathsf{t}_n\}(\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$ and the temporal version of the principle of excluded middle $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n) \vee \{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ will be false, since nothing has been stored in t_n's archive of truth. The system **TM** is in this respect similar to Prior's Peircean system of tense logic (Prior 1967, 128–135), in which the principle of future excluded middle is not a valid formula. However, the important difference between these two systems is that in the Peircean system if an event will not happen, it necessarily will not happen; the **TM**-analogon, however, does not hold, since for an actual interval t_m and a non-actual interval t_n, it is true that $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_n)$ but false that $\{\mathsf{t}_m\}\Box\neg\mathsf{e}(\mathsf{t}_n)$ — $\{\mathsf{t}_m\}$ is added since in **TM** modal claims have to be assessed from an actual world (see Section 4.2).

Basically, **TM** accords with the last proposal, but it is crucial to explain why and how the answer, if t_m is earlier than t_n, depends on the times of truth-assessments of the relevant propositions stated at t_m and t_n. In **TM**, there are three possibilities, i.e., there are three cases which are incompatible due to the differences in the positions of the apex at the time-continuum: (1) both t_m and t_n lie in the imaginary part of the t-axis; (2) t_m lies in the real and t_n in the imaginary part of the t-axis; (3) both t_m and t_n lie in the real part of the t-axis.

(1) Let us distinguish between relative future statements and genuine future statements. If t_m is earlier than t_n, $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_n)$ are *relative future statements* which say respectively that e happens and does not happen on t_n and are stated on t_m, while $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ are *genuine future statements* which say that it holds on t_m, that it will be true on t_n that e happens, and respectively does not happen, on t_n. Now, when both t_m and t_n lie in the imaginary part of the time-continuum (see the figure below), all relative and genuine future statements are false, because there are no truths at empty intervals. *A fortiori*, even $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_n)$ is false, not because $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$ is true but because $\{\mathsf{t}_m\}A$ is false for any A. So, if $\mathsf{t}_m \prec \mathsf{t}_n$, in view of all future contingents, it is true that:

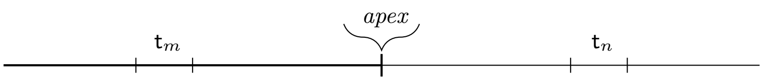
$$\forall t_m \forall t_n(\neg Act(t_m) \wedge \neg Act(t_n) \rightarrow \neg\{t_m\}\mathsf{e}(t_n));$$
$$\forall t_m \forall t_n(\neg Act(t_m) \wedge \neg Act(t_n) \rightarrow \neg\{t_m\}\neg\mathsf{e}(t_n));$$
$$\forall t_m \forall t_n(\neg Act(t_m) \wedge \neg Act(t_n) \rightarrow \neg\{t_m\}\{t_n\}\mathsf{e}(t_n));$$
$$\forall t_m \forall t_n(\neg Act(t_m) \wedge \neg Act(t_n) \rightarrow \neg\{t_m\}\{t_n\}\neg\mathsf{e}(t_n)).$$



(2) The previous case may seem trivial, since it turns out that for the falsehood of all future contingents, be they relative or genuine, it is sufficient that the earlier of the two intervals, t_m, is non-actual. A more interesting case is when t_m lies in the real and t_n in the imaginary part of the time continuum (see the figure below). Then, as in the first case, all genuine future statements are false, but the second of the two relative future statements is true, since it holds on t_m which is actual that nothing happens on t_n, which is non-actual. So, in this case it is true that

$$\forall t_m \forall t_n(Act(t_m) \wedge \neg Act(t_n) \rightarrow \neg\{t_m\}\mathsf{e}(t_n));$$
$$\forall t_m \forall t_n(Act(t_m) \wedge \neg Act(t_n) \rightarrow \{t_m\}\neg\mathsf{e}(t_n));$$

$$\forall t_m \forall t_n (Act(t_m) \wedge \neg Act(t_n) \to \neg\{t_m\}\{t_n\}\mathsf{e}(t_n));$$
$$\forall t_m \forall t_n (Act(t_m) \wedge \neg Act(t_n) \to \neg\{t_m\}\{t_n\}\neg\mathsf{e}(t_n)).$$



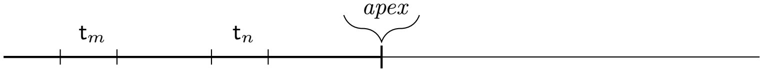
(3) The third case, in which both t_m and t_n lie in the real part of the time-continuum, is the most interesting one, because the assessment of $\mathsf{e}(\mathsf{t}_n)$ at t_n is not trivial and depends on whether e really happens on t_n or not. So,

$$\forall t_m \forall t_n (Act(t_m) \wedge Act(t_n) \to \{t_m\}\{t_n\}(\mathsf{e}(t_n) \vee \neg\mathsf{e}(t_n)))$$

and then

$$\forall t_m \forall t_n (Act(t_m) \wedge Act(t_n) \to (\{t_n\}\mathsf{e}(t_n) \to \{t_m\}\{t_n\}\mathsf{e}(t_n)))$$
$$\forall t_m \forall t_n (Act(t_m) \wedge Act(t_n) \to (\{t_n\}\neg\mathsf{e}(t_n) \to \{t_m\}\{t_n\}\neg\mathsf{e}(t_n))).$$

The reason why in this case both genuine future contingents $\{\mathsf{t}_m\}$ $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ are either true or false is that they are stated under antecedents that say that t_m and t_n are *both* actual, so that, in accordance with *Leibniz's principle of truth mirroring* (see Section 3.1.3), what is true on t_n must also be true on t_m independently of the fact that t_m is earlier than t_n. If it is actually true that e happens on t_n, it is true on t_n that it was true on t_m that it would be true on t_n that e had happened on t_n. At this point **TM** accords with Ockhamism (see Section 2.2). But this does not mean that future contingents are true or false simpliciter, since under different antecedents, as in the cases (1) and (2), both $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ are false.



The three cases, (1), (2) and (3) are models which represent different phases of the development of the real world history. In (1) and (2), $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ are false, while in (3), either $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ or $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ is true depending on whether $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ or $\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ is true. So, in **TM**, contrary to Aristotelians, who accept truth-value gaps (van Fraassen 1966; Thomason 1970), the models (1) and (2) accord with the Russellian open futurists (Todd 2016), who claim that future contingents are all false due to the unreality of the future (Burgess 1978). At this point **TM** does not accord with Ockhamists (Rosenkranz 2012), since in both (1) and (2), future contingents are false, and not either true or false simpliciter. In (3), future contingents stated on t_m about what happens on t_n *are* either true or false, but only due to the fact that the actuality of t_n makes either $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ or $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ true (see theorems $\mathrm{Th}_{\mathbf{TM}}19$ and $\mathrm{Th}_{\mathbf{TM}}20$ in Appendix). Finally, as we shall see in Section 6, the three incompatible models, (1), (2) and (3), could be incorporated in one and the same history of the real world, if "utterance-truth were relativized to the context at which the utterance is being assessed" (MacFarlane 2003, p. 327).

Let us mention that it is not difficult to see, on the basis of (1)–(3), what the situation would look like if the apex lay within t_m or t_n.

5.2. *Necessitas per accidens*

Being true when t_n is non-actual, $\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ *remain true* also when some world is actualized on t_n (see axiom $\mathrm{A}_{\mathbf{TM}}16$ in Appendix). But if t_n is actual, either $\{\mathsf{t}_n\}\Box\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_n\}\Box\neg\mathsf{e}(\mathsf{t}_n)$ is true (see axioms $\mathrm{A}_{\mathbf{TM}}14$ and $\mathrm{A}_{\mathbf{TM}}15$ in Appendix).[6] For, as Aristotle put it, "what is, necessarily is, when it is, and what is not, necessarily is not, when it is not", which is a kind of conditional necessity that was later, as already mentioned, called *necessitas per accidens*. However, the fact that $\mathsf{e}(\mathsf{t}_n)$, which is possible on t_m, has become false on t_n (and so ceased to be possible on t_n) does not mean that at t_n it ceases to be true that what is impossible on t_n was possible on earlier intervals. Namely, if $\neg\mathsf{e}(\mathsf{t}_n)$ is true on t_n, $\{\mathsf{t}_n\}\Box\neg\mathsf{e}(\mathsf{t}_n)$ is true, but $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_n)$ is true as well (see theorem $\mathrm{Th}_{\mathbf{TM}}13$ in Appendix). This example shows how **TM** enables us to say not only that something *can be so-and-so* but also that something *could have been otherwise than it actually is.* The following example concerns the phenomenon that is complementary to this but more intriguing, because it concerns *the revision of the archives of truths.*

[6] If t_m and t_n are both actual and t_n precedes t_m, then if $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$, it will also hold that $\{\mathsf{t}_m\}\Box\mathsf{e}(\mathsf{t}_n)$ as a consequence of $\mathrm{A}_{\mathbf{TM}}14$ (see Appendix). This is an important difference between **TM** and Prior's Ockhamist system (Prior 1967, pp. 122–137), in which the principle of the necessity of past is not valid. We believe that it is the advantage of our system to be able to avoid logical determinism without sacrifizing Aristotle's and Ockham's intuition of 'contingently acquired necessity' of past events.

