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Applications of Relating Semantics. From non-classical logics to philosophy of science

Abstract. Here, we discuss logical, philosophical and technical problems associated to relating logic and relating semantics. To do so, we proceed in three steps. The first step is devoted to providing an introduction to both relating logic and relating semantics. We discuss this problem on the example of different languages. Second, we address some of the main research directions and their philosophical applications to non-classical logics, particularly to connexive logics. Third, we discuss some technical problems related to relating semantics, and its application to philosophy of science, language and pragmatics.

Keywords: relating logic; relating semantics; non-classical logic; Boolean connexive logic; modal Boolean connexive logic; philosophy of science; pragmatics

1. Introduction: relating semantics and relating logic

In the paper [Jarmużek and Paoli, 2021] the term *Relating Logic* (henceforth, RL) is defined as a term describing logic of relating connectives — similarly as Modal Logic is a logic of modal operators.[1]

In turn, the basic idea behind a relating connectives is that the logical value of a given complex proposition is the result of two things:

(i) the logical values of the main components of this complex proposition; supplemented with

(ii) a valuation of the relation between these components.

[1] However, it is worth noting tha this definition was already considered in [Jarmużek, 2021; Jarmużek and Klonowski, 2021, submitted].

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The element (ii) is a formal representation of an intensional relation that emerges from the connection of several simpler propositions into one more complex proposition.

Speaking formally, let $\varphi_1, \ldots, \varphi_n$ be propositions with some fixed logical values and let c be an n-ary relating connective. Then the logical value of complex sentence $\mathsf{c}(\varphi_1, \ldots, \varphi_n)$ depends not only on the logical values of $\varphi_1, \ldots, \varphi_n$, but additionally on the value of the connection between $\varphi_1, \ldots, \varphi_n$.

So, it depends on an additional valuation of pairs (n-tuples) that is the part of the overall process of evaluation of the logical values of complex propositions built with relating connectives. This way we can form logical systems to deal with reasoning about non-logical relationships.[2]

On the other hand, we can look at when the semantics for a given language is called *relating semantics*. A semantics for the language is a *relating semantics* [see Jarmużek and Paoli, 2021] iff at least for one connective c_i the valuation of all complex propositions of the form $c_i(\varphi_1, \ldots, \varphi_j)$, where j is the arity of c_i, in a world w requires not only valuations of pairs $(\varphi_1, w), \ldots, (\varphi_j, w)$ in some set of logical values, but also a valuation of j-tuples $((\varphi_1, \ldots, \varphi_j), w)$ in some domain for values for connections [see Jarmużek, 2021; Jarmużek and Klonowski, 2021]. A valuation of j-tuples $((\varphi_1, \ldots, \varphi_j), w)$ can in a formal semantics represent various logical or non-logical relationships between $\varphi_1, \ldots, \varphi_j$ in a world w, for example:

- content relationships, for example, the relatedness relation,
- analytical relationships,
- causalities,
- temporal orderings,
- preference orderings,
- logical consequences of some logic,

among many others.

A function with a co-domain containing values for connections is used to evaluate either a relationship between $\varphi_1, \ldots, \varphi_j$ or a relationship between some objects to which we refer by means of $\varphi_1, \ldots, \varphi_j$ — for example, events, facts or states of affairs — in the relating semantics.

[2] Some interesting cases are described, for example, in [Jarmużek and Paoli, 2021; Jarmużek and Klonowski, 2021; Paoli, 2007; Ledda et al., 2019], mainly in the implicational fragment of the defined language.

To move to a more operational level, let us define a quite rich relating language. This is a language of Classical Mono-Relating Logics (shortly CMRL).[3] Let the language consist of countably many propositional variables Var, *classical connectives*: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, relating counterparts of binary classical connectives (*standard binary relating connectives*): $\wedge^{\mathrm{w}}$, $\vee^{\mathrm{w}}$, $\rightarrow^{\mathrm{w}}$, $\leftrightarrow^{\mathrm{w}}$ and parentheses), (.[4]

The set of all formulas is defined in the standard way and denoted by $\mathsf{For}^{\mathrm{w}}$. By For we denote the set of formulas of Classical Propositional Logic (so built only with Var, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and the brackets). This set is usually set to define various non-classical logics. Obviously we have: $\mathsf{For} \subset \mathsf{For}^{\mathrm{w}}$. By **CPL** we denote the set of all tautologies of Classical Propositional Logic included in For.

