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Axiomatization of BLRI Determined by Limited Positive Relational Properties

Abstract. In the paper a generalised method for obtaining an adequate axiomatic system for any relating logic expressed in the language with Boolean connectives and relating implication (**BLRI**), determined by the limited positive relational properties is studied. The method of defining axiomatic systems for logics of a given type is called an algorithm, since the analysis allows for any logic determined by the limited positive relational properties to define the adequate axiomatic system automatically, step-by-step. We prove in the paper that the algorithm really works and we show how it can be applied to **BLRI**.

Keywords: algorithm α; Boolean logics with relating implication; BLRI; relating logic, relating semantics

1. Introduction

In this study, we present a generalised method for obtaining a sound and complete axiomatic system for any relating logic expressed in a language with Boolean connectives and relating implication defined by the so-called limited positive relational properties.

Relational conditions of the analysed in the paper kind take the form of a general conditional with an antecedent in the form of relational expression conjunctions, i.e., expressions built with binary predicate and variables running over formulas, and a consequent in the form of a relational expression. Multiple examples of such properties can be found in [Epstein, 1990; Jarmużek and Klonowski, 2021, submitted-a; Jarmużek and Malinowski, 2019a], where it has been shown how relating semantics, with the appropriate conditions for the considered type, can allow for

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analysing implication that takes into account content relations of the expressions, causal implication, and connexive implication.

We call the method of obtaining axiomatic systems, for logics of a given type, the α *algorithm*, since our analysis allows for any logic of a given type to determine step-by-step the adequate axiomatic system. The proof of completeness of axiomatic systems obtained by applying the α algorithm that we will present is a modification of Henkin-style completeness proofs for zero-order logic. Such proofs, for various types of relating logic, were presented in [Epstein, 1979, 1990; Klonowski, 2019, 2021a].[1] All of those cases, however, made use of the fact of the expressivity of the relating relation in the language of the analysed logic. Our proof does not use the expressivity of the relating relation. By means of an appropriate transformation, we will show how to transform the relational conditions that determine a given logic into axioms. In addition to axioms, in some cases, we must additionally consider a rule that allows us to transform axioms in a way that corresponds to possible relational conditions that can be deduced from the explicitly given initial conditions.

The paper consists of an introduction, eight sections, and a conclusion. In Section 1, we introduce the language of the analysed logics, the necessary notations, and the notion of Boolean logic with relating implication. For the latter, we also define the type of relational condition of interest. In Section 2, we define an axiomatic system and use examples to describe and demonstrate the α algorithm. In Sections 3 and 4, we will deal respectively with the proof of soundness and completeness of the axiomatic systems obtained using α algorithm. In the Sections 5 and 6 we study general forms of soundness and completeness. Section 7 is devoted to the problem of cardinality of BLRI logical systems and the problem of their determination. In Section 8 we propose a translation into classical logic and investigate metalogical issues such as completeness and interpolation.

2. Boolean logics with relating implication

In this paper, we will focus on a certain family of Boolean logics with relating implications. In general, by Boolean logics with relating implication **BLRI**), we mean relating logics with classical negation, conjunction, and disjunction in which there is only one relating connective —

[1] Constructive proofs were examined in [Paoli, 1996; Klonowski, 2018].

the relating implication. That is, an implication whose interpretation necessitates taking into account hypothetical relationships between the antecedent and the consequent.

The language (object-language) of Boolean logic with relating implication is a language consisting of propositional variables and the following connectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$ and implication $\rightarrow$, as well as brackets:), (. The set of propositional variables is denoted by Var. Let us define the set of (object-language) formulas in a standard way and denote it by For. The metavariables $A, B, C, D, A_1, B_1, C_1, D_1 \ldots$ will ranging over set For. Thus, by means of these variables and the assumed connectives we can determine formulas schemata. The set of metavariables is denoted by VAR and the set of formulas schemata is denoted by FOR. By means of F, G, H, I, F_1, $G_1, H_1, I_1 \ldots$ we will represent any formulas schemata, i.e. these variables ranging over the set FOR. In turn, $X, Y, Z, X_1, Y_1, Z_1, \ldots$ will ranging over the power set of the set For, i.e. $\mathcal{P}(\mathsf{For})$.

We will omit brackets in formulas according to the standard convention of biding strength and define the following abbreviations: $\lambda \supset \pi := \neg\lambda \vee \pi$ and $\lambda \equiv \pi := (\neg\lambda \vee \pi) \wedge (\neg\pi \vee \lambda)$. By $\lambda[\pi/\sigma]$ we denote a formula obtained by the replacement of all occurrences of π by σ in λ.

In our discussion, we will sometimes use the iterated conjunction $\lambda_1 \wedge \ldots \wedge \lambda_n$, which we will also write as follows: $\bigwedge_{i=1}^{n} \lambda_i$. The conjunction obtained from $\bigwedge_{i=1}^{n} \lambda_i$ by excluding the conjuncts $\lambda_{j_1}, \ldots, \lambda_{j_m}$, where $m < n$, will be written as follows: $\bigwedge_{i=1, i \neq j_1, \ldots, j_m}^{n} \lambda_i$. Let us assume that if $n = 1$, formula $\lambda_1 \wedge \ldots \wedge \lambda_{n-1} \supset \pi$ (or $\bigwedge_{i=1}^{n-1} \lambda_i \supset \pi$) is just π.

A *model* of the analysed object-language is an ordered pair $\langle v, R \rangle$, such that, $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ is a classical valuation and $R \subseteq \mathsf{For} \times \mathsf{For}$ is a binary relation, called *relational relation*.[2] In this paper, we will use the notation $R(A, B)$ and $\sim R(A, B)$, to express, that A is in the relation R to B, and A is not in the relation R to B, respectively. The relational symbol R can be used to state various relations between denotations of sentences.

A formula $A \in \mathsf{For}$ is *true in model* $\mathfrak{M} = \langle v, R \rangle$ (in symb.: $\mathfrak{M} \models A$; $\mathfrak{M} \not\models A$, if false) iff for every $B, C \in \mathsf{For}$:

$v(A) = 1$,	if $A \in \mathsf{Var}$
$\mathfrak{M} \not\models B$,	if $A = \neg B$
$\mathfrak{M} \models B$ and $\mathfrak{M} \models C$,	if $A = B \wedge C$

[2] For a discussion on the relation R see [Estrada-González et al., 2021].

$$\mathfrak{M} \models B \text{ or } \mathfrak{M} \models C, \qquad \text{if } A = B \vee C$$
$$[\mathfrak{M} \not\models B \text{ or } \mathfrak{M} \models C] \text{ and } R(B, C), \qquad \text{if } A = B \to C.$$

Let $X \subseteq \mathsf{For}$. We will write $\mathfrak{M} \models X$ instead of, for every $A \in X$, $\mathfrak{M} \models A$.[3]

We adopt standard definitions of semantic consequence relations and valid formulas. Let $X \cup \{A\} \subseteq \mathsf{For}$ and $\mathbf{M}$ be a set of all models. Then:

- A is a *semantic consequence of* (*entailed by*) X *in the set* $\mathbf{N} \subseteq \mathbf{M}$ (in symb.: $X \models_{\mathbf{N}} A$) iff for every $\mathfrak{M} \in \mathbf{N}$, if $\mathfrak{M} \models X$ then $\mathfrak{M} \models A$.
- A is a *valid formula in the set* $\mathbf{N} \subseteq \mathbf{M}$ (in symb.: $\models_{\mathbf{N}} A$) iff $\emptyset \models_{\mathbf{N}} A$.[4]

By *logic*, we will mean an ordered pair consisting of a set of formulas and a semantic consequence relation closed under any uniform substitution. Since we will focus on one set of formulas, set For, we can identify logic with the semantic consequence relation. *A Boolean logic with a relating implication* is any logic $\models \subseteq \mathcal{P}(\mathsf{For}) \times \mathsf{For}$.

In this paper, we will focus on Boolean logics with relating implications, determined by sets of models satisfying some kind of relational conditions.[5] The conditions we will focus on will be the *limited positive relational properties* (LPR). In order to define properties of this type we use: metalogical connectives: implication ($\Rightarrow$), conjunction (and), disjunction (or), and atomic expressions built with the relational symbol R, formulas schemata, and brackets. Since there is no metalogical negation among the considered connectives, we call the analysed properties positive. We interpret the indicated metalogical connectives classically. The set of formulas constructed by the given metalogical symbols excluding implication is defined in a standard way and denoted by Ex. Variables $\varphi, \psi, \chi, \varphi_1, \psi_1, \chi_1 \ldots$ are ranging over the set Ex. We adopt similar as for above presented languages conventions for metalogical conjunction iterations. In this case, instead of: φ_1 and $\ldots$ and φ_n, we sometimes use the following notation: $\text{AND}_{i=1}^{n}\varphi_i$. Moreover, we assume similar conventions for replacing a formula within a given formula and use similar notation as for above presented languages. We have the following examples of expressions that are LPR:

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[3] We will say that a formula schema is false when there is a formula in the form of that schema that is false.

[4] We will say that a formula schema, in particular an axiom schema, is a semantic consequence of (derived from) a set (resp. valid) in a set of models if any formula of the form, of that schema, is a semantic consequence of (derived from) a given set (resp. valid) in that set of models.

[5] In other words, we analyse only those logics that we can define by models whose relations satisfy any fixed conditions of a certain type.

$$R(A, A) \tag{LPR1}$$
$$R(\neg A, A) \tag{LPR2}$$
$$R(A, \neg A) \tag{LPR3}$$
$$R(A, B) \text{ and } R(B, C) \Rightarrow R(A, C) \tag{LPR4}$$
$$R(A, B) \Rightarrow R(\neg A, \neg B) \tag{LPR5}$$
$$R(A, B) \text{ or } R(A, C) \Rightarrow R(A, B \vee C) \tag{LPR6}$$
$$R(A \wedge B, C \wedge D) \text{ or } R(A \wedge B, C \vee D) \Rightarrow R(A, C) \text{ and } R(B, D). \tag{LPR7}$$

We can now introduce the definition of the relational properties of interest. We say that φ is a *simplified limited positive relational property* (sLPR) iff there are $F, G \in \mathsf{FOR}$ such that either $\varphi = R(F, G)$ or there is $n \in N$ such that $\varphi = R(F_1, F_2)$ and $\ldots$ and $R(F_{2n-1}, F_{2n}) \Rightarrow R(F, G)$. Thus, any sLPR is of the following form:

$$\text{AND}_{i=1}^{n-1} R(F_{2i-1}, F_{2i}) \Rightarrow R(F_{n+1}, F_{n+2}),$$

for some $n \in \mathbb{N}$. The first five expressions of the LPR examples given above, i.e. (LPR1)–(LPR5), are sLPR. Let us consider the subsequent examples, i.e. (LPR6)–(LPR7). Clearly, the condition (LPR6) has the following corresponding sLPR conditions:

$$R(A, B) \Rightarrow R(A, B \vee C) \tag{LPR6.1}$$
$$R(A, C) \Rightarrow R(A, B \vee C), \tag{LPR6.2}$$

and (LPR7) has the following corresponding sLPR conditions:

$$R(A \wedge B, C \wedge D) \Rightarrow R(A, C) \tag{LPR7.1}$$
$$R(A \wedge B, C \wedge D) \Rightarrow R(B, D) \tag{LPR7.2}$$
$$R(A \wedge B, C \vee D) \Rightarrow R(A, C) \tag{LPR7.3}$$
$$R(A \wedge B, C \vee D) \Rightarrow R(B, D). \tag{LPR7.4}$$

In the paper we study these LPR properties that are reducible to some sets of sLPR properties.