5.3. Revision of the archives of truths

According to *Leibniz's principle of truth mirroring*, which we have adopted, if e really happens on t_n, i.e., if $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ becomes true, this truth has to be a part of any actual archive of truths, and so also of the archive of truths on t_m, which is earlier than t_n. Does this mean that $\{\mathsf{t}_m\}\neg\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$, which was *true* before t_n has become actual, has to *change* its truth value and become *false*? Fortunately not, for otherwise we would have to adopt a communist-style revision of history! $\{\mathsf{t}_m\}\neg\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ remains true in spite of the fact that t_m's archive of truths has been *completed* through the addition of $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$, for, as we saw in Section 4.2, the prefixing of a temporal operator is generally not a trivial matter, since it may affect the truth-value of the ensuing complex formula. Notice that $\{\mathsf{t}_m\}\neg\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ are true under *incompatible antecedents*, the former when t_n was non-actual, the latter when it has become actual *and* $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ true. However, if, when t_n has become actual it is $\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ that is true, then $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ will be true (see theorem $\mathrm{Th}_{\mathbf{TM}}20$ in Appendix). But in this case too, the formula added to t_m's archive says something *new*, since $\{\mathsf{t}_m\}\neg\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ differ in view of the position of negation, the former being true because of the supposed non-actuality of t_n, the latter because e does not happen on the supposedly actual t_n. For $\{\mathsf{t}_m\}\neg\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ to be true it is sufficient that t_m is earlier than t_n, while for $\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$ to be true it is also necessary to suppose that t_n is actual.

Let us illustrate what is said in ordinary language by using the sea battle example once again. Though today it is only possible that the sea battle will happen tomorrow (as well as that it will not happen), if the sea battle really happens tomorrow, it will be true tomorrow that it was true the day before that it would be true the day after that the sea battle had happened that day (see theorem $\mathrm{Th}_{\mathbf{TM}}19$ in Appendix). Similarly, if the sea battle does not happen tomorrow, it will be true tomorrow that it was true the day before that it would be true the day after that the sea battle had not happened that day (see theorem $\mathrm{Th}_{\mathbf{TM}}20$ in Appendix).

5.4. The prediction-retrodiction asymmetry

The consistency of the archives of truths revision with the denial of logical determinism can hardly be questioned (the relevant theorems proved in **TM** are cited in Appendix). But if there is still certain intuitive uneasiness about seemingly conflicting ideas, we can remove it by explaining the asymmetry between prediction and retrodiction. The reason why we cannot know the truth about future sea battle is not a matter of *epistemological* fact. Namely, in the given case, the impossibility of knowledge is completely based on the *logico-ontological* fact that *there is nothing to be known*, since there are different possible ways that lead to this or that outcome. We may *guess* that it will be so-and-so, but to guess is not to know. Everything depends on deliberations, decisions and coincidences of events that are not yet actual. However, once the sea battle really happens or really fails to happen, there is just *one single path* that history has laid out to this moment (which Belnap and Green 1994 call 'thin red line'), and we can explain (in principle at least) *how* it has come into being by taking into account *actual* deliberations, decisions, coincidences, etc.

6. The flow and the arrow of time as objective characteristics of the real world and its history

We can now explain in which sense the real part of the time continuum is a model of **TM**, while the real world history is a continuously ordered set of its parts as historically privileged models of **TM**, which will then be shown to imply the objectivity of the flow of time and the arrow of time as the *inter*-model and the *intra*-model characteristic of the real world history, respectively.

6.1. The set of continuously ordered equivalence classes of isomorphic models of TM

Any endless one-dimensional continuum that has an apex of the real part as a boundary between its real and its imaginary part (see figure 1, where the real part contains predicates denoting elementary events actually distributed over its intervals in such a way that all the axioms of **TM** hold in regard to well-formed formulae of **TM**, represents a *model of* **TM**. Then, given that there can be indefinitely many mutually incompatible distributions of elementary events over the real part of the continuum (see Section 4.3), there are indefinitely many different models of **TM** but *isomorphic* with regard to a *common apex*. At the same time, any two models with different apexes are *non-isomorphic*, since in such a case there are intervals that are *actual* in one of them and *non-actual* in the other. This means that, while all the possible developments of real world history that *end at the same instant* are represented by isomorphic models, any two possible histories that end at different instants must be represented by some two non-isomorphic models.

Since any instant can be an apex of the real part of the time-continuum, and the set of instants is continuously ordered (which can be expressed in the interval-based system of the linear continuum as well—see Section 4, the set of equivalences classes of models with the same apex can be shown to be continuously ordered as well (for the formal proof see Appendix).

6.2. The flow of time as the transition between the privileged models of TM

Given that one of the underlying assumptions of **TM** is that there is just one real world (see Section 3.1), there is just one real world history in view of any given apex as the boundary between the real and the imaginary part of the time continuum. Consequently, there must be *one privileged model* from the equivalence class of isomorphic models that represents the real world in view of a given apex. From a formal point of view, this is a consequence of the fact that **TM** implies *necessitas per accidens* (see Section 5.2). Though it could have been otherwise, what is, necessarily is. The distribution of elementary events over the *real* part of the time continuum is *necessarily such as it is.* So, for any given instant, there must be a *unique* real world history that ends at it. Then, though any instant is represented through an equivalence class of models regardless of the history that has paved the way to it, there is always a privileged world line that represents *the* history of *the* real world *up to the given instant.*

As mentioned in Section 5.4, the privileged world line corresponds to what Belnap and Green (1994) call *thin red line* (TRL). They are aware, however, that the idea that there is “a Thin Red Line given once and for all” will not do the job. By criticizing such an idea, MacFarlane says that “positing a thin red line amounts to giving up objective indeterminism” since this would imply that, “looking down on the tree of branching histories, God can see that given the past and the context of utterance, only one continuation remains in play: the one marked with the thin red line" (2003, p. 325). That's why Belnap and Green replaced TRL with "a *context-dependent* theory asserting that each possible moment m determines its own Thin Red Line TRL(m)" (Belnap and Green 1994, p. 380). This is exactly what we have in our system **TM**, where each equivalence class of isomorphic models has one and the same apex, which, as the boundary between the real and the imaginary part of the time continuum, determines a unique privileged model, i.e., a unique TRL. But it is important to notice that in **TM**, in any given privileged model it is possible to speak about any privileged model that represents an earlier stage of the real world history, in which the apex lies *within* the TRL associated with the given model, since **TM** enables us to say not only that something *can be so-and-so* but also that something *could have been otherwise than it actually is.*

Any given model that should represent the history of the real world up to a certain instant must contain, as its proper subset, the set of actual worlds that represents a segment of the real world history that ends earlier, where all the *factual* truths about events in a model which represents only a part of the history are *preserved* in the privileged model that describes the whole real world history up to the given instant. Then, the *development* of the real world history can be viewed as a continuous transition from one privileged model to the others such that each of them represents the real world history up to a certain instant *as if* it ended at that instant. *The flow of time* can be then understood as nothing else but such a continuous transition from one privileged model to the other such models as a consequence of the development of the real world history, where each of the previous models represents an earlier development of the real world history up to a certain instant.

Now, since the real world, in regard to a given apex as a model of **TM**, contains in itself all its segments as the models with different apexes, there can be no real world without its *history*, meaning that any instant earlier than the given apex must have been an apex with regard to a privileged model of **TM**. So, the *transition* from one model to the others must be something *objective*, and insofar the *flow of time* as a consequence of the necessity of such a transition must be an *objective characteristic* of the *real world history*, with which the real world itself is *identical* with regard to any given apex.

6.3. The development of the real world history as occurrence of actual events and the continuous transition between the privileged models of TM

A rather intriguing question concerns the reconciliation of the fact that, starting from an apex, the further development of the real world history consists in the addition of new actual events ascribed to *intervals* and *not instants* to the imaginary part of the continuum, with the fact that the *transition* between the related **TM** models is to be developing in accordance with the Cantorian *instant-based* system of the continuum. How are all *instants* after the given apex to be apexes of some **TM** model if any elementary event after the given apex is to be actualized on an *interval*?

The first expected answer to the above question might seem to be that according to Prior's axiom ($A_{\mathbf{TM}}6$) it holds for any elementary event that if it happened on an interval it also happened on any of its subintervals, so that for any instant of the Cantorian instant-based continuum holds that it is the end-instant of an interval on which an event happened, meaning that it can be an apex of the real part of the time continuum. However, given that in ($A_{\mathbf{TM}}6$) the implication does not hold in the opposite direction, it does not follow that there must be the actualization of an event on a subinterval *before* the interval itself has become actual. Though it is true that if it was raining on an interval it was raining on all of its subintervals, it does not mean that it was raining on some very, very short interval of the interval on which it was raining *before* it was raining on the given interval itself (see Section 4.1). However, according to axiom ($A_{\mathbf{TM}}7$), there is specially designated event ε whose occurrence on an interval is a necessary condition of its actuality. So for ε it is not possible, as it is in the case of raining, that starting from a given apex an interval be actualized without ε's occurrence on it.

A highly interesting consequence of the above explanation is that, though events always happen on intervals, so that the real world history does not develop according to the Cantorian instant-based system of the linear continuum, any set of models between two non-isomorphic **TM** models *is* continuous in accordance with the Cantorian instant-based system. To put it in the way in which Whitehead did, the fact that there is no continuity of becoming does not mean that there cannot be the becoming of continuity (in the Cantorian sense) (Whitehead 1929, p. 53). The question is only how this is to be understood concretely.