We assume also the set of substitution functions. By *substitution* we mean any function $\mathsf{s}\colon \mathsf{For}^{\mathrm{w}} \longrightarrow \mathsf{For}^{\mathrm{w}}$ preserving structure of propositions, so for any $\varphi, \psi \in \mathsf{For}^{\mathrm{w}}$:

- $\mathsf{s}(\neg\varphi) = \neg\mathsf{s}(\varphi)$
- $\mathsf{s}(\varphi * \psi) = \mathsf{s}(\varphi) * \mathsf{s}(\psi)$, where $*$ is a binary connective.

The set of all substitutions will be denoted by Su. A subset X of $\mathsf{For}^{\mathrm{w}}$ is *closed under substitutions* iff $\mathsf{s}(X) \subseteq X$, for any $\mathsf{s} \in \mathsf{Su}$.

We assume in addition that $\boldsymbol{R} = \{R\colon R \subseteq \mathsf{For}^{\mathrm{w}} \times \mathsf{For}^{\mathrm{w}}\}$. So $\boldsymbol{R}$ is the class of all binary relations defined on $\mathsf{For}^{\mathrm{w}}$. Let $R \subseteq \mathsf{For}^{\mathrm{w}} \times \mathsf{For}^{\mathrm{w}}$. By $\sim R$ we understand the complement of R, so $\sim R = \{\langle\varphi, \psi\rangle\colon \varphi, \psi \in \mathsf{For}^{\mathrm{w}}$ and $\langle\varphi, \psi\rangle \notin R\}$. Now we can introduce the notion of model of CMRL.

A *model* of CMRL is an ordered pair $\langle v, R\rangle$ such that:

- $v \in \{1, 0\}^{\mathsf{Var}}$ is a valuation of propositional variables
- $R \subseteq \mathsf{For}^{\mathrm{w}} \times \mathsf{For}^{\mathrm{w}}$ is a binary relation.

A valuation v (a relation R) of model $\mathfrak{M}$ is denoted by $v_{\mathfrak{M}}$ (resp. $R_{\mathfrak{M}}$). A *class of all models over* $\mathbf{Q} \subseteq \boldsymbol{R}$ is the class of all models over R, for every $R \in \mathbf{Q}$. Such a class of models is denoted by $\boldsymbol{M}\mathbf{Q}$. The class of

[3] The language was extensively examined in [Klonowski, 2019; Jarmużek and Klonowski, submitted]. Here we discuss some of the notions and issues presented there.

[4] Somebody may want to add a relating negation which requires going beyond the mono-relating approach. On the basis of CMRLs, we can simulate it by defining: $\varphi \rightarrow^{\mathrm{w}} \bot$, where $\bot$ is the falsity constant [see more proposals in Jarmużek, 2021]. In this context, there is also an interesting discussion about the incorporation of relation R into the object language [see Estrada-González et al., 2021].

all models is denoted by $\boldsymbol{MR}$. We define now the truth (and falsity) of formula in a model.

Let $\mathfrak{M} \in \boldsymbol{MR}$ and $\varphi \in \mathsf{For}^{\mathrm{w}}$. Formula φ is *true in* $\mathfrak{M}$ (in symb.: $\mathfrak{M} \models \varphi$; and $\mathfrak{M} \not\models \varphi$, if false) iff for every $\psi, \chi \in \mathsf{For}^{\mathrm{w}}$:

- for propositional variables and formulas built by classical connectives:

$$
\begin{array}{ll}
v_{\mathfrak{M}}(\varphi) = 1, & \text{if } \varphi \in \mathsf{Var} \\
\mathfrak{M} \not\models \psi, & \text{if } \varphi = \neg\psi \\
\mathfrak{M} \models \psi \text{ and } \mathfrak{M} \models \chi, & \text{if } \varphi = \psi \wedge \chi \\
\mathfrak{M} \models \psi \text{ or } \mathfrak{M} \models \chi, & \text{if } \varphi = \psi \vee \chi \\
\mathfrak{M} \not\models \psi \text{ or } \mathfrak{M} \models \chi, & \text{if } \varphi = \psi \rightarrow \chi \\
\mathfrak{M} \models \psi \text{ iff } \mathfrak{M} \models \chi, & \text{if } \varphi = \psi \leftrightarrow \chi
\end{array}
$$

- for formulas built by relating connectives:

$$
\begin{array}{ll}
[\mathfrak{M} \models \psi \text{ and } \mathfrak{M} \models \chi] \text{ and } R_{\mathfrak{M}}(\psi, \chi), & \text{if } \varphi = \psi \wedge^{\mathrm{w}} \chi \\
[\mathfrak{M} \models \psi \text{ or } \mathfrak{M} \models \chi] \text{ and } R_{\mathfrak{M}}(\psi, \chi), & \text{if } \varphi = \psi \vee^{\mathrm{w}} \chi \\
[\mathfrak{M} \not\models \psi \text{ or } \mathfrak{M} \models \chi] \text{ and } R_{\mathfrak{M}}(\psi, \chi), & \text{if } \varphi = \psi \rightarrow^{\mathrm{w}} \chi \\
[\mathfrak{M} \models \psi \text{ iff } \mathfrak{M} \models \chi] \text{ and } R_{\mathfrak{M}}(\psi, \chi), & \text{if } \varphi = \psi \leftrightarrow^{\mathrm{w}} \chi.
\end{array}
$$

Otherwise, φ is *false in* $\mathfrak{M}$. Moreover, for any $\Sigma \subseteq \mathsf{For}^{\mathrm{w}}$, we will write $\mathfrak{M} \models \Sigma$ instead of $\forall_{\varphi \in \Sigma} \mathfrak{M} \models \varphi$ and state that Σ is *true in* model $\mathfrak{M}$.

In the case of classical connectives, we have the classical extensional interpretation. In the case of relating connectives apart from extensional conditions, we have also the requirement of being related by both sentences.

We can also define a notion of truth with respect to a relation without a valuation, so in respect to all possible valuations of variables. Let $R \in \boldsymbol{R}$ and $\varphi \in \mathsf{For}$. Formula φ is *true with respect to* R (in symb.: $R \models \varphi$; and $R \not\models \varphi$, if otherwise) iff $\forall_{v \in \{1,0\}^{\mathsf{Var}}} \langle v, R \rangle \models \varphi$. Otherwise φ is *false with respect to* R. Let $\Sigma \subseteq \mathsf{For}^{\mathrm{w}}$, we will write $R \models \Sigma$ instead of $\forall_{\varphi \in \Sigma} R \models \varphi$ and state that Σ is *true in* relation R.

The definitions of truth in a model and the truth in a relation allow to define two different notions of semantic consequence relations over classes of semantic structures and result in two different ways of determining the notion of a valid formula. First one is the semantic consequence relation over models. Let $\boldsymbol{M} \subseteq \boldsymbol{MR}$ and $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}^{\mathrm{w}}$. Then:

- φ is *a semantic consequence of Σ over $\boldsymbol{M}$* (in symb.: $\Sigma \models_{\boldsymbol{M}} \varphi$) iff $\forall_{\mathfrak{M} \in \boldsymbol{M}}(\mathfrak{M} \models \Sigma \Rightarrow \mathfrak{M} \models \varphi)$
- φ is *valid in $\boldsymbol{M}$* (in symb.: $\models_{\boldsymbol{M}} \varphi$) iff $\emptyset \models_{\boldsymbol{M}} \varphi$.