3. Axiomatic systems

At a later stage, we will refer to the notion of derivability. We shall now define the notion of an axiomatic system of Boolean logic with relating implication. For this purpose, we will use the set of classically valid formulas expressed in the new language, denoted as BL.

By *axiomatic system* (*of BLRI*) we shall mean the set of formulas $X \subseteq \mathsf{For}$ satisfying the following conditions:

- $\mathsf{BL} \subseteq X$,
- X contains any formula of the form:

$$(A \to B) \supset (A \supset B) \tag{E→}$$

- X is closed under the rule of material detachment:

$$\frac{\begin{array}{l} A \\ A \supset B \end{array}}{B} \tag{MD}$$

i.e., for any $A, B \in \mathsf{For}$, if $A \supset B, A \in X$, then $B \in X$.[6]

The schema (E→) allows us to eliminate the relating implication or weaken the relating implication to the abbreviation classically equivalent to the material implication $\supset$. Note that because axiomatic system contains formulas of the form (E→) and is closed under (MD), it is also closed under the Modus Ponens rule:

$$\frac{\begin{array}{l} A \\ A \to B \end{array}}{B} \tag{MP}$$

Let X be an axiomatic system and $Y \cup \{A\} \subseteq \mathsf{For}$. Then:

- A is *thesis based on the system* X iff $A \in X$
- A is *syntactic consequence* (*derivable from*) Y *based on system* X (in symb.: $Y \vdash_X A$) iff exists $n \in \mathbb{N}$ such that $B_1, \ldots, B_n \in Y$ and $\bigwedge_{i=1}^{n} B_i \supset A \in X$.[7]

We can state that $A \in X$ iff $\emptyset \vdash_X A$ iff $X \vdash_X A$.

Let us denote the least axiomatic system by $\mathbf{W}_\to$. In turn, let us denote the least axiomatic system containing all formulas of the form of the schemata $(Ax_1), \ldots, (Ax_n)$ as $\mathbf{W}_\to \oplus \{(Ax_1), \ldots, (Ax_n)\}$.

[6] If a set X contains all formulas of the form of some schema F, we shall refer to them as $F \in X$ or $(x) \in X$, if (x) denotes F.

[7] We shall say that a formula schema, in particular an axiom schema, is a theorem (the corresponding syntactic consequence of (is derived from) some set) in some system when any formula of the form of that schema is a theorem (the corresponding syntactic consequence of (is derived from) a given set) in that system. We shall then use the same notation as for formulas.

4. Algorithm α – moving from relational conditions to axioms and inference rules

In this section, we will define a method for transforming arbitrary sLPR into schemata of formulas that will serve as schemata of axioms of logics defined by given relational conditions.

We define function α that transforms any sLPR into a formula schema (an axiom schema), for any $n \in \mathbb{N}$:

$$\alpha(\mathrm{AND}_{i=1}^{n-1} R(F_{2i-1}, F_{2i}) \Rightarrow R(F_{n+1}, F_{n+2})) := \\ \bigwedge_{i=1}^{n-1}(F_{2i-1} \to F_{2i}) \supset (F_{n+1} \to F_{n+2}) \vee (F_{n+1} \wedge \neg F_{n+2}).$$

Let us consider some examples, transforming (LPR1)–(LPR7). Condition (LPR1) can be transformed into the following formula:

$$(A \to A) \vee (A \wedge \neg A). \tag{A1}$$

Since (A1) $\equiv (A \to A) \in \mathsf{BL}$, (A1) can be reduced to the following form:

$$A \to A.$$

(LPR2) and (LPR3) can be transformed into the following schemata:

$$(\neg A \to A) \vee (\neg A \wedge \neg A) \tag{A2}$$

$$(A \to \neg A) \vee (A \wedge \neg\neg A). \tag{A3}$$

Once again, by BL we can reduce (A2) and (A3) to the following:

$$(\neg A \to A) \vee \neg A$$

$$(A \to \neg A) \vee A,$$

since (A2) $\equiv (\neg A \to A) \vee \neg A \in \mathsf{BL}$ and (A3) $\equiv (A \to \neg A) \vee A \in \mathsf{BL}$.

Condition (LPR4) can be transformed in the following way:

$$(A \to B) \wedge (B \to C) \supset (A \to C) \vee (A \wedge \neg C). \tag{A4}$$

In this case, we can also make a reduction, this time to the following schema:

$$(A \to B) \wedge (B \to C) \supset (A \to C).$$

However, in addition to classical logic, we need to apply (E$\to$) and (MD), i.e., (A4) $\equiv (A \to B) \wedge (B \to C) \supset (A \to C) \in \mathbf{W}_{\to}$.

Condition (LPR5) is transformed in the following way.

$$(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge \neg\neg B). \tag{A5}$$

In this case, we can only make the following minor reduction:

$$(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge B),$$

resulting in (A5) $\equiv (A \to B) \to (\neg A \to \neg B) \vee (\neg A \wedge B) \in \mathsf{BL}$.

(LPR6) can be transformed indirectly using (LPR6.1) and (LPR6.2):

$$(A \to B) \supset ((A \to B \vee C) \vee (A \wedge \neg(B \vee C)) \tag{A6.1}$$

$$(A \to C) \supset ((A \to B \vee C) \vee (A \wedge \neg(B \vee C)). \tag{A6.2}$$

Once again referring to the classical logic, (E$\to$) and (MD), (A6.1) and (A6.2) can be modified in the following way:

$$(A \to B) \supset (A \to B \vee C)$$
$$(A \to C) \supset (A \to B \vee C),$$

i.e., (A6.1) $\equiv (A \to B) \supset (A \to B \vee C) \in \mathbf{W}_{\to}$ and (A6.2) $\equiv (A \to C) \supset (A \to B \vee C) \in \mathbf{W}_{\to}$.

Similarly as for (LPR7), we can transform (LPR7.1)–(LPR7.4):

$$(A \wedge B \to C \wedge D) \supset (A \to C) \vee (A \wedge \neg C) \tag{A7.1}$$

$$(A \wedge B \to C \wedge D) \supset (B \to D) \vee (B \wedge \neg D) \tag{A7.2}$$

$$(A \wedge B \to C \vee D) \supset (A \to C) \vee (A \wedge \neg C) \tag{A7.3}$$

$$(A \wedge B \to C \vee D) \supset (B \to D) \vee (B \wedge \neg D). \tag{A7.4}$$

In this case, we are unable to make any reductions similar to those described above.

Let us note that, from the point of view of function α, (LPR1), (LPR4) and (LPR6) are similar in some respects. Namely, in the given cases, we have been able to reduce the axiom schemata obtained with function α to schemata in which only relating implications exist, either in the schema itself or in the antecedent and consequent of the schema, and in which there is no alternative with a single member of the form of a relating implication. In this way, we obtained schemata describing the well-known laws of various implications: reflexivity, transitivity, and the introduction of alternatives.

However, it should be stressed that the axiom schemata obtained by α do not always provide a complete axiomatization of a given logic. A problem may arise when we start to consider logics determined by several relational conditions from which a new condition (or conditions) can be deduced which, when transformed by α, allows us to obtain a valid schema (or schema) which we will not derive using the axioms obtained with α. For example, note that if a relation satisfies (LPR2)–(LPR4), it also satisfies (LPR1). However, from the set of formulas in the form of schemata (A2)–(A4) we will not derive (A1) on the basis of $\mathbf{W}_{\rightarrow}$. We can easily show that (A1) is independent of (A2)–(A4). For this purpose, it suffices to consider the classical matrices for $\neg, \wedge, \vee$, and the matrix 1 for $\rightarrow$.

$\rightarrow$	1	0
1	1	0
0	1	0

Table 1. Matrix for $\rightarrow$

Under the given interpretation, all elements of the set BL, as well as (E$\rightarrow$), are true, and so is (MD). Furthermore, (A2)–(A4) will also be true. In turn, the schema (A1) is false, ($p \rightarrow p$ is false if p is assigned 0).

Similarly, if a relation satisfies (LPR5), it also satisfies the following condition:

$$R(A, B) \Rightarrow R(\neg\neg A, \neg\neg B).$$

The condition given allows the validity of the following schema to be demonstrated:

$$(A \rightarrow B) \supset (\neg\neg A \rightarrow \neg\neg B). \tag{A5.1}$$

However, from (A5) we will not be able to derive the schema (A5.1) on the ground of $\mathbf{W}_{\rightarrow}$. The schema (A5.1) is indeed independent from (A5). Let us consider a relating model $\langle v, R\rangle$ such that for any $A \in \mathsf{Var}$, $v(A) = 1$ and for any $A, B \in \mathsf{For}$, $R(A, B)$ iff for some $C \in \mathsf{For}$, $A = C \wedge \neg C$ and $B = C \vee \neg C$. Such a model satisfies the following condition:

$$R(A, B) \Rightarrow R(\neg A, \neg B) \text{ or } (\langle v, R\rangle \not\models A \text{ and } \langle v, R\rangle \models B).$$

In the given model, all elements of the set BL, (E$\rightarrow$) and (MD) are true. The schema (A5) is true as well. In turn, the schema (A5.1) is false. Namely, since $\sim R(\neg\neg(p \wedge \neg p), \neg\neg(p \vee \neg p))$, then the formula $(p \wedge \neg p \rightarrow p \vee \neg p) \supset (\neg\neg(p \wedge \neg p) \rightarrow \neg\neg(p \vee \neg p))$ is false.