Though *no instant* in the real world history can be understood as a *part* of it, since any event happens on an interval, *any* instant within the real world history *can be* understood as an apex of the real world history at which it *could have ended.* That is why *the flow of time* is to be understood only as the *inter-model transition* between the models of **TM** as a *consequence* of the development of the real world history. A useful analogy can be the case of drawing a line (see Arsenijević 2016). Though no segment of the line can be drawn by drawing the points it should contain, every segment contains all the points of the continuum as places at which we *could have stopped* drawing the line. And while we could have stopped doing something at an *instant*, if we didn't stop, we must have continued doing it for *some period* of time, however short.

6.4. The arrow of time

One might think that what we have established to be the flow of time automatically solves the problem of the *arrow* of time, since the arrow of time can be understood as nothing else but the *direction* in which time flows as a consequence of the development of the world history. However, the problem is not that easy.

As we have mentioned in the Introduction, Wheeler and Feynman (1949) argued that nothing would be lost in the physicists' description of the world if the relations *earlier than* and *later than* were systematically interchanged, on the basis of which Grünbaum (1967) concluded that the talk about direction of time must be mind-dependent insofar as it requires a choice between the two alternatives. True, the world Grünbaum had in mind is deterministic, while the real world according to **TM** is not. But it still remains the question of why according to **TM** there is *the* flow of time, and not just *a* flow of time in one direction or the other according to our choice.

Now, it can be said that, by looking at figure 1, we can say that the direction of the flow of time must be from the left to the right, since the real world history develops from the real towards the imaginary part of the time continuum, and not the other way round. But, however correct such a conclusion may be, it is based on the *inessential* assumption that *there is no* beginning of the real world history, so that *FIGURE 1* is not symmetric in view of the given apex. However, as we have mentioned in Section 3.1, the system **TM** can easily be adjusted so as to be in accordance with *Big Bang* cosmology (it is shown in Appendix how it can be done formally). Let us denote such a revised system by **TMbb**. Then, we get, instead of figure 1, figure 4 as the representation of a **TMbb** model, in which there are *two* end-points of the real part of the time continuum, with the remaining question of *which* of the two is to be taken as *the* apex that determines *the* direction of time.

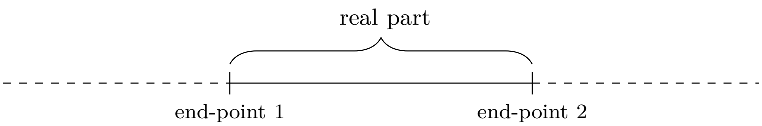


Figure 4.

Since we surely do not want that the direction of time depends on an inessential assumption, we have to find some *intra-model* characteristic of **TMbb** related to the *essential* assumptions of **TM** that would imply the necessity of the existence of *the* flow of time as the objective one. This will lead us to the central point of the whole preceding investigation, which concerns the relation between time and the in-the-world-inherent modalities.

The real part of the time continuum contains all the actual intervals along which the elementary events that make them actual are distributed. But the distribution can be equally well described by starting from the end-point 1 as well as by starting from the end-point 2. So, the *factual truths* about the real world history cannot be used for determining direction in which the *inter-model* transition between the continuously ordered set of the privileged models of **TM** is objectively ordered. This means that, if there is any way to show that there is *the* arrow of time, we must take into account the *modal truths* concerning the occurrence of elementary events along the real part of the time continuum.

Fortunately, **TM** enables us to speak not only of what it can be so-and-so but also of what *could have been otherwise* than it actually was (see Section 5.2), because all the modal truths remain *preserved* through the development of the real world history in spite of the archives of truths revision (see Section 5.3). So, we ought to be able to detect the objective arrow of time by considering the modal truths of the real part of the time continuum alone.

Let us suppose that an event e really happened on an interval t_n of the real part of the time continuum and that t_m is an interval earlier than t_n ($\mathsf{t}_m \prec \mathsf{t}_n$). Then $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ will be true, as a consequence of the theorem $\mathrm{Th}_{\mathbf{TM}}19$ that says that $\forall t_m \forall t_n((t_m \prec t_n \wedge \mathsf{E}(t_n)) \rightarrow \{t_n\}\{t_m\}\{t_n\}\mathsf{E}(t_n))$. But under the same supposition – that e really happened on t_n – $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ is true as well, which means that the fact that $\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ is true at the time when only t_m but not t_n was actual has been 'remembered' at t_n and remained true also at the time when t_n has become actual and $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ true.

Now, while *factual truths* are *insensitive* to the direction of the flow of time, so that we can choose any of the two alternatives, it is not so with *modal truths*. The formula $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\Diamond\neg\mathsf{e}(\mathsf{t}_n)$ in the above example ceases to be true if we in **TMbb** systematically interchange the relations *earlier than* and *later than*. Of course, the interchange is not supposed to be only notational, so that $\succ$ and $\prec$ start to denote the precedence and the succession relation respectively, but such that $\prec$ and $\succ$ are still to be interpreted as the precedence and the succession relation respectively. This means that, though we can choose which of the two end-points in *FIGURE 7* will be taken to represent the beginning of the real world history, the end-point 1 can be taken to represent it only if we suppose that it is *earlier* than the end-point 2, while it can be taken to be the end-point 2 only if we suppose that it is the end-point 2 which is earlier than the end-point 1. It is then clear why the *modal truths* determine the *objective direction* of the distribution of elementary events over the real part of the time continuum.

As is explained in Section 3.1.1, one of the basic assumptions present in the construction of **TM** is that the real world contains *real possibilites* as its 'part and parcel' (as Belnap put it). It is now clear of what importance it is for establishing the objectivity of the *arrow of time*. However odd Grünbaum's radical view may seem to be, according to which the *arrow of time* is mind-dependent, we think that his conclusion is based on a perfectly sound methodology. If we are not Newtonians, we have to find a characteristic of the world constitution, if there is any such characteristic, which is responsible for the fact that the time continuum is directed in *just this and not another way*. In view of **TM**, this characteristic is the objectivity of the difference between the *pastness* and *futurity* which the presence of *real possibilities* in any model of **TM** requires.

Finally, it is important to notice that for the objectivity of the arrow of time that **TM** implies it is sufficient to assume that there is just one single event that happens on some t_n, whose occurrence is not necessary. If the world is supposed to be determinintistic due to the causal chains governed by physical laws, the occurrence of such an event at some t_n will contaminate the necessity of indefinitely many truths about events which are to happen after $\mathtt{t}_n$ as the result of the meetings of different causal chains. This is so not only in view of indefinitely many propositions to be stated after $\mathtt{t}_n$ but also in view of indefinitely many propositions stated before $\mathtt{t}_n$. And then, curiously enough, before such a possible indeterministic event happens at all, it is so in view of *any* proposition stated at some interval, since it is not predetermined at *which* place and at *which* time the indeterministic event will occur, if it occurs at all.

6.5. The tense and the tenseless theory of time

As to the dispute between tensers and detensers, we may conclude that due to the *objectivity of the flow of time* the *tenseless theory of time* is wrong in regard to a non-deterministic world at least. However, the *tense theory of time implied by* **TM** is radically different from standard tense theories of time. Let us illustrate this through the following three points.

1. The objectivity of the flow of time is not, as it is according to any variant of the Newtonian or the Kantian theory of time, a consequence of the alleged fact that it is a property of time as such, independently of the constitution of the real world that exists in space and time. That is why, in any model of **TM** and **TMbb**, time cannot be said to flow along the imaginary parts of the time continuum, be they the part after the apex in figure 1 or the part before the end-point 1 or the part after the end-point 2 in figure 4.

2. The tense theory of time implied by **TM** is also radically different from those tense theories according to which, as in Fine 2005, there are tensed facts *at the basic level of reality.* For, curiously enough, the system **TM** is formulated in a *tenseless language,* which *does not* contain Prioresque tense operators that would, when prefixed to a sentence, make the related proposition true of false (Prior 1967). Temporal operators are *not* tense operators. Any formula, either true or false, that says something about what happened on an interval of the *real part* of the time continuum contains nothing that could *directly refer* to *tomorrow* or *yesterday*, though $\mathtt{t}_m$ and $\mathtt{t}_n$ may denote intervals that are days such that $\mathtt{t}_n$ is *after* $\mathtt{t}_m$ and $\mathtt{t}_m$ *before* $\mathtt{t}_n$, so that, at $\mathtt{t}_m$, $\mathtt{t}_n$ is *tomorrow*, whereas, at $\mathtt{t}_n$, $\mathtt{t}_m$ is *yesterday.* So, the difference between tenses is not a *direct* consequence of the flow of time. Time cannot be said to be flowing over the real part of the time continuum either, though it can be

said that it *has flown* over it (see the ingenious formulation of Diodorus Chronus in Sextus Empiricus *Adv. Math.* X 85, where he establishes that in reagard to locomotion it cannot be said that "even one thing is in motion (κινείται), but only that it has been in motion (κεκίνηται)"[7]).