It is worth noting that in the definition of the semantic consequence relation over models the set of models $\boldsymbol{M}$ does not have to be determined by any class of relations $\mathbf{Q} \subseteq \boldsymbol{R}$ in a sense that the class of models is $\{\langle v, R\rangle \colon v \in \{1,0\}^{\mathsf{Var}} \,\&\, R \in \mathbf{Q}\}$. Such class of models can determine interesting relating logics [see, e.g., Jarmużek and Klonowski, submitted]. The inverse property holds: every class of relations determines uniquely a class of models. In [Jarmużek and Klonowski, 2021, submitted] there was also proposed the semantic consequence relation over relations defined as follows. Let $\mathbf{Q} \subseteq \boldsymbol{R}$ and $\Sigma \cup \{\varphi\} \subseteq \mathsf{For}^{\mathrm{w}}$. Then:

- φ is *a semantic consequence of Σ over $\mathbf{Q}$* (in symb.: $\Sigma \models_{\mathbf{Q}} \varphi$) iff $\forall_{R \in \mathbf{Q}}(R \models \Sigma \Rightarrow R \models \varphi)$
- φ is *valid in $\mathbf{Q}$* (in symb.: $\models_{\mathbf{Q}} \varphi$) iff $\emptyset \models_{\mathbf{Q}} \varphi$.

In [Klonowski, 2019; Jarmużek and Klonowski, 2021] three kinds of properties determining subsets of the set of all relations $\boldsymbol{R}$ were defined. Namely, horizontal relations, vertical relations and relations diagonal. The latter are intersections of the previous two. The potentially determine different logical systems.

Many other interesting results are given in these works. We will list some of them. As we mentioned, for example, the fact that models define more logics than sets of relations is proved. An interesting division was also introduced there:

(a) logics, i.e., systems closed under functions from the set Su,
(b) quasi-logics, i.e., systems not closed under all functions from the set Su, but under some special subsets of Su.

For example, the set of all anti-symmetric relations determine the system of logic which is not closed under all functions from the set Su, however closed under all injective substitutions. Below, we have an axiom that uniquely determines the anti-symmetric class of relations. The axiom assumes non-injective substitutions:

$$\neg(\varphi \to^{\mathrm{w}} \psi \vee \varphi \vee^{\mathrm{w}} \psi) \vee \neg(\psi \to^{\mathrm{w}} \varphi \vee \psi \vee^{\mathrm{w}} \varphi), \text{ where } \varphi \neq \psi.$$

Note, however, that the property of antisymmetry requires two relations to be expressed: relation R and the relation of identity $=$. Thus, any logic defined by the above axiom is not mono-relating.

Further results concern the definability of relating connectives. The paper shows that the relating disjunction is the only relating connective that is undefinable. The definitions of other relating connectives are also indicated.

The most important result of this work is however to show that finite sets of conditions imposed on relations define logic that can be axiomatized, because in the language of CMRL it is possible to define relations independently of valuations. At the end of the work, the lattice order of axiomatized logics is described, defining the joint and meet operations that make it up.

To be honest, we do not have to consider the entire CMRL language. We get very interesting results when we consider only **CPL** language, but interpretating at least one connective c as a relating connective c^{w}.

So let us say we are working with the set For — it is **CPL** language, but some of the connectives can be interpreted as relating connectives. Therefore, they require interpretation of the connection of arguments. It is worth noting that when the relation R is universal, it can be omitted in the conditions of truth of a given connective. As a consequence, even **CPL** can be treated as a relating logic determined by the universal relation [see, e.g., Jarmużek and Klonowski, submitted].

When we consider some logic $\boldsymbol{L}$ defined in the set For, we usually call it subclassical when $\boldsymbol{L} \setminus \mathbf{CPL} = \emptyset$. On the other hand, when $\boldsymbol{L} \setminus \mathbf{CPL} \neq \emptyset$, logic is called anti-classical (or more often contra-classical). That is, it contains at least one formula that is contingent or counter-tautology.

The use of relating semantics seems to be a good step in defining systems of logic with the property $\boldsymbol{L} \setminus \mathbf{CPL} \neq \emptyset$. In the next section we present a case of the accepted starting points of application of relating semantics to non-classical logics.