As a result, we introduce the following rules that will allow us to use the obtained axiom schemata to prove the formulas that we will obtain using α from the conditions that determine the logic in question. For all $n, m \in \mathbb{N}$:

$$\frac{\begin{array}{l} A_1 \wedge \ldots \wedge A_{n-1} \supset ((A_n \to A_{n+1}) \vee (A_n \wedge \neg A_{n+1})) \\ B_1 \wedge \ldots \wedge B_m \supset B_{m+1} \end{array}}{B_1 \wedge \ldots \wedge B_m[B_i / A_1 \wedge \cdots \wedge A_{n-1}] \supset B_{n+1},} \tag{Rα}$$

where $B_i = A_n \to A_{n+1}$, for some $i \leqslant m$.

The applicability of the rule (Rα) must be limited accordingly. The following inference based on (Rα) shows that our rule need not always lead from thesis to thesis (the premises are theses, while the conclusion may not be a thesis):

$$\frac{\begin{array}{l} (p \to q) \supset (\neg p \vee p \to p \wedge \neg p) \vee ((\neg p \vee p) \wedge \neg(p \wedge \neg p)) \\ (\neg p \vee p \to p \wedge \neg p) \supset (q \to p) \vee (q \wedge \neg p) \end{array}}{(p \to q) \supset (q \to p) \vee (q \wedge \neg p)}$$

However, the (Rα) will not allow falsity if we apply it to substitutions of axioms obtained by α function and/or formulas obtained by application of (Rα). Thus, in practice, our rule can be applied if the subformulas of premises are of the following form:

- for any $i \leqslant n-1$, there are $C_i, D_i \in \mathsf{For}$ such that $A_i := C_i \to D_i$,
- for any $i \leqslant m+1$, there are $C_i, D_i \in \mathsf{For}$ such that $B_i := C_i \to D_i$.

Let us also remind that if $n = 1$, then $(A_1 \wedge \cdots \wedge A_{n-1}) \supset (A_n \to A_{n+1}) \vee (A_n \wedge \neg A_{n+1}) = (A_n \to A_{n+1}) \vee (A_n \wedge \neg A_{n+1})$. Therefore, rule (R$\alpha$) can also be formulated in the following way:

$$\frac{\begin{array}{l} \bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i}) \supset (A_{n+1} \to A_{n+2}) \vee (A_{n+1} \wedge \neg A_{n+2}) \\ \bigwedge_{i=1}^{m}(B_{2i-1} \to B_{2i}) \supset (B_{m+1} \to B_{m+2}) \vee (B_{m+1} \wedge \neg B_{m+2}) \end{array}}{\begin{array}{l} \bigwedge_{i=1}^{m}(B_{2i-1} \to B_{2i})[B_{2j-1} \to B_{2j} / \bigwedge_{i=1}^{n-1}(A_{2i-1} \to A_{2i})] \supset \\ (B_{m+1} \to B_{m+2}) \vee (B_{m+1} \wedge \neg B_{m+2}), \end{array}}$$

where $B_{2j-1} \to B_{2j} = A_{n+1} \to A_{n+2}$, for some $j \leqslant m$.

Let us consider some examples of rules that are special cases of the rule (Rα). For example, we have the following rules:

$$\frac{\begin{array}{l} (\neg A \to A) \vee (\neg A \wedge \neg A) \\ (\neg A \to A) \wedge (A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A) \end{array}}{(A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A)} \tag{R1}$$

$$\frac{\begin{array}{l}(A \to \neg A) \vee (A \wedge \neg\neg A) \\ (A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A)\end{array}}{(A \to A) \vee (A \wedge \neg A)} \quad \text{(R2)}$$

Using (R1), (A2) and (A4), we derive the following schema:

$$(A \to \neg A) \supset (A \to A) \vee (A \wedge \neg A).$$

Moreover, using additionally (R2) and (A3), we can derive (A1). Thus, we can conclude that (A1) $\notin \mathbf{W}_{\to} \oplus \{$(A2), (A3), (A4)$\}$ but (A1) $\in \mathbf{W}_{\to} \oplus \{$(A2), (A3), (A4); (R$\alpha$)$\}$.

Let us further consider the following rule:

$$\frac{\begin{array}{l}(A \to B) \supset (\neg A \to \neg B) \vee (\neg A \wedge \neg\neg B) \\ (\neg A \to \neg B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B)\end{array}}{(A \to B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B)} \quad \text{(R3)}$$

Using (R3) and twice (A5) we shall derive the following schema:

$$(A \to B) \supset (\neg\neg A \to \neg\neg B) \vee (\neg\neg A \wedge \neg\neg\neg B),$$

which using classical logic can be reduced to (A5.1). Therefore, we can conclude that (A5.1) $\notin \mathbf{W}_{\to} \oplus \{$(A5)$\}$, but (A5.1) $\in \mathbf{W}_{\to} \oplus \{$(A5); (R$\alpha$)$\}$.

Due to the indicated restriction, we need to introduce a specific notion of the set closed under (Rα). Let X be an axiomatic system whose only axiom schemata obtained with α function are $(Ax_1), \ldots, (Ax_n)$. A *closure under* (Rα) *with respect to* X (denoted by: $\mathrm{R}\alpha(X)$) is a set of all formulas A such that there is a sequence $(A_1, \ldots, A_n)$ $(n \in \mathbb{N})$ such that:

- $A_n = A$,
- for any $i \leqslant n$ at least one of the following conditions holds:
  - A_i is of the form of (Ax_1) or ... or (Ax_n),
  - there are $l, k \leqslant i$ such that A_i is obtained from A_l, A_k by (Rα).

We say that X is *axiomatically closed under* (Rα) iff $\mathrm{R}\alpha(X) \subseteq X$. Let as denote the least axiomatic system containing the axiom schemata $(F_1), \ldots, (F_n)$ and axiomatically closed under (Rα) as $\mathbf{W}_{\to} \oplus \{(Ax_1), \ldots, (Ax_n); (\mathrm{R}\alpha)\}$.

5. Soundness theorem

In order to prove the soundness theorem, we show that by means of rule (Rα) we can capture all properties of relations that can be derived from some initial, assumed properties that determine a given set of relations.

Lemma 5.1. *Let $\varphi_1, \ldots, \varphi_n$ be sLRP and $X = \mathbf{W}_{\rightarrow} \oplus \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n);$ (Rα)$\}$ be an axiomatic system. Then, for any relation R satisfying the conditions $\varphi_1, \ldots, \varphi_n$, for any $A \in R\alpha(X)$, there is $m \in \mathbb{N}$ such that:*

1. $A = \bigwedge_{i=1}^{m-1}(A_{2i-1} \rightarrow A_{2i}) \supset (A_{m+1} \rightarrow A_{m+2}) \vee (A_{m+1} \wedge \neg A_{m+2})$,
2. $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i}) \Rightarrow R(A_{m+1}, A_{m+2})$ *holds.*

Proof. By definition of R$\alpha(X)$, for any $A \in \mathrm{R}\alpha(X)$ there is a sequence $(B_1, \ldots, B_k)$ such that the indicated conditions are met. We conduct the inductive proof on the number k.

Base case. Let $k = 1$. Thus, there is $j \leqslant n$ such that A is of the form of $\alpha(\varphi_j)$. We have $\varphi_j = \mathrm{AND}_{i=1}^{m-1} R(F_{2m-1}, F_{2m}) \Rightarrow R(F_{m+1}, F_{m+2})$, for some $m \in \mathbb{N}$. Thus, $\alpha(\varphi_j) = \bigwedge_{i=1}^{m-1}(F_{2i-1} \rightarrow F_{2i}) \supset (F_{m+1} \rightarrow F_{m+2}) \vee (F_{m+1} \wedge \neg F_{m+2})$. Therefore, $A = \bigwedge_{i=1}^{m-1}(A_{2i-1} \rightarrow A_{2i}) \supset (A_{m+1_1} \rightarrow A_{m+1_2}) \vee (A_{m+1} \wedge \neg A_{m+2})$ and $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i}) \Rightarrow R(A_{m+1}, A_{m+2})$ holds.

Inductive hypothesis. Let $1 \leqslant o < k$. Suppose for any $j \leqslant o$, there is $m \in \mathbb{N}$ such that $B_j = \bigwedge_{i=1}^{m-1}(A_{2i-1} \rightarrow A_{2i}) \supset (A_{m+1} \rightarrow A_{m+2}) \vee (A_{m+1} \wedge \neg A_{m+2})$ and $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i}) \Rightarrow R(A_{m+1}, A_{m+2})$ holds.

Inductive step. Let $k = o + 1$. Suppose A is obtained by application of α to φ_j, for some $j \leqslant n$. So A is of the form $\alpha(\varphi_j)$. Then we reason as in the base case.

Suppose A is obtained from B_{j_1}, B_{j_2} by the application of rule (Rα). Thus, by inductive hypothesis, we have that there are m_1, m_2 such that:

1. $B_{j_1} = \bigwedge_{i=1}^{m_1-1}(C_{2i-1} \rightarrow C_{2i}) \supset (C_{m_1+1} \rightarrow C_{m_1+2}) \vee (C_{m_1+1} \wedge \neg C_{m_1+2})$,
2. $B_{j_2} = \bigwedge_{i=1}^{m_2}(D_{2i-1} \rightarrow D_{2i}) \supset (D_{m_2+1} \rightarrow D_{m_2+2}) \vee (D_{m_2+1} \wedge \neg D_{m_2+2})$,
3. $\mathrm{AND}_{i=1}^{m_1-1} R(C_{2i-1}, C_{2i}) \Rightarrow R(C_{m_1+1}, C_{m_1+2})$ holds
4. $\mathrm{AND}_{i=1}^{m_2} R(D_{2i-1}, D_{2i}) \Rightarrow R(D_{m_2+1}, D_{m_2+2})$ holds,

where $D_{2j-1} \rightarrow D_{2j} = C_{m_1+1} \rightarrow C_{m_1+2}$, for some $j \leqslant m_2$. Moreover, since A is obtained by (Rα), $A = \bigwedge_{i=1}^{m_2}(D_{2i-1} \rightarrow D_{2i})[D_{2j-1} \rightarrow D_{2j} / \bigwedge_{i=1}^{m_1-1}(C_{2i-1} \rightarrow C_{2i})] \supset (D_{m_2+1} \rightarrow D_{m_2+2}) \vee (D_{m_2+1} \wedge D_{m_2+2})$.