3. The reason why it can be said that time *has flown* over the real part of the time continuum is that *every* point of this part of the time continuum *could have been* the apex of a privileged model of **TM** at which the real world history ended, so that the real part represents always the history of the real world as the continuous transition between the privileged **TM** models up to its apex as the boudary between the real and the imaginary part of the time continuum (see Section 6.3). But it is of crucial significance to notice that the apex is the end-instant and not the beginning of the real world history—which can be illustrated by using the **TMbb** variant of **TM**—due to the fact that *modal truths* concern the *in-the-world-inherent modalities*, which determine at the *basic* level *the arrow of time*. The succeeding privileged models contain stored information about the segments of the real world history which ended earlier. Thus, as in the succeeding privileged models it is no longer possible for some events to happen on certain earlier intervals, as it was in the preceding ones, the flow of time has been steadily manifested as the *killing off* of the in-the-world-inherent possibilities (which corresponds exactly to the model that Storrs McCall (2004, pp. 1–19) prefers among several known cosmic models). The truthmakers and the falsemakers of *tensed sentences* in ordinary language are, consequently, *tenseless modal facts* and the way they are *distributed* over the time continuum. Metaphorically speaking, the *arrow* of time can be read off from any model as a *reflection* of the process of killing off of the real possibilities—a *frozen image* of time that *has flown* betwen separate privileged models.

Appendix

The syntax of the system TM

The language of **TM** contains:

- individual constants: $\mathtt{t}_1, \mathtt{t}_2, \ldots, \mathtt{t}_i, \ldots$;
- individual variables: $t_1, t_2, \ldots, t_i, \ldots$;

[7] It should be noted that κεκίνηται is in the *perfect tense*, which in Greek means not only that the motion happened in the past but also that the result of it is completely brought about. In English this is better expressed through the *present perfect tense*, as it is done in the translation we have used (see Sextus Empiricus 2012).

- predicate constants: ε, e, e_1, e_2, ..., e_i, ...;
- relation symbols: $=$, $\prec$, $\succ$, $\lhd$, $\cap$ and $\subset$, for the identity, the precedence, the succession, the abutment, the overlapping and the inclusion relation, respectively;
- connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$;
- quantifiers: $\forall$ and $\exists$;
- modal operators: $\Box$ and $\Diamond$;
- temporal (but not tense) operators: $\{\mathsf{t}_1\}$, $\{\mathsf{t}_2\}$, ..., $\{\mathsf{t}_i\}$, ...;
- auxiliary symbols: (,).

Moreover, we will use $t_n \neq t_m$, $t_n \not\prec t_m$ and $t_n \not\succ t_m$ as abbreviations for $\neg t_n = t_m$, $\neg t_n \prec t_m$ and $\neg t_n \succ t_m$, respectively. With regard to relation symbols, $=$ and $\prec$ can be taken as primitive, while others can be defined in the following way:

Definition 1. $t_m \succ t_n =_{\mathrm{df}} t_n \prec t_m$

Definition 2. $t_m \lhd t_n =_{\mathrm{df}} t_m \prec t_n \wedge \neg\exists t_k(t_m \prec t_k \wedge t_k \prec t_n)$

Definition 3. $t_m \cap t_n =_{\mathrm{df}} t_m \neq t_n \wedge t_m \not\prec t_n \wedge t_n \not\prec t_m \wedge$
$\exists t_k \exists t_l (t_k \lhd t_n \wedge t_k \not\prec t_m \wedge t_m \lhd t_l \wedge t_n \not\prec t_l)$

Definition 4. $t_m \subset t_n =_{\mathrm{df}} t_m \neq t_n \wedge t_m \not\prec t_n \wedge t_n \not\prec t_m \wedge$
$\forall t_k(t_k \lhd t_n \rightarrow t_k \prec t_m) \wedge \forall t_l(t_n \lhd t_l \rightarrow t_m \prec t_l)$

Elementary wffs are:

1. $\mathsf{t}_i R \mathsf{t}_j$, and $t_i R t_j$ where t_i and t_j are individual constants and t_i and t_j are individual variables, and $R \in \{=, \prec, \succ, \lhd, \cap, \subset\}$;
2. $\mathsf{E}(\mathsf{t}_j)$, where E is a predicate constant and t_j is an individual constant or variable;
3. nothing else.

Wffs are:

1. elementary wffs;
2. if A and B are wffs, then $\neg A$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$ are wffs;
3. if A_1, A_2, ..., A_i are wffs, $\bigwedge_{1 \leq i < \omega} A_i$ is also a wff;
4. if A is a wff and t_m an individual variable, $\forall t_m\, A$ and $\exists t_m\, A$ are also wffs;
5. if A is a wff, then $\Diamond A$ and $\Box A$ are also wffs;
6. if A is a wff and t_m an individual symbol, $\{\mathsf{t}_m\}A$ is also a wff;
7. nothing else.

Inference rules

We will use the following inference rules: modus ponens, generalization and temporal necessitation:

$$\frac{\vdash A}{\vdash \forall t_m(\{t_m\}A \to \{t_m\}\Box A)}$$

The axioms of the temporal part of TM

($\mathrm{A}_{\mathbf{T}}1$) $\forall t_n \ t_n \nprec t_n$

($\mathrm{A}_{\mathbf{T}}2$) $\forall t_k \forall t_l \forall t_m \forall t_n((t_k \prec t_m \wedge t_l \prec t_n) \to (t_k \prec t_n \vee t_l \prec t_m))$

($\mathrm{A}_{\mathbf{T}}3$) $\forall t_m \forall t_n(t_m \prec t_n \to (t_m \lhd t_n \vee \exists t_l(t_m \lhd t_l \wedge t_l \lhd t_n)))$

($\mathrm{A}_{\mathbf{T}}4$) $\forall t_k \forall t_l \forall t_m \forall t_n((t_k \lhd t_m \wedge t_k \lhd t_n \wedge t_l \lhd t_m) \to t_l \lhd t_n)$

($\mathrm{A}_{\mathbf{T}}5$) $\forall t_k \forall t_l \forall t_m \forall t_n((t_k \lhd t_l \wedge t_l \lhd t_n \wedge t_k \lhd t_m \wedge t_m \lhd t_n) \to t_l = t_m)$

($\mathrm{A}_{\mathbf{T}}6$) $\forall t_m \exists t_n \ t_m \prec t_n$

($\mathrm{A}_{\mathbf{T}}7$) $\forall t_m \exists t_n \ t_n \prec t_m$

($\mathrm{A}_{\mathbf{T}}8$) $\forall t_m \exists t_n \ t_n \subset t_m$

($\mathrm{A}_{\mathbf{T}}9$) $\forall t_1 \forall t_2 \ldots \forall t_i \ldots (\exists t'(\bigwedge_{1 \leqslant i < \omega} t_i \prec t') \to \exists t''(\bigwedge_{1 \leq i < \omega} t_i \prec t'' \wedge \neg\exists t'''(\bigwedge_{1 \leqslant i < \omega} t_i \prec t''' \wedge t''' \prec t'')))$

($\mathrm{A}_{\mathbf{T}}10$) $\forall t_1 \forall t_2 \ldots \forall t_i \ldots (\exists t'(\bigwedge_{1 \leqslant i < \omega} t_i \succ t') \to \exists t''(\bigwedge_{1 \leqslant i < \omega} t_i \succ t'' \wedge \neg\exists t'''(\bigwedge_{1 \leq i < \omega} t_i \succ t''' \wedge t''' \succ t'')))$

The axioms of the temporal-modal part of TM

The temporal-modal part consists of 17 axiom schemas, with E as a schematic letter for which predicate constants can be substituted. The new predicate and the two new relations that appear in these schemas are defined in the following way, where $Act(t_m)$ means that t_m is actual. Definition 6 says when an interval ends earlier than or simoultaneously with some other interval, while Definition 7 says when the first of some two intervals ends before the other does.

Definition 5. $Act(t_m) =_{\mathrm{df}} \{t_m\}(\mathsf{E}(t_m) \vee \neg\mathsf{E}(t_m))$

Definition 6. $t_m \leq t_n =_{\mathrm{df}} t_m \prec t_n \vee t_m \subset t_n \vee t_m \cap t_n \vee \exists t_k(t_m \lhd t_k \wedge t_n \lhd t_k)$

Definition 7. $t_m < t_n =_{\mathrm{df}} t_m \leq t_n \wedge \neg\exists t_k(t_m \lhd t_k \wedge t_n \lhd t_k)$

($\mathrm{A}_{\mathbf{TM}}1$) $\forall t_m \forall t_n(Act(t_n) \to (\mathsf{E}(t_m) \leftrightarrow \{t_n\}\mathsf{E}(t_m))$

This axiom secures that everything that happened be considered true at any actual world (when reading the equivalence from the left to the right), and that everything that happened can be read off from the truths of any actual world (when reading the equivalence from the right to the left).

(A$_{\mathbf{TM}}$2) $\forall t_m \forall t_n(Act(t_n) \rightarrow (\{t_m\}A \leftrightarrow \{t_n\}\{t_m\}A))$

Concerning the iteration of temporal operators, the axiom provides that all the truths of an actual world hold of it at any other actual world.

(A$_{\mathbf{TM}}$3) $\forall t_n(Act(t_n) \rightarrow (\neg\{t_n\}A \leftrightarrow \{t_n\}\neg A))$

The axiom guarantees the consistency and the completeness of any actual world, where the implication from the right to the left (in the consequent) precludes that both a formula and its negation are true at any actual world, while the implication from the left to the right secures that one of them is.