2. Application of relating semantics to intensional phenomena

The main aim of the *1st Workshop On Relating Logic* (1st WRL) was to create an international community of logicians, that explores the potential of RL and relating semantics.[5] In the scope of the workshop, the idea of defining non-classical logics (including counter-classical) via relating-semantics was also included. During the workshop various cases of using semantics to analyze and define intensional phenomena logic was often analyzed. We will discuss some of them.

Let us introduce some basic knowledge on connexive logics. Connexive logic is based on the theses set forth by Aristotle and Boethius, which only use negation and implication connectives. What is more, these theses are inconsistent to the classical logic. Therefore, in connexive logic we must interpret at least one of these connectives in a non-classical manner.

There is an idea behind connexive logic that proposition φ has nothing in common with proposition $\neg\varphi$ in terms of the content. Similarly, if φ has a common content with ψ, it cannot have any common content with $\neg\psi$, and vice versa, if φ has a common content with $\neg\psi$, it cannot have any common content with ψ. A necessary condition for occurrence or truthness of implication is a common content of premiss and conclusion (the implication antecedent and consequent, respectively). Such intuitions form the motivation for connexive logics. The roots of connexive logic date back to the ancient times.

In the formal language which features at least two connectives, unary: $\neg$, referred to as *negation*, and binary: $\rightarrow$, referred to as *implication*, the concept of connexiviy is expressed by the requirement of occurrence of the following theses in a logic:

(A1) $\neg(\varphi \rightarrow \neg\varphi)$

(A2) $\neg(\neg\varphi \rightarrow \varphi)$

(B1) $(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)$

(B2) $(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$.

Formulas (A1) and (A2) are referred to as Aristotle's Theses, while (B1) and (B2) as Boethius' Theses. Since none of formulas (A1), (A2), (B1) and (B2) is a thesis of the classical logic, and at the same time the classical logic is Post-complete, then having attached any of those to the classical logic, we would produce an inconsistent logic. Thus, if we

[5] The 1st Workshop on Relating Logic took place in September 25–26, 2020, see: `https://www.filozofia.umk.pl/en/department-of-logic/call-for-workshop-on-relating-logic/12`. More on the workshop, see [Jarmużek and Paoli, 2021].

comprehend the negation and implication in a classical manner, we will produce an inconsistent logic. As a consequence, in any connexive logic the implication or the negation is non-classical. On the other hand if we do not assume the classical logic as a background formulas (A1), (A2), (B1) and (B2) can be independent, which was shown in [Jarmużek and Malinowski, 2019a].

The denomination of *connexive logic* is to promise some special connection, relation between the formulas or premisses and conclusions. There are strong indications that by dint of application of relating semantics can accurately and directly express the relations between the propositions that lie in the heart of connexive logic. These relations do not have to follow from a common lingual form, but — as mentioned before — from a content similarity that is not always expressible in a logical form. So it can be for instance so that propositions φ and ψ are related in terms of content, even if they do not include a single common propositional letter. The relating semantics make this relation expressible.

In order to establish which of models are appropriate to define connexive logics the following conditions were specified:

- R is (a1) iff for all $\varphi \in \mathsf{For}$, $\sim R(\varphi, \neg\varphi)$
- R is (a2) iff for all $\varphi \in \mathsf{For}$, $\sim R(\neg\varphi, \varphi)$
- R is (b1) iff for all $\varphi, \psi \in \mathsf{For}$:
  - if $R(\varphi, \psi)$, then $\sim R(\varphi, \neg\psi)$
  - $R(\varphi \to \psi, \neg(\varphi \to \neg\psi))$
- R is (b2) iff for all $\varphi, \psi \in \mathsf{For}$:
  - if $R(\varphi, \psi)$, then $\sim R(\varphi, \neg\psi)$
  - $R(\varphi \to \neg\psi, \neg(\varphi \to \psi))$.

On a model level, the relevant conditions from that definition exclude the connection of specific propositions. For instance, if relation R has a property (a1), then for none proposition φ it is so that it is connected with its negation, i.e. proposition $\neg A$. The same applies to the other conditions. In accordance with their notation, on the semantic level they conform to the theses of Aristotle and Boethius.