Since $D_{2j-1} \to D_{2j} = C_{m_1+1} \to C_{m_1+2}$, $D_{2j-1} = C_{m_1+1}$ and $D_{2j} = C_{m_1+2}$. Thus, $\mathrm{AND}_{i=1}^{m_2} R(D_{2i-1}, D_{2i})[R(D_{2j-1}, D_{2j})/\mathrm{AND}_{i=1}^{m_1-1} R(C_{2i-1}, C_{2i})] \Rightarrow R(D_{m_2+1}, D_{m_2+2})$ holds. Therefore, after renumerating those sequences, we have: $A = \bigwedge_{i=1}^{m-1}(A_{2i-1} \to A_{2i}) \supset (A_{m+1} \to A_{m+2}) \vee (A_{m+1} \wedge \neg A_{m+2})$ and $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i}) \Rightarrow R(A_{m+1}, A_{m+2})$ holds, for some $m \in \mathbb{N}$. ⊣

We can now proceed to show that any axiom system obtained by using the algorithm α is consistent. We carry out the proof in the standard way, by showing that the axiom schemata and inference rules preserve the validity of the models of the logics in question.

Theorem 5.2. *Let $\models$ be any Boolean logic with a relating implication determined by $\varphi_1, \ldots, \varphi_n$ being sLPR and let $X = \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n); (\mathrm{R}\alpha)\}$ be an axiomatic system. Then, for any $Y \cup \{A\} \subseteq \mathsf{For}$, if $Y \vdash_X A$, then $Y \models A$.*

Proof. All classical tautologies of Boolean logic are valid, and (MD) preserves validity in any Boolean logic with relating implication as well. It is also easy to see that in any Boolean logic with relating implication, (E$\to$) is also valid.

Let $v \colon \mathsf{Var} \longrightarrow \{1, 0\}$ be a valuation and $R \subseteq \mathsf{For} \times \mathsf{For}$ be any binary relation. By Lemma 5.1, if $A \in \mathrm{R}\alpha(Y)$, then there is $m \in \mathbb{N}$ such that $A = \bigwedge_{i=1}^{m-1}(A_{2i-1} \to A_{2i}) \supset (A_{m+1} \to A_{m+2}) \vee (A_{m+1} \wedge \neg A_{m+2})$ and $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i}) \Rightarrow R(A_{m+1}, A_{m+2})$ holds. Suppose that $\langle v, R\rangle \models \bigwedge_{i=1}^{m-1}(A_{2i-1} \to A_{2i})$. Then, $\mathrm{AND}_{i=1}^{m-1} R(A_{2i-1}, A_{2i})$ holds. Thus, $R(A_{m+1}, A_{m+2})$ holds. We have that either $\langle v, R\rangle \models A_{m+1} \supset A_{m+2}$ or $\langle v, R\rangle \models A_{m+1} \wedge \neg A_{m+2}$. Therefore, $\langle v, R\rangle \models (A_{m+1} \to A_{m+2}) \vee (A_{m+1} \wedge \neg A_{m+2})$. ⊣

6. Completeness theorem

For completeness analysis, we introduce the standard notions of consistent and maximally consistent sets. Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Then:

- Y is X-*consistent* iff $Y \nvdash_X p \wedge \neg p$,
- Y is X-*inconsistent* iff Y is not X-consistent.

The following fact is also the standard one:

FACT 6.1. *Let X be an axiomatic system and $Y \cup \{A\} \subseteq \mathsf{For}$. Then, $Y \cup \{\neg A\}$ is X-consistent iff $Y \nvdash_X A$.*

The notion of maximal X-consistent set is defined in the standard way. Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Y is *maximal X-consistent* iff the following conditions are satisfied:

- Y is X-consistent,
- for every $Z \subseteq \mathsf{For}$, if $Y \subset Z$, then Z is X-inconsistent.

A set of all maximal X-consistent sets is denoted by $\mathsf{Max}(X)$. Maximally consistent sets are obviously theories, i.e., they are closed on the relation of logical consequence.

FACT 6.2. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. For any $A \in \mathsf{For}$, $A \in Y$ iff $Y \vdash_X A$.*

By Fact 6.2 and the axiomatic system definition, we can show that the maximal X-consistent sets are saturated with respect to connectives $\neg, \wedge, \vee$.

FACT 6.3. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then, for every $A, B \in \mathsf{For}$:*

1. *$\neg A \in Y$ iff $A \notin Y$,*
2. *$A \wedge B \in Y$ iff $A \in Y$ and $B \in Y$,*
3. *$A \vee B \in Y$ iff $A \in Y$ or $B \in Y$.*

The next theorem is Lindenbaum's lemma, which we will use in our proof of completeness:

FACT 6.4. *Let X be an axiomatic system and $Y \subseteq \mathsf{For}$. Then, if Y is X-consistent, then there is $Z \subseteq \mathsf{For}$ such that $Y \subseteq Z$ and $Z \in \mathsf{Max}(X)$.*

Let us now proceed to define the canonical model. Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. We define the valuation of the propositional variables in the following way, for any $A \in \mathsf{Var}$:

$$v_Y(A) = \begin{cases} 1, & \text{if } A \in Y, \\ 0, & \text{if } A \notin Y. \end{cases}$$

We now define a sequence of relations $(R_n)_{n \in \mathbb{N}}$:

- $R_1(A, B)$ iff $A \to B \in Y$,
- $R_{n+1}(A, B)$ iff at least one of the following holds:
  - $\langle A, B \rangle \in \bigcup_{i \leqslant n} \mathsf{R}_i$,

- there is $m \in \mathbb{N}$ such that $\langle v_Y, \bigcup_{i \leqslant n} R_i \rangle \models \bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i})$ and $\bigwedge_{i=1}^{m}(C_{2i-1} \to C_{2i}) \supset ((A \to B) \vee (A \wedge \neg B)) \in \mathrm{R}\alpha(X)$,
- $(A \to B) \vee (A \wedge \neg B) \in \mathrm{R}\alpha(X)$.

Let us denote the sum of the defined sequence of relations in the following way: $\breve{R}_Y := \bigcup_{n \in \mathbb{N}} R_n$.

The *canonical model determined with respect to* Y (in short: Y*-model*, in symb.: $\mathfrak{M}_Y$) is the model $\langle v_Y, \breve{R}_Y \rangle$. For any $n \in \mathbb{N}$, the model $\langle v_Y, R_n \rangle$ we call the *canonical* n*-model determined with respect to* Y (in short: n*-model*, in symb.: $\mathfrak{M}_Y^n$). Let us notice that $\mathfrak{M}_Y^n = \langle v_Y, \bigcup_{m \leqslant n} R_m \rangle$, because for any $n \in \mathbb{N}$, $R_n \subseteq R_{n+1}$.

Notice that $\mathrm{R}\alpha(\mathbf{W}_{\to}) = \emptyset$ and for any $Y \in \mathsf{Max}(\mathbf{W}_{\to})$, $(R_n)_{n \in \mathbb{N}} = R_1$, therefore $\mathfrak{M}_Y = \mathfrak{M}_Y^1$.

Note also that the 1-model has the property that any formula belonging to the maximally consistent set with respect to which a given canonical model is determined is true in the 1-model.

FACT 6.5 (Klonowski, 2021a). *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then, for any $A \in \mathsf{For}$, $\mathfrak{M}_Y^1 \models A$ iff $A \in Y$.*

We will now show that the same formulas are true in any canonical model as in the 1-model.

LEMMA 6.6. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then, for any $n \in \mathbb{N}$, for any $A \in \mathsf{For}$, $\mathfrak{M}_Y^1 \models A$ iff $\mathfrak{M}_Y^n \models A$.*

PROOF. Let X be an axiomatic system such that $\mathrm{R}\alpha(X) \neq \emptyset$. We use a double induction on number n and complexity of formulas. We start with $n = 2$.

Initial step. Let $n = 2$.

1.1. *Base case.* Let $A \in \mathsf{For}$, where the complexity of A is equal to 1. Then, $A \in \mathsf{Var}$. By the definition of $\mathfrak{M}_Y^1$ and $\mathfrak{M}_Y^2$, $\mathfrak{M}_Y^1 \models A$ iff $\mathfrak{M}_Y^2 \models A$.

1.2. *Inductive hypothesis.* Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, if complexity A is not bigger than m, then $\mathfrak{M}_Y^1 \models A$ iff $\mathfrak{M}_Y^2 \models A$.

1.3. *Inductive step.* Let $A \in \mathsf{For}$, where complexity of A is equal to $m+1$. If $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then by the inductive hypothesis 1.2, $\mathfrak{M}_Y^1 \models A$ iff $\mathfrak{M}_Y^2 \models A$. Let us consider the case where $A = B \to C$.

„$\Rightarrow$" Suppose that $\mathfrak{M}_Y^1 \models A \to B$. Then, by the definition of truth in a model, ($\mathfrak{M}_Y^1 \not\models B$ or $\mathfrak{M}_Y^1 \models C$) and $R_1(B, C)$. By inductive hypothesis 1.2, $\mathfrak{M}_Y^2 \not\models B$ or $\mathfrak{M}_Y^2 \models C$. By the definition of $(R_n)_{n \in \mathbb{N}}$, $R_2(B, C)$. Thus, by the definition of truth in a model, $\mathfrak{M}_Y^2 \models B \to C$.

„⇐" Suppose that $\mathfrak{M}^2_Y \models B \to C$. Then, by the definition of truth in a model ($\mathfrak{M}^2_Y \not\models B$ or $\mathfrak{M}^2_Y \models C$) and $R_2(B, C)$. By the inductive hypothesis 1.2, $\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$. Suppose $\sim R_1(B, C)$. Then, by the definition of $(R_n)_{n\in\mathbb{N}}$, we have the following possibilities:

(a) there are $k, l \in \mathbb{N}$ such that $D_1, \ldots, D_{2k} \in \mathsf{For}$, $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^k (D_{2i-1} \to D_{2i})$ and $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$,

(b) $((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$.

In case (a), since $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$, then $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in Y$. Therefore, by Fact 6.5, $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C))$. Therefore, since $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^k (D_{2i-1} \to D_{2i})$, we have that $\mathfrak{M}^1_Y \models (B \to C) \vee (B \wedge \neg C)$. But, by the inductive hypothesis 1.2, $\mathfrak{M}^1_Y \not\models B \wedge \neg C$. Therefore $\mathfrak{M}^1_Y \models B \to C$. By the definition of truth in a model, $R_1(B, C)$. In case (b), we reason similarly as in case (a).

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^n_Y \models A$.

Inductive step. Let us consider model $\mathfrak{M}^{n+1}_Y$. We show by induction that for any $A \in \mathsf{For}$, $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$.

2.1. *Base case.* We reason similarly as in 1.1.

2.2. *Inductive hypothesis.* Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, if complexity A is not bigger than m, then $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$.