(A$_{\mathbf{TM}}$4) $\forall t_n((A \rightarrow B) \rightarrow (\{t_n\}A \rightarrow \{t_n\}B))$

The axiom secures that any actual world be closed for the implication, i.e., that if something is true at the world actualized on a certain interval, all its logical consequences must also be true at that world.

(A$_{\mathbf{TM}}$5) $\forall t_n(\{t_n\}A \rightarrow (\{t_n\}B \rightarrow \{t_n\}(A \wedge B)))$

The axiom provides for the validity of temporal adjunction.

(A$_{\mathbf{TM}}$6) $\forall t_m \forall t_n(t_m \subset t_n \rightarrow (\mathsf{E}(t_n) \rightarrow \mathsf{E}(t_m))$

This is the axiom suggested by Prior, which implicitly defines the holistic character of elementary events.

(A$_{\mathbf{TM}}$7) $\forall t_n(Act(t_n) \rightarrow \varepsilon(t_n))$

The axiom states that the occurrence of the specially designated event ε on an interval is a necessary condition for the actuality of that interval.

(A$_{\mathbf{TM}}$8) $\forall t_n(\neg Act(t_n) \rightarrow \neg\mathsf{E}(t_n))$

The axiom states that the occurrence of the specially designated event ε on an interval is a necessary condition for the actuality of that interval.

(A$_{\mathbf{TM}}$9) $\forall t_m \exists t_k \exists t_n(Act(t_k) \wedge t_k \prec t_m \wedge t_m \prec t_n \wedge \neg Act(t_n))$

The axiom says that for any time interval there is an earlier interval on which a world is actualized, and a later one on which no world is actualized.

(A$_{\mathbf{TM}}$10) $\forall t_m(Act(t_m) \rightarrow \exists t_n(t_m \leq t_n \wedge \forall t_k(Act(t_k) \leftrightarrow t_k \leq t_n)))$

The axiom implicitly introduces the absolute presentness as the abutment place of a set of intervals that have no part on which nothing happened and the set of intervals abutting them such that nothing happened on any of their parts.

(A$_{\mathbf{TM}}$11) $\forall t_m(\{t_m\}\Box A \rightarrow \forall t_n((Act(t_n) \wedge t_m \leq t_n) \rightarrow \{t_n\}A))$

The axiom claims that everything necessarily true at an actual world is true at every world actualized simultaneously or later, i.e., that all the worlds actualized simultaneously with or later than the given world are accessible from it.

(A$_{\mathbf{TM}}$12) $\forall t_m(\{t_m\}\Box(A \rightarrow B) \rightarrow (\{t_m\}\Box A \rightarrow \{t_m\}\Box B))$

The axiom states that if at an actual world A is necessarily true and strictly implies B, then B must be necessarily true at that world as well.

(A$_{\mathbf{TM}}$13) $\forall t_m(\{t_m\}\Box A \rightarrow \{t_m\}\Box\Box A)$

The axiom says that everything necessarily true at an actual world is by necessity so, namely, that everything true in all the possible worlds accessible from the given actual world must also hold in every possible world accessible from these worlds.

(A$_{\mathbf{TM}}$14) $\forall t_m(\{t_m\}\mathsf{E}(t_m) \rightarrow \forall t_n((t_m \leq t_n \wedge Act(t_n)) \rightarrow \{t_n\}\Box\mathsf{E}(t_m)))$

(A$_{\mathbf{TM}}$15) $\forall t_m(\{t_m\}\neg\mathsf{E}(t_m) \rightarrow \forall t_n((t_m \leq t_n \wedge Act(t_n)) \rightarrow \{t_n\}\Box\neg\mathsf{E}(t_m)))$

Taken together, the axioms 14 and 15 state the unalterability of the past.

(A$_{\mathbf{TM}}$16) $\forall t_m(Act(t_m) \rightarrow \forall t_n(t_m \prec t_n \rightarrow \{t_m\}(\Diamond\mathsf{E}(t_n) \wedge \Diamond\neg\mathsf{E}(t_n))))$

The axiom says that for any actual world, it holds that it is both possible for an event to happen and not to happen on a later interval, independently of whether there is a world actualized on that later interval or not.

(A$_{\mathbf{TM}}$17) $\forall t_m\forall t_n((Act(t_m) \wedge t_m \cap t_n) \rightarrow (\{t_m\}\Diamond\mathsf{E}(t_n) \leftrightarrow \forall t_k((t_k \subset t_m \wedge t_k \subset t_n) \rightarrow \mathsf{E}(t_k))))$

The axiom says that, in the case of two overlapping intervals, the possibility for obtaining an event on the interval which ends later at the world actualized on the interval with an earlier ending depends on obtaining the event on the intersection of the two intervals.

Some important theorems of TM

(Th$_{\mathbf{TM}}$1) $\forall t_n(Act(t_n) \leftrightarrow \varepsilon(t_n))$

The theorem claims that the occurrence of the designated event ε on an interval is both a necessary and a sufficient condition for the actuality of that interval.

Proof. The necessity is given by (A$_{\mathbf{TM}}$7) and the sufficiency follows by the counterposition from (A$_{\mathbf{TM}}$8). ⊣

(Th$_{\mathbf{TM}}$2) $\forall t_n(\mathsf{E}(t_n) \rightarrow \varepsilon(t_n))$

The theorem says that the occurrence of any event on an interval is accompanied by the occurrence of the designated event.

Proof. Suppose that an event e happens on an interval t_n, i.e., that (1) $\mathsf{e}(\mathsf{t}_n)$ holds. From (1) and (A$_{\mathbf{TM}}$8), it follows that (2) $Act(\mathsf{t}_n)$, which together with (Th$_{\mathbf{TM}}$1) implies that (3) $\varepsilon(\mathsf{t}_n)$. ⊣

(Th$_{\mathbf{TM}}$3) $\forall t_n(\neg Act(t_n) \rightarrow \neg\{t_n\}A)$

The theorem states that non-actual intervals have no history: no truths have been archived on them.

Proof. Suppose that an interval t_n is not actual—(1) $\neg Act(\mathsf{t}_n)$—and that its archive contains a formula A, i.e. (2) $\{\mathsf{t}_n\}A$. Since every formula implies any logical truth, it follows that, for an event e, (3) $A \rightarrow (\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$. As every archive is closed under implication (see (A$_{\mathbf{TM}}$4), (2) and (3) imply (4) $\{\mathsf{t}_n\}(\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$, which means that t_n must be actual, contrary to supposition (1). ⊣

(Th$_{\mathbf{TM}}$4) $\forall t_n(\{t_n\}A \rightarrow \neg\{t_n\}\neg A)$

The theorem claims the qualified auto-duality of the temporal operator.

Proof. It is straightforwardly derivable from (A$_{\mathbf{TM}}$3). ⊣

(Th$_{\mathbf{TM}}$5) $\forall t_n(\mathsf{E}(t_n) \leftrightarrow \{t_n\}\mathsf{E}(t_n))$

The theorem says that an event happened on an interval if and only if it is registered in the interval's archive that it did.

Proof. If t_n is an actual interval, the theorem follows directly from ($\mathrm{A}_{\mathbf{TM}}1$). If not, then, for a given event e, neither $\mathsf{e}(\mathsf{t}_n)$ (see $\mathrm{A}_{\mathbf{TM}}8$) nor $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ (see $\mathrm{Th}_{\mathbf{TM}}3$) are true, which means that $\mathsf{e}(\mathsf{t}_n)$ and $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$ are equivalent. ⊣

($\mathrm{Th}_{\mathbf{TM}}6$) $\quad \forall t_n(\{t_n\}(A \vee B) \leftrightarrow (\{t_n\}A \vee \{t_n\}B))$

So the temporal operator distributes over disjunction.

Proof. Firstly, we shall prove the implication from the left to the right. Suppose that, for an interval t_n, (1) $\{\mathsf{t}_n\}(A \vee B)$ holds and, further, that (2) $\neg\{\mathsf{t}_n\}A$ and (3) $\neg\{\mathsf{t}_n\}B$ are true as well. According to ($\mathrm{Th}_{\mathbf{TM}}3$), (1) implies that t_n is actual, i.e., that (4) $Act(\mathsf{t}_n)$. By applying ($\mathrm{A}_{\mathbf{TM}}3$) to (4) and (2), we get (5) $\{\mathsf{t}_n\}\neg A$, and, analogously, to (4) and (3), (6) $\{\mathsf{t}_n\}\neg B$. Temporal adjunction (see $\mathrm{A}_{\mathbf{TM}}5$) of (5) and (6) gives (7) $\{\mathsf{t}_n\}(\neg A \wedge \neg B)$, which together with the logical truth $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$ and ($\mathrm{A}_{\mathbf{TM}}4$) implies (8) $\{\mathsf{t}_n\}\neg(A \vee B)$. From (4) and (8), it follows by ($\mathrm{A}_{\mathbf{TM}}3$) that (9) $\neg\{\mathsf{t}_n\}(A \vee B)$, which contradicts the supposition (1). Therefore, (2) and (3) cannot both be true, so either $\{\mathsf{t}_n\}A$ or $\{\mathsf{t}_n\}B$ holds. Now, to prove the reverse implication, suppose that (10) $\{\mathsf{t}_n\}A \vee \{\mathsf{t}_n\}B$ is true of an interval t_n. The first disjunct in (10) implies (11) $\{\mathsf{t}_n\}(A \vee B)$ due to the truth of $A \rightarrow (A \vee B)$ and ($\mathrm{A}_{\mathbf{TM}}4$), while the second leads to the same conclusion via $B \rightarrow (A \vee B)$ and ($\mathrm{A}_{\mathbf{TM}}4$). ⊣

($\mathrm{Th}_{\mathbf{TM}}7$) $\quad \forall t_n(\{t_n\}(A \wedge B) \leftrightarrow (\{t_n\}A \wedge \{t_n\}B))$

So the temporal operator distributes over conjunction.