From the definition, it almost directly follows that conditions (a1), (a2), (b1), (b2) imposed on models from some class are sufficient for occurrence of the respective theses: (A1), (A2), (B1), (B2). For any binary relation R on For we obtain:

(1) R is (a1) $\Rightarrow R \models \neg(\varphi \to \neg\varphi)$
(2) R is (a2) $\Rightarrow R \models \neg(\neg\varphi \to \varphi)$
(3) R is (b1) $\Rightarrow R \models (\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$
(4) R is (b2) $\Rightarrow R \models (\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$.

A theorem with implications the other way around is not true which is easy to demonstrate by providing relevant countermodels for each case.

Still, the authors would also like to get converse implications. This would enable a transition from the syntactic formulation of connexive logic, i.e., adoption of formulas (A1), (A2), (B1), (B2) as axioms, to the relevant classes of models within the set of all models. There are probably various ways to receive a converse theorem. The authors offered, however, a rather intuitive way, which is probably also minimalistic. They adopted one more property of the relating relation: closure under negation. Let $R \subseteq \mathsf{For}^2$.

(c1) R is *closed under negation* iff for all φ, $\psi \in \mathsf{For}$, $R(\varphi, \psi) \Rightarrow R(\neg\varphi, \neg\psi)$.

The closure under negation is a minimal condition which preserves the connection of two propositions and their negations in terms of content. For, in accordance with (c1) it is so that if two propositions φ and ψ are connected: $R(\varphi, \psi)$, then their negations are also connected $R(\neg\varphi, \neg\psi)$ which seems reasonable. We can also consider a stronger condition, reinforcing (c1) to an equivalence. But condition (c1) is sufficient to get a single equivalence among conditions (a1), (a2), (b1), (b2) with the theses of Aristotle and Boethius that are relevant. The examination of the reinforced (c1) as well as the other conditions producing a similar effect as (c1) were left for further studies.

Condition (c1) features an interesting property. When imposed on models it will produce new theses. If R is closed under negation then $R \models \neg(((\varphi \to \psi) \wedge \neg\psi) \wedge \neg(\neg\varphi \to \neg\psi))$. But the converse implication does not hold. However, also the adoption of conditions (a1), (a2), (b1), (b2) to the relating relation R does not warrant that the below formula occurs as a tautology: $R \models \neg(((\varphi \to \psi) \wedge \neg\psi) \wedge \neg(\neg\varphi \to \neg\psi))$.

Putting the former propositions we get two conclusions. Firstly, an addition of condition (c1) to define the class of the relating models brings new laws that are not generated by conditions (a1), (a2), (b1), (b2). Secondly, condition (c1) does not follow from those conditions. The adoption of the condition of closure under negation produces a theorem on the correspondence for the Aristotle's and Boethius' theses. If a binary relation R on For satisfy (c1), then:

- R is (a1) $\iff R \models \neg(\varphi \to \neg\varphi)$
- R is (a2) $\iff R \models \neg(\neg\varphi \to \varphi)$
- R is (b1) $\iff R \models (\varphi \to \psi) \to \neg(\varphi \to \neg\psi)$
- R is (b2) $\iff R \models (\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$.

In the paper [Jarmużek and Malinowski, 2019a] thirty-two logics by determining respective classes of relations R were constructed. In two of them Aristotelian and Boethian laws hold and at the same time negation, conjunction and disjunction preserve Boolean meaning. That is why for such logics the name *Boolean connexive logics* was proposed.

In the presented paper connexive logic was understood in a very general way, just as any set of sentences closed under substitution and modus ponens containing Aristotle's and Boethian laws. This way structural properties of a broad spectrum of sentential connexive logics was possible to investigate.

Let us stress again that relating semantics in a general way was proposed in the paper [Jarmużek and Kaczkowski, 2014].[6] Its main notion — a relating relation — can be equipped with a large quantity of philosophical and not only philosophical motivations and interpretations. Two formulas can be related by R in many ways. For example they could be related analytically, causally, thematically, temporally, etc. In the presented paper relating semantics is directly applied to connexive implication. As a consequence, in this approach connexive implication is true iff its antecedent is false or its consequent is true and simultaneously both are connected in some way. In the semantics this connection is expressed by the relating relation.