2.3. *Inductive step.* Let $A \in \mathsf{For}$, where complexity of A is equal to $m + 1$. As in 1.3, if $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then by the inductive hypothesis 2.2, $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$. Let us consider a case where $A = B \to C$.

„⇒" As in 1.3 „⇒". By the definition of $(R_n)_{n\in\mathbb{N}}$, we obtain that if $R_n(B, C)$, then $R_{n+1}(B, C)$.

„⇐" We reason similarly as in 1.3 „⇐". Suppose $\mathfrak{M}^{n+1}_Y \models B \to C$. Then, by the definition of truth in a model, ($\mathfrak{M}^{n+1}_Y \not\models B$ or $\mathfrak{M}^{n+1}_Y \models C$) and $R_{n+1}(B, C)$. By the inductive hypothesis, $\mathfrak{M}^n_Y \not\models B$ or $\mathfrak{M}^n_Y \models C$. Suppose $\sim R_n(B, C)$. Then, by the definition of $(R_n)_{n\in\mathbb{N}}$, we have the following possibilities:

(a) there are $k, l \in \mathbb{N}$ such that $D_1, \ldots, D_{2k} \in \mathsf{For}$, $\mathfrak{M}^n_Y \models \bigwedge_{i=1}^k (D_{2i-1} \to D_{2i})$ and $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$,

(b) $((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$.

In case (a), $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in \mathrm{R}\alpha(X)$, so $\bigwedge_{i=1}^k (D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C)) \in Y$. Thus, by Fact 6.5,

$\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i}) \supset ((B \to C) \vee (B \wedge \neg C))$. Therefore, if $\mathfrak{M}^1_Y \models \bigwedge_{i=1}^{k}(D_{2i-1} \to D_{2i})$, then $\mathfrak{M}^1_Y \models (B \to C) \vee (B \wedge \neg C)$. Thus, by the main inductive hypothesis, $\mathfrak{M}^n_Y \models (B \to C) \vee (B \wedge \neg C)$. But, by the inductive hypothesis 2.2, $\mathfrak{M}^n_Y \not\models B \wedge \neg C$. Therefore, $\mathfrak{M}^n_Y \models B \to C$. This, by the definition of truth in a model, $R_n(B, C)$. In case (b), we reason similarly as in case (a).

Since for any $A \in \mathsf{For}$, $\mathfrak{M}^n_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$, then by inductive hypothesis for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}^{n+1}_Y \models A$. ⊣

By Lemma 6.6 we can prove the following fact:

Lemma 6.7. *Let X be an axiomatic system and $Y \in \mathsf{Max}(X)$. Then for any $A \in \mathsf{For}$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.*

Proof. *Base case.* Let $A \in \mathsf{For}$, where complexity of A is equal to 1, therefore $A \in \mathsf{Var}$. By the definition of $\mathfrak{M}^1_Y$ and $\mathfrak{M}_Y$, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.

Inductive hypothesis. Let $m \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, if complexity A is no bigger than m, then $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$.

Inductive step. Let $A \in \mathsf{For}$, where complexity of A is equal to $m+1$. If $A = \neg B$ or $A = B * C$, where $* \in \{\wedge, \vee\}$, then by the inductive hypothesis, $\mathfrak{M}^1_Y \models A$ iff $\mathfrak{M}_Y \models A$. Let us consider a case $A = B \to C$.

„⇒" Suppose that $\mathfrak{M}^1_Y \models A \to B$. Then, by the definition of truth in a model, ($\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$) and $R_1(B, C)$. By the inductive hypothesis, $\mathfrak{M}_Y \not\models B$ or $\mathfrak{M}_Y \models C$. By the definition of $(R_n)_{n \in \mathbb{N}}$, $\breve{R}_Y(B, C)$. Therefore, by the definition of truth in a model, $\mathfrak{M}_Y \models B \to C$.

„⇐" Suppose that $\mathfrak{M}_Y \models B \to C$. Then, by the definition of truth in a model, ($\mathfrak{M}_Y \not\models B$ or $\mathfrak{M}_Y \models C$) and $\breve{R}_Y(B, C)$. By the inductive hypothesis, $\mathfrak{M}^1_Y \not\models B$ or $\mathfrak{M}^1_Y \models C$. If $\breve{R}_Y(B, C)$, then there is $n \in \mathbb{N}$ such that $R_n(B, C)$. By Lemma 6.6, $\mathfrak{M}^n_Y \not\models B$ or $\mathfrak{M}^n_Y \models C$. Therefore, by the definition of truth in a model, $\mathfrak{M}^n_Y \models B \to C$. Once again by Lemma 6.6, $\mathfrak{M}^1_Y \models B \to C$. ⊣

We will now show that the canonical model determined with respect to a given axiomatic system satisfies the relevant relational conditions.

Lemma 6.8. *Let $\varphi_1, \ldots, \varphi_n$ be sLPR and $X = \mathbf{W}_{\to} \oplus \{\alpha(\varphi_1), \ldots, \alpha(\varphi_n); (\mathtt{R}\alpha)\}$ be an axiomatic system. Then, for any $Y \in \mathsf{Max}(X)$, $\breve{R}_Y$ satisfies conditions $\varphi_1, \ldots, \varphi_n$.*

Proof. Let φ be any of conditions $\varphi_1, \ldots, \varphi_n$. Suppose that φ has the following form $R(F, G)$ for some $F, G \in \mathsf{FOR}$. Then, $\alpha(\varphi)$ has the following form:

$$(F \to G) \vee (F \wedge \neg G).$$

Let F and G have $n \geqslant 1$ and $m \geqslant 1$ variables, respectively. Let $A_1, \ldots, A_n \in \mathsf{For}$ and $B_1, \ldots, B_m \in \mathsf{For}$. By F', G' we denote formulas of object-language obtained from F by replacing i-th variable in F by the formula A_i and G by replacing i-th variable in G by the formula B_i, respectively. Hence, $(F' \to G') \vee (F' \wedge \neg G') \in \mathrm{R}\alpha(X)$. Thus, by the definition of $(R_n)_{n\in\mathbb{N}}$, for any $j > 1$, $R_j(F', G')$. Therefore, $\breve{R}_Y(F', G')$.

Suppose that φ has the following form:

$$\mathrm{And}_{i=1}^n R(F_{2i-1}, F_i) \Rightarrow R(G, H),$$

for some $F_1, \ldots, F_{2n} \in \mathsf{FOR}$ $(n \in \mathbb{N})$ and some $G, H \in \mathsf{FOR}$. Then, $\alpha(\varphi)$ has the following form:

$$\bigwedge_{i=1}^n (F_{2i-1} \to F_{2i}) \supset ((G \to H) \vee (G \wedge \neg H)).$$

Let F_i, G and H have $m_i \geqslant 1$, $j \geqslant 1$ and $k \geqslant 1$ variables, respectively. Let $A_{1_1}, \ldots, A_{m_1}, \ldots, A_{1_n}, \ldots, A_{m_n}, B_1, \ldots, B_j, C_1, \ldots, C_k \in \mathsf{For}$. Let $F'_i, G', H' \in \mathsf{For}$ be formulas obtained from F_i, G and H by replacing the l-th variable in F_i with A_{m_i}, l-th variable in G with B_l and l-th variable in H with C_l, respectively. Hence:

$$\bigwedge_{i=1}^n (F'_{2i-1} \to F'_{2i}) \supset ((G' \to H') \vee (G' \wedge \neg H')). \tag{1}$$

is an element of $\mathrm{R}\alpha(X)$.

Suppose that $\langle F'_1, F'_2\rangle, \ldots, \langle F'_{2n-1}, F'_{2n}\rangle \in \breve{R}_Y$. We consider the following possibilities:

(a) for any $i \leqslant n$, $\langle F'_{2i-1}, F'_{2i}\rangle \in R_1$,
(b) for any $i \leqslant n$, $\langle F'_{2i-1}, F'_{2i}\rangle \notin R_1$,
(c) there is $i \leqslant n$ such that $\langle F'_{2i-1}, F'_{2i}\rangle \in R_1$ and there is $i \leqslant n$ such that $\langle F'_{2i-1}, F'_{2i}\rangle \notin R_1$.

Let us consider case (a). Let $i \leqslant n$. If $\langle F'_{2i-1}, F'_{2i}\rangle \in R_1$, then by the definition of $(R_n)_{n\in\mathbb{N}}$, $F'_{2i-1} \to F'_{2i} \in Y$. Therefore, by Fact 6.3, $\bigwedge_{i=1}^n (F'_{2i-1} \to F'_{2i}) \in Y$. Thus, by Fact 6.3, $(G' \to H') \vee (G' \wedge \neg H') \in Y$. Suppose that $G' \wedge \neg H' \notin Y$. Then, by Fact 6.3, $G' \to H' \in Y$. Thus, by the definition of $(R_n)_{n\in\mathbb{N}}$, $R_1(G', H')$. Therefore, $\breve{R}_Y(G', H')$. Suppose that $G' \wedge \neg H' \in Y$. Since $\bigwedge_{i=1}^n (F'_{2i-1} \to F'_{2i}) \in Y$, so by Fact 6.5, $\mathfrak{M}^1_Y \models$

$\bigwedge_{i=1}^{n}(F'_{2i-1} \to F'_{2i})$. Thus, by Lemma 6.6, $\mathfrak{M}^2_Y \models \bigwedge_{i=1}^{n}(F'_{2i-1} \to F'_{2i})$. Thus, by the definition of $(R_n)_{n\in\mathbb{N}}$, $R^2_Y(G', H')$. Thus $\breve{R}_Y(G', H')$.

Let us consider case (b). For any $i \leqslant n$, there is the least l_i such that $\langle F'_{2i-1}, F'_{2i}\rangle \in R_{l_i}$ and at least one of following possibilities holds:

(b1) there is $s_i \in \mathbb{N}$ such that $C_{1_i}, \dots, C_{2s_i} \in$ For, $\langle v_Y, \bigcup_{t<l_i} R_t\rangle \models \bigwedge_{t=1_i}^{s_i}(C_{2t-1} \to C_{2t})$ and $\bigwedge_{t=1_i}^{s_i}(C_{2t-1} \to C_{2t}) \supset ((F'_{2i-1} \to F'_{2i}) \vee (F'_{2i-1} \wedge \neg F'_{2i})) \in \mathrm{R}\alpha(X)$,

(b2) $(F'_{2i-1} \to F'_{2i}) \vee (F'_{2i-1} \wedge \neg F'_{2i}) \in \mathrm{R}\alpha(X)$.