Proof. The left-to-right implication is directly derivable from ($\mathrm{A}_{\mathbf{TM}}4$) and that a conjunction implies any of its conjuncts, while the converse implication is a straightforward consequence of ($\mathrm{A}_{\mathbf{TM}}5$). ⊣

($\mathrm{Th}_{\mathbf{TM}}8$) $\quad \forall t_m \forall t_n((t_m \leq t_n \wedge Act(t_n)) \rightarrow Act(t_m))$

The theorem claims that any interval that ends simultaneously with or before an actual interval must be actual itself.

Proof. Let t_m and t_n be two intervals of which (1)$\mathsf{t}_m \leq \mathsf{t}_n \wedge Act(\mathsf{t}_n)$ is true. The second conjunct in (1), together with ($\mathrm{A}_{\mathbf{TM}}10$), implies that there is an interval t_k such that (2) $\mathsf{t}_n \leq \mathsf{t}_k \wedge \forall \mathsf{t}_l(Act(\mathsf{t}_l) \leftrightarrow \mathsf{t}_l \leq \mathsf{t}_k)$. From the temporal component of the system **TM** and (*Def.* 2), it follows that the binary relation on the domain of intervals $\leq$ must be transitive, which, together with the first conjuncts in (1) and (2), gives (3) $\mathsf{t}_m \leq \mathsf{t}_k$. Considering (3) in relation to the second conjunct in (2), we infer that (4) $Act(\mathsf{t}_m)$. ⊣

(Th$_{\mathbf{TM}}$9) $\forall t_n(\{t_n\}\Box A \rightarrow \{t_n\}A)$

The theorem states that if A is necessarily true of an actual world, it must be true of that world.

Proof. This is a special instance of (A$_{\mathbf{TM}}$11). ⊣

(Th$_{\mathbf{TM}}$10) $\forall t_n(\{t_n\}A \rightarrow \forall t_m(t_m \leq t_n \rightarrow \{t_m\}\Diamond A))$

The theorem says that if an interval's archive contains a formula A, then every archive that belongs to an interval with an earlier ending includes the possibility of A.

Proof. Suppose that (1) $\{\mathsf{t}_n\}A$ is true for an interval t_n. According to (Th$_{\mathbf{TM}}$3), t_n must be actual, i.e. (2) $Act(\mathsf{t}_n)$. Let t_m be an interval such that (3) $\mathsf{t}_m \leq \mathsf{t}_n$. From (Th$_{\mathbf{TM}}$8), it follows that (4) $Act(\mathsf{t}_m)$. Now, suppose that (5) $\neg\{\mathsf{t}_m\}\Diamond A$. Then, (4) and (5) imply that (6) $\{\mathsf{t}_m\}\Box\neg A$, which, together with (A$_{\mathbf{TM}}$11), gives (7) $\{\mathsf{t}_n\}\neg A$, which contradicts supposition (1). ⊣

(Th$_{\mathbf{TM}}$11) $\forall t_n(Act(t_n) \rightarrow \{t_n\}\neg(\Diamond\mathsf{E}(t_n) \wedge \Diamond\neg\mathsf{E}(t_n)))$

If a world is actualized on an interval, its archive cannot contain the proposition that it is both possible that a certain event occurs and that it does not occur on that interval.

Proof. Consider an event e and suppose that an interval t_n is actual. From (*Def.* 1), it follows (1) $\{\mathsf{t}_n\}(\mathsf{e}(\mathsf{t}_n) \vee \neg\mathsf{e}(\mathsf{t}_n))$. By (Th$_{\mathbf{TM}}$6), we get (2) $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n) \vee \{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$. Now, suppose that the first disjunct in (2)—(3) $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$—holds. (A$_{\mathbf{TM}}$14) gives (4) $\{\mathsf{t}_n\}\Box\mathsf{e}(\mathsf{t}_n)$. Since (5) $\Box\mathsf{e}(\mathsf{t}_n) \rightarrow (\Box\mathsf{e}(\mathsf{t}_n) \vee \Box\neg\mathsf{e}(\mathsf{t}_n))$, by (A$_{\mathbf{TM}}$4), from (4) and (5), we get (6) $\{\mathsf{t}_n\}(\Box\mathsf{e}(\mathsf{t}_n) \vee \Box\neg\mathsf{e}(\mathsf{t}_n))$. So, if (3) is true, (6) must be true as well. Suppose now that the second disjunct in (2) holds, i.e., that (7) $\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$. Then, by (A$_{\mathbf{TM}}$15) we get (8) $\{\mathsf{t}_n\}\Box\neg\mathsf{e}(\mathsf{t}_n)$. As (9) $\Box\neg\mathsf{e}(\mathsf{t}_n) \rightarrow (\Box\mathsf{e}(\mathsf{t}_n) \vee \Box\neg\mathsf{e}(\mathsf{t}_n))$, (A$_{\mathbf{TM}}$4) gives (6) again. So, independently of which disjunct in (2) is true, (6) holds. Finally, by substituting $\Box A$ with $\neg\Diamond\neg A$ in (6) and by applying one of De Morgan's laws, we derive (9) $\{\mathsf{t}_n\}\neg(\Diamond\mathsf{e}(\mathsf{t}_n) \wedge \Diamond\neg\mathsf{e}(\mathsf{t}_n))$. ⊣

(Th$_{\mathbf{TM}}$12) $\forall t_m\forall t_n((Act(t_m) \wedge \neg Act(t_n)) \rightarrow (\{t_m\}\Diamond\mathsf{E}(t_n) \leftrightarrow \forall t_k((t_k \subset t_m \wedge t_k \subset t_n) \rightarrow \mathsf{E}(t_k))))$

The theorem says that at a world actualized on a given interval any event might happen on an interval on which no world has yet actualized, provided that the event has already happened on their intersection.

PROOF. Suppose that t_m is actual and t_n not, i.e. (1) $Act(\mathsf{t}_m) \wedge \neg Act(\mathsf{t}_n)$. From the first conjunct in (1) and ($\mathrm{A}_{\mathbf{TM}}10$), it follows that there is an interval t_k such that (2) $\mathsf{t}_m \leq \mathsf{t}_k$ and (3) $\forall t_l(Act(t_l) \leftrightarrow t_l \leq \mathsf{t}_k)$. Now, (3), with the second conjunct in (1), gives (4) $\mathsf{t}_k < \mathsf{t}_n$. If t_m and t_n have an empty intersection, the theorem follows from (1), (4) and ($\mathrm{A}_{\mathbf{TM}}16$); if not, it can be derived from the same suppositions and ($\mathrm{A}_{\mathbf{TM}}17$). ⊣

($\mathrm{Th}_{\mathbf{TM}}13$) $\forall t_k \forall t_m \forall t_n((t_k \prec t_m \wedge t_m \prec t_n \wedge Act(t_n) \wedge \neg \mathsf{E}(t_m)) \rightarrow (\{t_n\}\Box\neg\mathsf{E}(t_m) \wedge \{t_n\}\{t_k\}\Diamond\mathsf{E}(t_m)))$

This theorem reveals the conditional character of necessity, which medieval logicians called *necessitas per accidens*, by showing how real possibilities are cancelled by the flow of time.

PROOF. Suppose that the antecedent is true for intervals t_k, t_m and t_n and for an event e. According to the temporal component of **TM**, from (1) $\mathsf{t}_k \prec \mathsf{t}_m \wedge \mathsf{t}_m \prec \mathsf{t}_n$, it follows that (2) $\mathsf{t}_k \prec \mathsf{t}_n$. Since t_n is actual, t_k and t_m must be actual as well (see $\mathrm{Th}_{\mathbf{TM}}8$). So, ($\mathrm{A}_{\mathbf{TM}}1$) and the third conjunct of the antecedent, $\neg\mathsf{e}(\mathsf{t}_m)$, give (3) $\neg\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_m)$. As t_m is actual, its archive must be complete (see $\mathrm{A}_{\mathbf{TM}}3$), from which it follows that (4) $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_m)$. Now, on one side, since $\mathsf{t}_m \prec \mathsf{t}_n$ and t_n is actual, by ($\mathrm{A}_{\mathbf{TM}}15$) we get (5) $\{\mathsf{t}_n\}\Box\neg\mathsf{e}(\mathsf{t}_m)$. On the other side, as t_k is actual and $\mathsf{t}_k \prec \mathsf{t}_m$, ($\mathrm{A}_{\mathbf{TM}}16$) gives (6) $\{\mathsf{t}_k\}(\Diamond\mathsf{e}(\mathsf{t}_m) \wedge \Diamond\neg\mathsf{e}(\mathsf{t}_m))$. Further, from (7) $(\Diamond\mathsf{e}(\mathsf{t}_m) \wedge \Diamond\neg\mathsf{e}(\mathsf{t}_m)) \rightarrow \Diamond\mathsf{e}(\mathsf{t}_m)$ and ($\mathrm{A}_{\mathbf{TM}}4$) applied to the record corresponding to t_k, we get (8) $\{\mathsf{t}_k\}\Diamond\mathsf{e}(\mathsf{t}_m)$. Finally, ($\mathrm{A}_{\mathbf{TM}}2$) gives (9) $\{\mathsf{t}_n\}\{\mathsf{t}_k\}\Diamond\mathsf{e}(\mathsf{t}_m)$. ⊣

($\mathrm{Th}_{\mathbf{TM}}14$) $\forall t_m \forall t_n((Act(t_m) \wedge \neg Act(t_n)) \rightarrow \neg\{t_m\}\{t_n\}A)$

So no archive corresponding to an actual interval contains a statement about recording anything in a non-actual interval's archive.