In the next paper [Jarmużek and Malinowski, 2019b] investigations initiated in the former paper are continued with generalizing the results to the area of modal logics. In a natural way by *modal Boolean connexive logics* is meant a logic formulated in the sentential language with implication, classical negation, classical disjunction, classical conjunction,

[6] But the first systems of relating logic was propose in [Epstein, 1979, 1990]; for the historical issues, see [Klonowski, 2021b]. The article argues that, apart from the person of R. Epstein, also D. Walton [1979a,b] is an indispensable researcher for the development of relating semantics. More on various variants of this kind of implication can be found in [Paoli et al., 2021; Ledda et al., 2019; Paoli, 2007, 1993].

necessity and possibility operators satisfying Aristotle's and Boethian laws. So Boolean connexive logics are examined in the language with modal operators: $\Box$, $\Diamond$ are investigated. In such logics negation, conjunction and disjunction behave in a classical, Boolean way, again. Only implication is non-classical.

These logics are constructed by mixing relating semantics with possible worlds. This way we obtain connexive counterparts of basic normal modal logics. However, most of their traditional axioms formulated in terms of modalities and implication do not hold anymore without additional constraints, since our implication is weaker than material implication.

Expressing this in a short way: modal Boolean connexive logic is a Boolean connexive logic defined in a modal language. The semantics considered there is a kind of combination of possible worlds semantics and relating semantics. As a consequence we have two types of binary relations: a relating relation between formulas determining a meaning of implication and an accessibility relation on possible worlds defining modal operators. There appears that both kinds of relations influence each other and limit to some extent traditional modal laws. A motivation for considering this particular combination of two semantics is natural. Since we considered in connexive logics without possible worlds, here we extend our ideas to possible worlds framework.

Any models that mix relations with possible worlds can be named *modalized models* [see Jarmużek and Malinowski, 2019b; Jarmużek and Klonowski, 2020; Jarmużek and Paoli, 2021]). During the 1st WRL, the following problems were identified as some of the most important for the understanding of RL:

1. problem α: axiomatization of logics defined by relating semantics (by given classes of valuations/relations);
2. problem β: relating semantics for logics defined as some set of formulas closed under some rules of inference;
3. problem γ: defining philosophical logics by relating semantics (reduction of various logical connectives to relating connectives);
4. problem δ: relationships between relating semantics and other kinds of formal semantics (problem of reduction);
5. problem η: combining relating semantics with other kinds of formal semantics.

One of them, the problem η, was combining relating semantics with other kinds of formal semantics. So, the examination of modalized models is a special instance of the problem η.

By using these modalized models a number of connexive counterparts of basic normal modal logics was defined. However, most of their traditional axioms formulated in terms of modalities and implication do not hold anymore, since the relating implication is weaker than material implication. To make them valid some additional constraints on relating semantics must be imposed, like in particular, that modal operators have no influence on being related. Most of the basic systems of Boolean and Modal Boolean connexive logics were axiomatized in [Klonowski, 2021a].

The articles selected for the second volume contain partial solutions to some of the problems listed above.

In the article "Relating semantics as compatibility semantics" (by Luis Estrada-González) a technical problem is addressed. The question the author is interested in there is whether the relating semantics for connexive logics currently available provide a good account of what connexivity is, or at least as good as some of the proposals already in circulation. He argues that relating semantics for Boolean connexive logics, in its current state, does not give a satisfactory account of connexivity, as it is merely designed to validate the connexive schemas. Nonetheless, following the initial ideas of Jarmużek and Malinowski, he suggests that such semantics can be understood as a version of the Incompatibility Approach, and that the relating semantics can be naturally enriched so as to make it virtually indistinguishable from the Incompatibility Approach. Said boldly, his claim is that, if relating semantics for Boolean connexive logics is going to be more than a means to validate the connexive schemas, and if the relating relation to model connexive logics is to be understood as connectedness, as Jarmużek and Malinowski want, the relating relation must have more properties than those currently allowed. Further arguments for those additional properties come from considering highly desirable properties of a conditional.