Let us assume that for $i_1, \dots, i_u \leqslant n$, (b1) holds, while for $i_{u+1}, \dots, i_{u+w} \leqslant n$, (b2) holds. Thus:

$$\bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t}) \supset ((F'_{2i_1-1} \to F'_{2i_1}) \vee (F'_{2i_1-1} \wedge \neg F'_{2i_1})) \qquad (i_1)$$

$$\vdots$$

$$\bigwedge_{t=1_{i_u}}^{s_{i_u}}(C_{2t-1} \to C_{2t}) \supset ((F'_{2i_u-1} \to F'_{2i_u}) \vee (F'_{2i_u-1} \wedge \neg F'_{2i_u})) \qquad (i_u)$$

are elements of $\mathrm{R}\alpha(X)$. Let us also consider the following formulas:

$$\begin{aligned}&\bigwedge_{t=1}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1} / \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset \\ &((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \qquad (i_1^+)$$

$$\vdots$$

$$\begin{aligned}&\bigwedge_{t=1}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1} / \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \dots \\ &[F'_{2i_{u-1}-1} \to F'_{2i_{u-1}} / \bigwedge_{t=1_{i_{u-1}}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset \\ &((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \qquad (i_{u-1}^+)$$

By ($\mathtt{R}\alpha$), from (i_1) and (1) we obtain (i_1^+). Thus, (i_1^+) is an element of $\mathrm{R}\alpha(X)$. By ($\mathtt{R}\alpha$), from (i_2) and (i_1^+) we obtain (i_2^+). Thus, (i_2^+) is an element of $\mathrm{R}\alpha(X)$. Let $2 < z < u-2$ and suppose that (i_{u-z}^+) is an element of $\mathrm{R}\alpha(X)$. By ($\mathtt{R}\alpha$), from ($i_{u-(z-1)}$) and (i_{u-z}^+) we obtain ($i_{u-(z-1)}^+$). Thus, ($i_{u-(z-1)}^+$) is an element of $\mathrm{R}\alpha(X)$. Therefore, (i_{u-1}^+) is an element of $\mathrm{R}\alpha(X)$. By ($\mathtt{R}\alpha$), from (i_u) and (i_{u-1}^+) we obtain the following formula:

$$\begin{aligned}&\bigwedge_{t=1}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1} / \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \dots \\ &[F'_{2i_u-1} \to F'_{2i_u} / \bigwedge_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset ((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \qquad (2)$$

Therefore, (2) is an element of $\mathrm{R}\alpha(X)$.

The following formulas:

$$(F'_{2i_{u+1}-1} \to F'_{2i_{u+1}}) \vee (F'_{2i_{u+1}-1} \wedge \neg F'_{2i_{u+1}}) \tag{i_{u+1}}$$

$$\vdots$$

$$(F'_{2i_{u+w}-1} \to F'_{2i_{u+w}}) \vee (F'_{2i_{u+w}-1} \wedge \neg F'_{2i_{u+w}}). \tag{i_{u+w}}$$

are also elements of $\mathrm{R}\alpha(X)$. Let us consider the following formulas:

$$\begin{aligned}&\bigwedge\nolimits_{t=1,t\neq i_{u+1}}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1}/\bigwedge\nolimits_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})]\ldots\\&[F'_{2i_u-1} \to F'_{2i_u}/\bigwedge\nolimits_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset ((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \tag{i^{+}_{u+1}}$$

$$\vdots$$

$$\begin{aligned}&\bigwedge\nolimits_{t=1,t\neq i_{u+1},\ldots,i_{u+(w-1)}}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1}/\bigwedge\nolimits_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})]\ldots\\&[F'_{2i_u-1} \to F'_{2i_u}/\bigwedge\nolimits_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset ((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \tag{$i^{+}_{u+(w-1)}$}$$

By ($\mathrm{R}\alpha$), from (i_{u+1}) and (2) we obtain (i^{+}_{u+1}). Thus, (i^{+}_{u+1}) is an element of $\mathrm{R}\alpha(X)$. By ($\mathrm{R}\alpha$), from (i_{u+2}) and (i^{+}_{u+1}) we obtain (i^{+}_{u+2}). Theus, (i^{+}_{u+2}) is an element of $\mathrm{R}\alpha(X)$. Let $2 < z < w-2$ ans suppose that ($i^{+}_{(u+(w-z)}$) is an element of $\mathrm{R}\alpha(X)$. By ($\mathrm{R}\alpha$), from ($i_{u+(w-(z-1))}$) and ($i^{+}_{u+(w-z)}$) we obtain ($i^{+}_{u+(w-(z-1))}$). Thus, ($i^{+}_{u+(w-(z-1))}$) is an element of $\mathrm{R}\alpha(X)$. Therefore, ($i^{+}_{u+(w-1)}$) is an element of $\mathrm{R}\alpha(X)$. By ($\mathrm{R}\alpha$), from (i_{u+w}) and ($i^{+}_{u+(w-1)}$) we obtain the following formula:

$$\begin{aligned}&\bigwedge\nolimits_{t=1,t\neq i_{u+1},\ldots,i_{u+w}}^{n}(F'_{2t-1} \to F'_{2t})[F'_{2i_1-1} \to F'_{2i_1}/\bigwedge\nolimits_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t})]\ldots\\&[F'_{2i_u-1} \to F'_{2i_u}/\bigwedge\nolimits_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1} \to C_{2t})] \supset ((G' \to H') \vee (G' \wedge \neg H'))\end{aligned} \tag{3}$$

Therefore, (3) belong to $\mathrm{R}\alpha(X)$. Note that the antecedent of (3) has only the subformulas: $\bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t}), \ldots, \bigwedge_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1} \to C_{2t})$.

Let us take the greatest indices among $i_1, \ldots, i_u$ and denote it as z. By Lemma 6.6, $\langle v_Y R_{l_z}\rangle \models \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}}(C_{2t-1} \to C_{2t})$. Moreover, (3) $\in \mathrm{R}\alpha(X)$, so by Fact 6.5, $\langle v_Y R_1\rangle \models$ (3). Thus, by Lemma 6.6, $\langle v_Y R_{l_z}\rangle \models$ (3). Therefore, by the definition of $(R_n)_{n\in\mathbb{N}}$, $\breve{R}_Y(G', H')$.

The cases when for all $i \leqslant n$ (b1) holds and the case when for all $i \leqslant n$ (b2) holds are simple modifications of the case we consider.

Let us consider case (c). Let $i_1, \ldots, i_{u+w}$ be all indices for which $\langle F'_{2i-1}, F'_{2i}\rangle \notin R_1$, while $i_{u+w+1}, \ldots, i_{u+w+z}$ for which $\langle F'_{2i-1}, F'_{2i}\rangle \in R_1$. As in (b), suppose that for $i_1, \ldots, i_u \leqslant n$, (b1) holds, and for $i_{u+1}, \ldots,$

$i_{u+w} \leqslant n$, (b2) holds. We reason similarly as in case (b) and obtain that the following formula in an element of $\mathrm{R}\alpha(X)$:

$$\begin{aligned}&\textstyle\bigwedge_{t=1,t\neq i_{u+1},\ldots,i_{u+w}}^{n}(F'_{2t-1}\to F'_{2t})[F'_{2i_1-1}\to F'_{2i_1}/\bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1}\to C_{2t})]\ldots\\&[F'_{2i_u-1}\to F'_{2i_u}/\textstyle\bigwedge_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1}\to C_{2t})]\supset((G'\to H')\vee(G'\wedge\neg H'))\end{aligned}\tag{4}$$

Note that the antecedent of (4) has only the subformulas:

$$\textstyle\bigwedge_{t=i_{u+w+1}}^{i_{u+w+z}}(F'_{2t-1}\to F'_{2t}),\bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1}\to C_{2t}),\ldots,\bigwedge_{t=1_{i_u}}^{s_{i_1}}(C_{2t-1}\to C_{2t}).$$

Let us take the greatest indices among $i_1,\ldots,i_u$ and denote it as g. By Lemma 6.6, $\langle v_Y R_{l_g}\rangle \models \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}}(C_{2t-1} \to C_{2t})$. If $\bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}}(F'_{2i-1} \to F'_{2i}) \in Y$, then by Fact 6.5, $\mathfrak{M}^1_Y \models \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}}(F'_{2i-1} \to F'_{2i})$. Thus, by Lemma 6.6, $\mathfrak{M}^{l_g}_Y \models \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}}(F'_{2i-1} \to F'_{2i})$. Hence, $\mathfrak{M}^{l_g}_Y \models \bigwedge_{t=1_{i_1}}^{s_{i_1}}(C_{2t-1} \to C_{2t}) \wedge \ldots \wedge \bigwedge_{t=1_{i_u}}^{s_{i_u}}(C_{2t-1} \to C_{2t}) \wedge \bigwedge_{s=u_{u+w+1}}^{i_{u+w+z}}(F'_{2i-1} \to F'_{2i})$. Since (4) $\in \mathrm{R}\alpha(X)$, so by Fact 6.5, $\mathfrak{M}^1_Y \models$ (4). Thus, by Lemma 6.6, $\mathfrak{M}^{l_g}_Y \models$ (4). Therefore, by the definition of $(R_n)_{n\in\mathbb{N}}$, $\breve{R}_Y(G', H')$. ⊣

Using the lemma 6.7 and the lemma 6.3 we can easily prove the completeness of any axiomatic system obtained by the α algorithm.

Theorem 6.9. *Let $\varphi_1,\ldots,\varphi_n$ be sLPR, $\models$ be a Boolean logic with relating implication determined by $\varphi_1,\ldots,\varphi_n$ and $X = \{\alpha(\varphi_1),\ldots,\alpha(\varphi_n); (\mathtt{R}\alpha)\}$ be an axiomatic system. Then, for any $Y \cup \{A\} \subseteq \mathsf{For}$, if $Y \models A$, then $Y \vdash_X A$.*

Proof. Suppose that $Y \nvdash_X A$. By Fact 6.1, $Y \cup \{\neg A\}$ is X-consistent. Therefore, by Fact 6.4, there is $Z \in \mathsf{Max}(X)$ such that $Y \cup \{\neg A\} \subseteq Z$. By Fact 6.5, $\mathfrak{M}^1_Z \models Y \cup \{\neg A\}$. By Lemma 6.7, $\mathfrak{M}_Z \models Y \cup \{\neg A\}$. Since $X = \{\alpha(\varphi_1),\ldots,\alpha(\varphi_n); (\mathtt{R}\alpha)\}$, by Lemma 6.8, $\breve{R}_Z$ satisfies $\varphi_1,\ldots,\varphi_n$. Therefore, $Y \not\models A$. ⊣

7. Cardinalities and udefiniability in BLRI

In the section we would like to make some observations about BLRI and relating semantics. In order:

- we want to determine how many BLRI systems can be defined in relating semantics,
- investigate how many different semantic structures can define a single BLRI logic,

- check whether there are BLRI logics that cannot be defined by relational properties.