PROOF. Let t_m and t_n be two intervals such that a world has been actualized on the former but not on the latter, i.e. (1) $Act(\mathsf{t}_m) \wedge \neg Act(\mathsf{t}_n)$. The second conjunct in (1) implies that (2) $\neg\{\mathsf{t}_n\}A$ by ($\mathrm{Th}_{\mathbf{TM}}3$). An application of ($\mathrm{A}_{\mathbf{TM}}2$) to (2), together with the first conjunct in (1), gives (3) $\neg\{\mathsf{t}_m\}\{\mathsf{t}_n\}A$. ⊣

($\mathrm{Th}_{\mathbf{TM}}15$) $\forall t_m \forall t_n((t_m \prec t_n \wedge \mathsf{E}(t_n)) \rightarrow \{t_n\}\{t_m\}\mathsf{E}(t_n))$

The theorem claims that if an event happens on an interval, it is true at that interval that at an earlier interval it is true that the event happens on the later one.

PROOF. Suppose that (1) $\mathsf{t}_m \prec \mathsf{t}_n \wedge \mathsf{e}(\mathsf{t}_n)$ holds for intervals t_m and t_n and an event e. From the second conjunct in (1) and ($\mathrm{A}_{\mathbf{TM}}8$), it follows that (2) $Act(\mathsf{t}_n)$, which, together with ($\mathrm{Th}_{\mathbf{TM}}8$), implies that (3) $Act(\mathsf{t}_m)$. By applying ($\mathrm{A}_{\mathbf{TM}}1$) to (3) and $\mathsf{e}(\mathsf{t}_n)$, we get (4) $\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$, and from (2), (4) and ($\mathrm{A}_{\mathbf{TM}}2$), we further derive (5) $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\mathsf{e}(\mathsf{t}_n)$. ⊣

($\mathrm{Th}_{\mathbf{TM}}16$) $\quad \forall t_m \forall t_n((Act(t_m) \wedge \neg Act(t_n)) \to \{t_m\}\neg\mathsf{E}(t_n))$

The theorem states that it is true at an actual world that nothing happens on a non-actual interval.

PROOF. Let t_m and t_n be two intervals such that (1) $Act(\mathsf{t}_m)$ and (2) $\neg Act(\mathsf{t}_n)$. By ($\mathrm{A}_{\mathbf{TM}}8$), (2) implies (3) $\neg\mathsf{e}(\mathsf{t}_n)$ for a given event e, from which, together with (1), it follows that (4) $\{\mathsf{t}_m\}\neg\mathsf{e}(\mathsf{t}_n)$ via ($\mathrm{A}_{\mathbf{TM}}1$). ⊣

($\mathrm{Th}_{\mathbf{TM}}17$) $\quad \forall t_m \forall t_n((Act(t_m) \wedge \neg Act(t_n)) \to \neg\{t_m\}\{t_n\}\mathsf{E}(t_n))$

The theorem says that no archive corresponding to an actual interval contains a statement about recording of the occurrence of an event on a non-actual interval in its archive.

PROOF. This is a direct corollary of ($\mathrm{Th}_{\mathbf{TM}}14$). ⊣

($\mathrm{Th}_{\mathbf{TM}}18$) $\quad \forall t_m \forall t_n((Act(t_m) \wedge \neg Act(t_n)) \to \neg\{t_m\}\{t_n\}\neg\mathsf{E}(t_n))$

The theorem claims that no archive corresponding to an actual interval contains a statement about the recording of the non-occurrence of an event on a non-actual interval in that interval's archive.

PROOF. It is a straightforward consequence of ($\mathrm{Th}_{\mathbf{TM}}14$). ⊣

($\mathrm{Th}_{\mathbf{TM}}19$) $\quad \forall t_m \forall t_n((t_m \prec t_n \wedge \mathsf{E}(t_n)) \to \{t_n\}\{t_m\}\{t_n\}\mathsf{E}(t_n))$

The theorem states that if an event happens on an interval, it is true at that interval that at an earlier interval it is true at the later one that the event happens on it.

PROOF. Suppose that t_m and t_n are intervals such that (1) $\mathsf{t}_m \prec \mathsf{t}_n$ and that an event e happened on t_n, i.e. (2) $\mathsf{e}(\mathsf{t}_n)$. ($\mathrm{A}_{\mathbf{TM}}8$), in relation to (2), gives us (3) $Act(\mathsf{t}_n)$, which, by (1) and ($\mathrm{Th}_{\mathbf{TM}}8$), implies (4) $Act(\mathsf{t}_m)$. From (2) and ($\mathrm{Th}_{\mathbf{TM}}5$), we get (5) $\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$. By two successive applications of ($\mathrm{A}_{\mathbf{TM}}2$), we finally derive (6) $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\mathsf{e}(\mathsf{t}_n)$. ⊣

$(\mathrm{Th_{\mathbf{TM}}}20)$ $\forall t_m \forall t_n((t_m \prec t_n \wedge Act(t_n) \wedge \neg \mathsf{E}(t_n)) \rightarrow \{t_n\}\{t_m\}\{t_n\}\neg \mathsf{E}(t_n))$

The theorem says that if an event failed to happen on an actual interval, it is true at that interval that at an earlier interval it is true that at the later one the event failed to happen.

PROOF. From the second and the third conjunct in the antecedent, we first derive (1) $\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$, by using $(\mathrm{A_{\mathbf{TM}}}1)$ and $(\mathrm{A_{\mathbf{TM}}}3)$, and then, just as in the proof of the previous theorem, by two applications of $(\mathrm{A_{\mathbf{TM}}}2)$, (2) $\{\mathsf{t}_n\}\{\mathsf{t}_m\}\{\mathsf{t}_n\}\neg\mathsf{e}(\mathsf{t}_n)$. ⊣

$(\mathrm{Th_{\mathbf{TM}}}21)$ $\forall t_m(\{t_m\}\Diamond\exists t_n \mathsf{E}(t_n) \rightarrow \{t_m\}\exists t_n \Diamond \mathsf{E}(t_n))$

The theorem is a temporal variant of the Barcan formula: it states that if it is true at a world actualized on an interval that there could be an interval on which an event happens, then, at the same actual world, it is also true that there is an interval on which the event could happen.

PROOF. Suppose that there is no world actualized on t_m, i.e., that (1) $\neg Act(\mathsf{t}_m)$. By $(\mathrm{Th_{\mathbf{TM}}}3)$, it follows, from (1), that (2) $\neg\{\mathsf{t}_m\}\Diamond\exists t_n\mathsf{E}(t_n)$, which straightforwardly implies (3) $\{\mathsf{t}_m\}\Diamond\exists t_n\mathsf{E}(t_n) \rightarrow \{\mathsf{t}_m\}\exists t_n\Diamond\mathsf{E}(t_n)$. Suppose now that t_m is actual, i.e. (4) $Act(\mathsf{t}_m)$. Then, $(\mathrm{A_{\mathbf{T}}}6)$ gives that there is an interval, call it t_k, such that (5) $\mathsf{t}_m \prec \mathsf{t}_k$. From (5) and $(\mathrm{A_{\mathbf{TM}}}16)$, we get (6) $\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_k)$. As (7) $\Diamond\mathsf{e}(\mathsf{t}_k) \rightarrow \exists t_n\Diamond\mathsf{e}(t_n)$ is a logical truth, it is implied by any formula, therefore (8) $\varepsilon(\mathsf{t}_m) \rightarrow (\Diamond\mathsf{e}(\mathsf{t}_k) \rightarrow \exists t_n\Diamond\mathsf{e}(t_n))$ holds. From (4), it follows, by $(\mathrm{Th_{\mathbf{TM}}}1)$, that (9) $\varepsilon(\mathsf{t}_m)$ must be true, which further gives (10) $\{\mathsf{t}_m\}\varepsilon(\mathsf{t}_m)$ via $(\mathrm{Th_{\mathbf{TM}}}5)$. (10) and $(\mathrm{A_{\mathbf{TM}}}4)$ together imply (11) $\{\mathsf{t}_m\}(\Diamond\mathsf{e}(\mathsf{t}_k) \rightarrow \exists t_n\Diamond\mathsf{e}(t_n))$. From (6) and (11), it can be derived that (12) $\{\mathsf{t}_m\}\exists t_n\Diamond\mathsf{e}(t_n)$, which implies (3). ⊣

$(\mathrm{Th_{\mathbf{TM}}}22)$ $\forall t_m(\{t_m\}\exists t_n\Diamond\mathsf{E}(t_n) \rightarrow \{t_m\}\Diamond\exists t_n\mathsf{E}(t_n))$

The theorem is a temporal variant of the converse Barcan formula: it claims that if it is true at a world actualized on an interval that there is an interval on which an event could happen, then, at the same actual world, it is also true that there could be an interval on which the event happens.