The subject of relating semantics for connexive logics is also developed in the paper "Relating semantics for hyper-connexive and totally connexive logics" (by Ricardo Arturo Nicolás-Francisco and Jacek Malinowski).

In this paper the authors present an interesting characterization of hyper-connexivity by means of relating semantics for Boolean connexive logics. They also show that the minimal Boolean connexive logic is

Abelardian, strongly consistent, Kapsner strong and antiparadox. But they give an example showing that minimal connexive logic is not simplificative. This shows that minimal Boolean connexive logics is not totally connexive.

The paper "Axiomatization of **BLRI** determined by limited positive relational properties" (by Jarmużek and Klonowski) is the study of a generalised method for obtaining a complete axiomatic system for any relating logic expressed in a language with Boolean connectives and relating implication, defined by the so-called limited positive relational properties.

Conditions of this kind take the form of a general conditional sentence with an antecedent in the form of conjunctions of relational expressions, i.e., expressions built with a binary predicate and variables over formulas, and a consequent in the form of a relational expression. Multiple examples of such properties can be found in [Epstein, 1990; Jarmużek and Klonowski, 2021; Jarmużek and Malinowski, 2019a], where it has been show how relating semantics, with the appropriate conditions for the considered type, can allow for analysing implication that takes into account content-relationships of the expressions, causal implication and connexive implication.

The method of obtaining axiomatic systems for logics of a given type is called the α *algorithm*, since the analysis allows for any logic of a given type to determine step-by-step the axiomatic system adequate for it. The proof of completeness of axiomatic systems obtained by applying the α algorithm that is presented constitutes a modification of Henkin-style completeness proofs for zero-order logics. Such proofs for various types of relating logics were presented in [Epstein, 1979, 1990; Klonowski, 2019].[7] However, all those cases used the fact of expressivity of the relating relation in the language of the analysed logic. The proof in the paper "Axiomatization of **BLRI** determined by limited positive relational properties" does not use expressivity of the relating relation. By means of an appropriate transformation, it is shown how to transform the relational conditions that determine a given logic into axioms. In addition to axioms, in some cases it must additionally be considered a rule that allows to transform axioms in a way that corresponds to deducing relational conditions from the given initial conditions.

[7] Constructive proofs were analysed in [Paoli, 1996; Klonowski, 2018].

Another possibility of applying relating semantics can be found in a debate from epistemology regarding the content of scientific understanding. The paper by Maria Martinez-Ordaz ("Scientific understanding meets relating logic") uses the philosophical grounds of relating semantics for explaining the primacy of the logical component of scientific understanding.

The question that the paper tackles is that of the difference between *understanding* and merely *knowing* in scientific contexts; and the answer points to a special kind of logical constraint. First, scientific understanding has been largely characterized as consisting of knowledge about relations of dependence. So, if one understands something, one can make all kinds of correct inferences about it. Second, the content of understanding is traditionally regarded to be *factive*, meaning that *legitimate* cases of understanding can only include true propositions that are known to be so.

The combination of these two assumptions gives the impression of understanding surpassing the limits of knowledge by possessing an importantly logical component — more than by being constrained by the truth value of its content. Unfortunately, for a long time, the logical side of understanding has been neglected by epistemologists of science.

The paper argues that the role that truth might play for this matter is actually less important and, especially, less decisive than the one of logic. Furthermore, it contends that the analysis of the basics of relating semantics can shed light on the logical constraints of scientific understanding in a significant way. In order to support such a claim, the author proceeds in four steps: first, she introduces the debate about the role that truth plays for the legitimacy of understanding.

Second, she explains that the debates on the truth value of the content of understanding neglect the crucial component of the phenomenon: its logical character; the key element to move forward the study of understanding would be an analysis of its logical constraints, which she contends to be content-related.

Third, the author introduces the philosophical basics of relating semantics. This section includes a brief recap of the motivations behind Epstein's programme and the later integrative views on Torunian Programme.

Fourth, the author tackles two main issues, first, how relating semantics can shed light, in a novel and unique way, on the study of scientific understanding; and second, the philosophical value of such an application for the relating semantics project.

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