7.1. Cardinality of the family of BLRI

The answer to the question how many BLRI logics there are is not surprising. We have a fact.

FACT 7.1. *There exists continuum of BLRI logics as sets of theses.*

PROOF. We can define uncountable many independent properties of relation R. Let us take for example two different subsets of natural numbers N_1, N_2. Then the properties $R(\bigwedge_{i=1}^{n} A_i, A)$, for $n \in N_j$, where $j \in \{1, 2\}$ and for all $i \leqslant n$, $A_i = A$, define two non-equivalent sets of axioms. Because we have uncountable many subsets of $\mathbb{N}$ then there exists continuum BLRI logical systems. ⊣

7.2. Cardinality of the set of relating semantics

We can show that a logic can be defined by infinitely many relating semantics. Let $N \subseteq \mathbb{N}$. We have the relational property:

$$R(A \vee \neg A, (B \wedge \neg B) \wedge (\bigwedge_{i=1}^{n} B_i))$$

for $n \in N$. We know that there exist uncountably many such properties. They define the sets of equivalent axioms of the following form:

$$((A\vee\neg A) \rightarrow ((B\wedge\neg B)\wedge(\bigwedge_{i=1}^{n} B_i)))\vee((A\vee\neg A)\wedge\neg((B\wedge\neg B)\wedge(\bigwedge_{i=1}^{n} B_i)))$$

So adding these properties to the set of LPR relations defining a given logic does not change that logic. So we can define a given BLRI logic in infinitely many (continuum) ways.

FACT 7.2. *Let $\models$ be BLRI a logic defined by some set of LPR. There exist uncountably many sets of LPR that define exactly the same logic.*

7.3. Undefinability of BRLI logic by means of relations

Each set of LPRs determines some logic, because it defines a set of models. But can every set of models that defines a logic be determined by some set of LPRs? The answer is negative.

FACT 7.3. *There exist BLRI logics that are not definable by sets of relational properties.*

Proof. Let $\mathbf{M}_1$ be the class of all models $\mathfrak{M} = \langle v, R\rangle$, where the relation R, for any $A \in \mathsf{For}$, fulfills the following condition:

$$R(A \wedge \neg A, A) \text{ iff } \langle v, R\rangle \models A. \tag{R^v1}$$

Notice that the condition (R^v1) makes the relation R in a model dependent from the valuation of variables. If we change the valuation, this changes the relation R. The relation R certainly defines a lot of models by adding different valuations v. However, not all models $\langle v, R\rangle$ obtained in that way satisfy the condition (R^v1). So, the set of models $\mathbf{M}_1$ can validate more formulas than the set of all models based on relations R. Below, we present such formulas:

$$\models_{\mathbf{M}_1} A \equiv ((A \wedge \neg A) \to A) \tag{$\mathbf{M}_1 1$}$$

$$\not\models_{\mathbf{M}_1} (A \wedge \neg A) \to A. \tag{$\mathbf{M}_1 2$}$$

For ($\mathbf{M}_1 1$), notice that the logical value of formula A in model $\langle v, R\rangle$ and $R(A \wedge \neg A, A)$ are interdependent. Whereas, the logical value of the formula $(A \wedge \neg A) \to A$ depends only on whether $R(A \wedge \neg A, A)$. For ($\mathbf{M}_1 2$), notice that if we take such model $\langle v, R\rangle$ that $v(p) = 0$, then $\sim R(p \wedge \neg p, p)$. Hence, $\langle v, R\rangle \not\models_{\mathbf{M}_1} (p \wedge \neg p) \to p$. On the other hand, for any set of relations $\mathbf{R}$, we have the following observations:

$$\text{for all } v, \text{for all } R \in \mathbf{R}, \langle v, R\rangle \models A \equiv ((A \wedge \neg A) \to A) \text{ iff } \mathbf{R} = \emptyset \tag{$\mathbf{R}1$}$$

$$\text{for all } v, \text{for all } R \in \emptyset, \langle v, R\rangle \models \neg(A \wedge \neg A) \to A). \tag{$\mathbf{R}2$}$$

So any logic defined with the set of relations $\mathbf{R}$ and containing $A \equiv ((A \wedge \neg A) \to A)$ is the trivial logic: the set of all formulas. On the other hand, the logic determined by the set of models $\mathbf{M}_1$ that contains the formula $A \equiv ((A \wedge \neg A) \to A)$ is not the trivial logic (see ($\mathbf{M}_1 2$)). ⊣

8. Towards an alternative approach to completeness theorem

In [Jarmużek, Klonowski and Kulicki, submitted], we discuss the methods of proving soundness and completeness by a translation of relating logic with only one intensional connective, relating conjunction, into classical logic.[8] The introduced method can be also applied for BLRI.

[8] Such a method, but for logics with the relating implication, was presented in the paper "Axiomatization of the smallest relating logic RF and strict relating implication" during *The 8th edition of Non-Classical Logic*, Łódź, 2016.

Let us consider the partition of Var into Var_1 and Var_2 such that Var_1 and Var_2 are countably infinite sets. Let $\mathsf{Var}_1 = \{p_n : n \in \mathbb{N}\}$ and we index elements of Var_2 by ordered pairs of $\mathsf{For} \times \mathsf{For}$. Thus, $\mathsf{Var}_2 = \{p^{\langle A,B\rangle} : A, B \in \mathsf{For}\}$. In the sequel, we consider a bijection $b\colon \mathsf{Var} \longrightarrow \mathsf{Var}_1$. We denote the set of formulas constructed by elements of $\mathsf{Var}_1 \cup \mathsf{Var}_2$ and $\neg, \wedge, \vee$ by Form. We assume similar conventions as for formulas of For. We can define classical logic on Form. A *classical valuation* is any function $V\colon \mathsf{Form} \longrightarrow \{1, 0\}$ satisfying the following conditions:

$$\begin{aligned} V(\neg A) = 1 &\text{ iff } V(A) = 0 \\ V(A \wedge B) = 1 &\text{ iff } V(A) = V(B) = 1 \\ V(A \vee B) = 1 &\text{ iff } V(A) = 1 \text{ or } V(B) = 1. \end{aligned}$$

Let Val be the set of all classical valuations. Any valuation $v\colon \mathsf{Var} \longrightarrow \{1, 0\}$ can be extended to a classical valuation in a unique way. We can define Classical Propositional Logic in the standard way: for any $X \cup \{A\} \subseteq \mathsf{Form}$, $X \models_{\mathsf{Val}} A$ iff for all $V \in \mathsf{Val}$, if for any $B \in X$, $V(B) = 1$, then $V(A) = 1$.

Let us now define the translation $\tau_1\colon \mathsf{For} \to \mathsf{Form}$ (cf. [Jarmużek, Klonowski and Kulicki, submitted]). For any $A \in \mathsf{For}$, we put:

$$\tau_1(A) := \begin{cases} b(A), & \text{if } A \in \mathsf{Var} \\ \neg\tau_1(B), & \text{if } A = \neg B \\ \tau_1(B) \wedge \tau_1(C), & \text{if } A = B \wedge C \\ \tau_1(B) \vee \tau_1(C), & \text{if } A = B \vee C \\ (\tau_1(B) \supset \tau_1(C)) \wedge p^{\langle A,B\rangle}, & \text{if } A = B \to C. \end{cases}$$

Let us also define the translation $\tau_2\colon \mathsf{Form} \longrightarrow \mathsf{For}$. For any $A \in \mathsf{Form}$, we put:

$$\tau_2(A) := \begin{cases} b^{-1}(A), & \text{if } A \in \mathsf{Var}_1 \\ (B \to C) \vee (B \wedge \neg C), & \text{if } A = p^{\langle B,C\rangle} \\ \neg\tau_2(B), & \text{if } A = \neg B \\ \tau_2(B) \wedge \tau_2(C), & \text{if } A = B \wedge C \\ \tau_2(B) \vee \tau_2(C), & \text{if } A = B \vee C. \end{cases}$$

Let us note the following fact about the composition of τ_1 and τ_2.

FACT 8.1. *For all $\mathfrak{M} \in \mathbf{M}$ and $A \in \mathsf{For}$, $\mathfrak{M} \models \tau_2(\tau_1(A))$ iff $\mathfrak{M} \models A$.*

PROOF. We use an induction on complexity of A.

Base case. Let $A \in \mathsf{For}$, where the complexity of is equal 1. Thus, $A \in \mathsf{Var}$. Thus, $\tau_2(\tau_1(A)) = \tau_2(b(A)) = b^{-1}(b(A)) = A$. Hence, $\mathfrak{M} \models \tau_2(\tau_1(A))$ iff $\mathfrak{M} \models A$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose that for any $A \in \mathsf{For}$, if the complexity of A is not greater than n, then $\mathfrak{M} \models \tau_2(\tau_1(A))$ iff $\mathfrak{M} \models A$.

Inductive step. Let $A \in \mathsf{For}$, where the complexity of is equal $n + 1$.

Let $A = \neg B$. By the inductive hypothesis and the definitions of τ_1 and τ_2, $\mathfrak{M} \models A$ iff $\mathfrak{M} \models \tau_2(\tau_1(\neg B))$ iff $\mathfrak{M} \models \tau_2(\neg\tau_1(B))$ iff $\mathfrak{M} \models \neg\tau_2(\tau_1(B))$ iff $\mathfrak{M} \models \neg B$.

Let $A = B * C$, where $* \in \{\wedge, \vee\}$. By the inductive hypothesis and the definitions of τ_1 and τ_2, $\mathfrak{M} \models \tau_2(\tau_1(A))$ iff $\mathfrak{M} \models \tau_2(\tau_1(B * C))$ iff $\mathfrak{M} \models \tau_2(\tau_1(B) * \tau_1(C))$ iff $\mathfrak{M} \models \tau_2(\tau_1(B)) * \tau_2(\tau_1(C))$ iff $\mathfrak{M} \models B * C$.

Let $A = B \to C$. By the inductive hypothesis and the definitions of τ_1 and τ_2, $\mathfrak{M} \models \tau_2(\tau_1(A))$ iff $\mathfrak{M} \models \tau_2(\tau_1(B \to C))$ iff $\mathfrak{M} \models \tau_2((\tau_1(B) \supset \tau_1(C)) \wedge p^{\langle B,C\rangle})$ iff $\mathfrak{M} \models (\tau_2(\tau_1(B)) \supset \tau_2(\tau_1(C))) \wedge \tau_2(p^{\langle B,C\rangle})$ iff $\mathfrak{M} \models (B \supset C) \wedge ((B \to C) \vee (B \wedge \neg C))$ iff $\mathfrak{M} \models B \to C$. ⊣

We have the following expected fact about translations and the introduced interpretations.