PROOF. Suppose that t_m is not an actual interval, i.e. (1) $\neg Act(\mathsf{t}_m)$. (1) implies, via $(\mathrm{Th_{\mathbf{TM}}}3)$, (2) $\neg\{\mathsf{t}_m\}\exists t_n\Diamond\mathsf{E}(t_n)$, whose direct consequence is (3) $\{\mathsf{t}_m\}\exists t_n\Diamond\mathsf{E}(t_n) \rightarrow \{\mathsf{t}_m\}\Diamond\exists t_n\mathsf{E}(t_n)$. Now suppose that (4) $Act(\mathsf{t}_m)$ holds. From $(\mathrm{A_{\mathbf{T}}}6)$, it follows that there is an interval t_k such that

(5)$\mathsf{t}_m \prec \mathsf{t}_k$. (5) and ($\mathrm{A}_{\mathbf{TM}}16$) imply (6) $\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_k)$. (7) $\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k)$ is a logical truth. (4) and ($\mathrm{Th}_{\mathbf{TM}}1$) give (8) $\varepsilon(\mathsf{t}_m)$, from which (9) $\{\mathsf{t}_m\}(\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k))$ can be derived by ($\mathrm{Th}_{\mathbf{TM}}5$), ($\mathrm{A}_{\mathbf{TM}}4$) and (10) $\varepsilon(\mathsf{t}_m) \rightarrow (\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k))$. By applying the rule TempNec to (7), we obtain (11) $\{\mathsf{t}_m\}(\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k)) \rightarrow \{\mathsf{t}_m\}\Box(\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k))$. From (9) and (11), by MP, we get (12) $\{\mathsf{t}_m\}\Box(\neg\exists t_n \mathsf{e}(t_n) \rightarrow \neg\mathsf{e}(\mathsf{t}_k))$. Suppose that (13) $\neg\{\mathsf{t}_m\}\Diamond\exists t_n \mathsf{e}(t_n)$ holds. Via (4) and ($\mathrm{A}_{\mathbf{TM}}3$), (13) implies (14) $\{\mathsf{t}_m\}\Box\neg\exists t_n \mathsf{e}(t_n)$. From (12) and (14), it follows, by ($\mathrm{A}_{\mathbf{TM}}12$), that (15) $\{\mathsf{t}_m\}\Box\neg\mathsf{e}(\mathsf{t}_k)$, which, via (4) and ($\mathrm{A}_{\mathbf{TM}}3$), implies (16) $\neg\{\mathsf{t}_m\}\Diamond\mathsf{e}(\mathsf{t}_k)$. Finally, (16) and (6) together yield contradiction, which completes the reductio of (13). ⊣

The proof that the set of equivalence classes of TM models with the same apex is continuously ordered

Let $\sim$ be a binary relation on the set of intervals defined in the following way:

$$t_m \sim t_n =_{\mathrm{df}} \exists t_k (t_m \lhd t_k \wedge t_n \lhd t_k),$$

where $\lhd$ is the abutment relation formally defined above. Two intervals stand in the relation $\sim$ if and only if they have a common ending. It can easily be shown that $\sim$ is an equivalence relation. Its reflexivity follows from the fact that for every interval there is an interval which abuts it. Axiom ($\mathrm{A}_{\mathbf{T}}6$) says that for any interval t_m there is a later interval t_n. According to ($\mathrm{A}_{\mathbf{T}}3$), t_n either itself abuts t_m or there is an interval t_l, which abuts t_m and is abutted by t_n. The symmetry of the $\sim$ relation is a straightforward consequence of the commutativity of conjunction. The transitivity follows from the axiom ($\mathrm{A}_{\mathbf{T}}4$): if $\mathsf{t}_m \sim \mathsf{t}_n$ and $\mathsf{t}_n \sim \mathsf{t}_l$, there are intervals t_k and t_p such that $\mathsf{t}_m \lhd \mathsf{t}_k, \mathsf{t}_n \lhd \mathsf{t}_k, \mathsf{t}_n \lhd \mathsf{t}_p$ and $\mathsf{t}_l \lhd \mathsf{t}_p$. The first three relations imply, via ($\mathrm{A}_{\mathbf{T}}4$), $\mathsf{t}_m \lhd \mathsf{t}_p$, which means that t_p abuts both t_m and t_l, and that, therefore, $\mathsf{t}_m \sim \mathsf{t}_l$. As an equivalence relation, $\sim$ divides the set of intervals I into equivalence classes, which are members of the corresponding quotient set $I/\sim$.

Now, in every **TM**-model the actuality predicate has to be interpreted as a non-empty set of intervals due to the axiom ($\mathrm{A}_{\mathbf{TM}}9$). The presence of a single actual interval is, according to the axiom ($\mathrm{A}_{\mathbf{TM}}10$), sufficient to determine the moment of absolute presentness. If a world has actualized on at least one interval, then there must be an interval – call it t_0 – such that all and only those intervals that end simultaneously with it or earlier are actual: the border point of t_0 is the apex of the world history. The very instant of presentness can be defined as a class of intervals which end when t_0 does, that is, $0 = [\mathsf{t}_0]$, where $[\mathsf{t}_0] \in I/\sim$. In any **TM**-model $\mathfrak{M}$, $0^{\mathfrak{M}}$ is a non-empty proper subset of $Act^{\mathfrak{M}}$. On the set of **TM**-models with a certain domain D, $\mathcal{M}$, we can define the binary relation $\approx$ in the following way: $\mathfrak{M}_i \approx \mathfrak{M}_j$ if and only if $0^{\mathfrak{M}_i} = 0^{\mathfrak{M}_j}$. It is obvious that $\approx$ is an equivalence relation. It partitions the set $\mathcal{M}$ into equivalence classes such that models in each equivalence class have the same dividing point between actualized and not yet actualized part of the time continuum. Just as every **TM**-model can be said to pick out a particular instant as an end of a certain real world history, the equivalence class to which the model belongs is best understood as the representation of that instant regardless of the history that has paved the way to it.

The quotient set $\mathcal{M}/\approx$, generated by the relation $\approx$ is linearly ordered by the relation $\leq_{\mathcal{M}}$ defined as follows: an equivalence class X stands in the relation $\leq_{\mathcal{M}}$ to an equivalence class Y if and only if the instant of presentness of any representative $\mathfrak{M}$ of X, $0^{\mathfrak{M}}$, is a subset of the set of actual intervals of any representative $\mathfrak{N}$ of Y, $Act^{\mathfrak{N}}$. If we denote the set of instants with $\mathcal{I}$, and the standard order on that set with $\leq_{\mathcal{I}}$, the structure $(\mathcal{M}/\approx, \leq_{\mathcal{M}})$ is isomorphic to the structure $(\mathcal{I}, \leq_{\mathcal{I}})$. As the set of instants is not only linearly but also continuously ordered, so must be the quotient set $\mathcal{M}/\approx$.

TMbb as a variant of TM adjusted to the Big Bang model of the universe

1. To the original system we add a definition of the binary relation $\lesssim$ on the set of intervals such that two intervals stand in this relation if and only if they have a common beginning or the first of the intervals begins earlier:

$$(\textit{Def. 8})\ \ t_m \lesssim t_n =_{\mathrm{df}} t_m \prec t_n \vee t_n \subset t_m \vee t_m \cap t_n \vee \exists t_k (t_k \lhd t_m \wedge t_k \lhd t_n)$$

2. Axioms ($\mathrm{A}_{\mathbf{TMbb}}1$) through ($\mathrm{A}_{\mathbf{TMbb}}8$) remain the same as the first eight axioms of **TM**.

3. Two new axioms, the ninth and the tenth axiom of **TMbb**, express the assumptions relevant for the Big Bang theory:

($\mathrm{A}_{\mathbf{TMbb}}9$) $\forall t_m \exists t_k \exists t_n (\neg Act(t_k) \wedge t_k \prec t_m \wedge t_m \prec t_n \wedge \neg Act(t_n))$:

($\mathrm{A}_{\mathbf{TMbb}}10$) $\forall t_m (Act(t_m) \rightarrow \exists t_k \exists t_n (t_k \lesssim t_m \wedge t_m \leq t_n \wedge \forall t_l (Act(t_l) \leftrightarrow (t_k \lesssim t_l \wedge t_l \leq t_n))))$.

4. Axioms ($A_{\mathbf{TMbb}}11$) through ($A_{\mathbf{TMbb}}17$) are identical to the corresponding axioms of **TM**.

In the system **TMbb** the real world history is taken to be continuous, just as in **TM**, but it has the absolute beginning.

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