LEMMA 8.2. 1. *For any* $V \in \mathsf{Val}$, *there is* $\mathfrak{M} \in \mathbf{M}$ *such that for any* $A \in \mathsf{For}$, $\mathfrak{M} \models A$ *iff* $V(\tau_1(A)) = 1$.

2. *For any* $\mathfrak{M} \in \mathbf{M}$, *there is* $V \in \mathsf{Val}$ *such that for any* $A \in \mathsf{For}$, $\mathfrak{M} \models A$ *iff* $V(\tau_1(A)) = 1$.
3. *For any* $\mathfrak{M} \in \mathbf{M}$, *there is* $V \in \mathsf{Val}$ *such that for any* $A \in \mathsf{Form}$, $V(A) = 1$ *iff* $\mathfrak{M} \models \tau_2(A)$.

PROOF. *Ad* 1. For $V \in \mathsf{Val}$, we define $\mathfrak{M}_V := \langle v_V, R_V\rangle$ by:

- for any $A \in \mathsf{Var}$, $v_V(A) = V(b(A))$,
- for all $A, B \in \mathsf{For}$, $R_V(A, B)$ iff $V(p^{\langle A,B\rangle}) = 1$.

We prove in the standard way, by induction, that for any $A \in \mathsf{For}$, $\mathfrak{M}_V \models A$ iff $V(\tau_1(A)) = 1$.

Ad 2. For $\mathfrak{M} = \langle v, R\rangle \in \mathbf{M}$, we define $v_{\mathfrak{M}} \colon \mathsf{Var}_1 \cup \mathsf{Var}_2 \longrightarrow \{1, 0\}$ by:

$$v_{\mathfrak{M}}(A) := \begin{cases} v(b^{-1}(A)), & \text{if } A \in \mathsf{Var}_1 \\ 1, & \text{if } A = p^{\langle B,C\rangle} \in \mathsf{Var}_2 \text{ and } R(B, C) \\ 0, & \text{if } A = p^{\langle B,C\rangle} \in \mathsf{Var}_2 \text{ and } {\sim}R(B, C). \end{cases}$$

We extend $v_{\mathfrak{M}}$ on Form to classical valuation $V_{\mathfrak{M}}$. We prove in the standard way, by induction, that for any $A \in \mathsf{For}$, $\mathfrak{M} \models A$ iff $V_{\mathfrak{M}}(\tau_1(A)) = 1$.

Ad 3. For $\mathfrak{M} = \langle v, R\rangle \in \mathbf{M}$, we define $v_{\mathfrak{M}} \colon \mathsf{Var}_1 \cup \mathsf{Var}_2 \longrightarrow \{1, 0\}$ by:

$$v_{\mathfrak{M}}(A) := \begin{cases} v(b^{-1}(A)), & \text{if } A \in \mathsf{Var}_1 \\ 1, & \text{if } A = p^{\langle B,C\rangle} \in \mathsf{Var}_2, \text{ either } R(B,C) \text{ or } \mathfrak{M} \models B \wedge \neg C \\ 0, & \text{if } A = p^{\langle B,C\rangle} \in \mathsf{Var}_2, {\sim}R(B,C) \text{ and } \mathfrak{M} \not\models B \wedge \neg C. \end{cases}$$

We extend $v_{\mathfrak{M}}$ on Form to classical valuation $V_{\mathfrak{M}}$. We prove, by induction, that for any $A \in \mathsf{Form}$, $V_{\mathfrak{M}}(A) = 1$ iff $\mathfrak{M} \models \tau_2(A)$. ⊣

By Lemma 8.2, we obtain the following theorem:

THEOREM 8.3. 1. *For any* $X \cup \{A\} \subseteq \mathsf{For}$, $X \models_{\mathbf{M}} A$ *iff* $\tau_1(X) \models_{\mathsf{Val}} \tau_1(A)$.
2. *For any* $X \cup \{A\} \subseteq \mathsf{Form}$, *if* $X \models_{\mathsf{Val}} A$, *then* $\tau_2(X) \models_{\mathbf{M}} \tau_2(A)$.

Note that the other direction of the second implication from Theorem 8.3 does not hold. Let $\tau_2(p_1) = q$ and $\tau_2(p_2) = r$. Then, $\neg p^{\langle q,r\rangle} \not\models_{\mathsf{Val}} \neg(p_1 \wedge \neg p_2)$, while $\neg((q \to r) \vee (q \wedge \neg r) \models_{\mathbf{M}} \neg(q \wedge \neg r)$.

By standard methods for classical logic we can prove:

THEOREM 8.4. *For any* $X \cup \{A\} \subseteq \mathsf{For}$, *if* $\tau_1(X) \models_{\mathsf{Val}} \tau_1(A)$, *then* $X \vdash_{\mathbf{W}_\to} A$.

PROOF. Suppose that $X \nvdash_{\mathbf{W}_\to} A$. By Fact 6.1, $X \cup \{\neg A\}$ is $\mathbf{W}_\to$-consistent. Therefore, by Fact 6.4, there is $Z \in \mathsf{Max}(\mathbf{W}_\to)$ such that $X \cup \{\neg A\} \subseteq Z$. For any $B \in \mathsf{Var}_1$, we put $v_Z(B) = 1$ iff for some $C \in \mathsf{For}$ such that $\tau_1(C) = B$, $C \in Z$. For any $B \in \mathsf{Var}_2$, we put $v_Z(B) = 1$ iff for some $C, D \in \mathsf{For}$ such that $\tau_1(C \to D) = B$, $C \to D \in Z$. Since Z is a maximally consistent set, we can extend v_Z to $V_Z \in \mathsf{Val}$. Therefore, $\tau_1(X) \not\models_{\mathsf{Val}} \tau_1(A)$. ⊣

There is also another way to relate classical logic defined on Form to BLRI. Take any adequate axiomatization of $\models_{\mathsf{Val}}$ and the provability relation $\vdash_{\mathsf{Val}}$, defined by this axiomatization.

THEOREM 8.5. *For any* $X \cup \{A\} \subseteq \mathsf{For}$, $\tau_1(X) \vdash_{\mathsf{Val}} \tau_1(A)$ *iff* $X \vdash_{\mathbf{W}_\to} A$.

PROOF. Let us observe that $\vdash_{\mathbf{W}_\to}$ contains all classical principles. So, if $\tau_1(X) \vdash_{\mathsf{Val}} \tau_1(A)$, then $X \vdash_{\mathbf{W}_\to} A$, for all $X \cup \{A\} \subseteq \mathsf{For}$. The additional axiom (E→) after translation changes into:

$$((\tau_1(A) \supset \tau_1(B)) \wedge p^{\langle A,B\rangle}) \supset (\tau_1(A) \supset \tau_1(B)).$$

So, this is just a classically valid formula. In consequence, what can be proved by means (E→), after the translation can be also proved in the classical logic. ⊣

By means of some proper restrictions of Val, one can define classical theories, i.e., classical logic with some extra axioms but not closed under substitution, that will be sound an complete for most of axiomatic systems of BLRI [see Jarmużek, Klonowski and Kulicki, submitted].

Let us also note that by Fact 8.1 and Theorem 8.3 many metalogical properties of classical logic are inherited by the least BLRI. For instance, the interpolation theorem holds in the obvious way. Let var be a function assigning to any formula the set of its variables.

Corollary 8.6 (Interpolation). *For any* $A, B \in \mathsf{For}$, *if* $\models_{\mathbf{M}} A \supset B$ *and* $\mathsf{var}(A) \cap \mathsf{var}(B) \neq \emptyset$, *then there is* $C \in \mathsf{For}$ *such that* $\mathsf{var}(C) \subseteq \mathsf{var}(A) \cap \mathsf{var}(B)$, $\models_{\mathbf{M}} A \supset C$ *and* $\models_{\mathbf{M}} C \supset B$.

Proof. Let $\models_{\mathbf{M}} A \supset B$ and $\mathsf{var}(A) \cap \mathsf{var}(B) \neq \emptyset$. Then $\models_{\mathsf{Val}} \tau_1(A) \supset \tau_2(B)$, by Theorem 8.3. Moreover, $\mathsf{var}(\tau_1(A)) \cap \mathsf{var}(\tau_1(B)) \neq \emptyset$. By the interpolation theorem for classical logic, there is $C \in \mathsf{Form}$ such that $\mathsf{var}(C) \subseteq \mathsf{var}(\tau_1(A)) \cap \mathsf{var}(\tau_1(B))$, $\models_{\mathsf{Val}} \tau_1(A) \supset C$ and $\models_{\mathsf{Val}} C \supset \tau_1(B)$. Thus, $\models_{\mathbf{M}} \tau_2(\tau_1(A)) \supset \tau_2(C)$ and $\models_{\mathbf{M}} \tau_2(C) \supset \tau_2(\tau_1(B))$, by Theorem 8.3. Therefore, $\models_{\mathbf{M}} A \supset \tau_2(C)$ and $\models_{\mathbf{M}} \tau_2(C) \supset B$, by Fact 8.1. ⊣

9. Summary

In our paper we studied a generalised method for obtaining an adequate axiomatic system for any relating logic expressed in the language with Boolean connectives and relating implication, determined by the limited positive relational properties.

The method of obtaining axiomatic systems for logics of a given type is called an algorithm, since the analysis allows for any logic of a given type (determined by the limited positive relational properties) to define the axiomatic system adequate for it. We call it the algorithm α.

The proof of completeness of axiomatic systems obtained by applying the α algorithm that we presented is a modification of Henkin-style completeness proofs for propositional logics. The proof in the paper does not use expressivity of the relating relation, since in many cases of limited relational properties the relation R is not expressible.

Our proposal is a partial answer to the problem formulated during the 1st Workshop On Relating Logic[9], called problem α: axiomatiza-

[9] The 1st Workshop On Relating Logic took place in September 25–26, 2020. More on the workshop, see [Jarmużek and Paoli, 2021].

tion of logical systems defined by relating semantics (by given classes of valuations/relations).

We call the answer *partial* because it concerns only the relating implication and the properties that are LPRs. To have more, we also need to consider other relating connectives and non-limited relational properties, including negative properties and disjunction in a consequent of implication. But our method works for all of these cases, so we take up this challenge in [Jarmużek and Klonowski, submitted-b].

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