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On Barrio, Lo Guercio, and Szmuc on Logics of Evidence and Truth

Abstract. The aim of this text is to reply to criticisms of the logics of evidence and truth and the epistemic approach to paraconsistency advanced by Barrio [2018], and Lo Guercio and Szmuc [2018]. We also clarify the notion of evidence that underlies the intended interpretation of these logics and is a central point of Barrio's and Lo Guercio & Szmuc's criticisms.

Keywords: logics of evidence and truth; paraconsistency; evidence; information

1. On logics of evidence and truth

In some recent papers we have proposed what we call an epistemic approach to paraconsistency and the corresponding formal systems, the logics of evidence and truth (*LET*s) [Carnielli and Rodrigues, 2015a, 2017a, 2021; Rodrigues et al., 2021]. The motivation is the pre-theoretical idea that in real-life reasoning we deal with positive and negative evidence, which can be conclusive or non-conclusive.[1] According to the intended interpretation of these logics, non-explosive contradictions are understood as conflicting non-conclusive evidence for both A and $\neg A$. Evidence is thought of as a notion weaker than truth in the sense that

[1] We prefer the expression 'non-conclusive evidence' to 'inconclusive evidence' because the former in our view is more effective in saying that that the evidence available, positive or negative, is *not* conclusive, as opposed to being conclusive. Inconclusive evidence may be read as evidence that is not sufficient to establish which one is the case between A and $\neg A$.

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there may be evidence for a proposition A even if A is not true. Positive and negative evidence, respectively evidence for truth and for falsity, are independent and non-complementary. The falsity of A is represented by $\neg A$, so negative evidence for A is identified with positive evidence for $\neg A$. It may happen that there is no evidence at all for a proposition A, and in this case excluded middle fails, and it may happen that there is conflicting evidence for A, and so explosion fails.

*LET*s are extensions of the logic of first-degree entailment (*FDE*, or Belnap-Dunn 4-valued logic) equipped with a classicality operator $\circ$ that recovers classical logic, in the following sense:

(1) $A, \neg A \nvdash B$, while $\circ A, A, \neg A \vdash B$;
(2) $\nvdash A \vee \neg A$, while $\circ A \vdash A \vee \neg A$;
(3) If $\circ A_1, \ldots, \circ A_n$ hold, then all formulas formed with $A_1, \ldots, A_n$ over $\{\neg, \wedge, \vee, \rightarrow\}$ have classical behavior.

From items (1) and (2), *LET*s are Logics of Formal Inconsistency and Undeterminedness [cf. Carnielli et al., 2020; Carnielli and Rodrigues, 2017a]. Item (3) expresses the fact that a formula $\circ A$ implies that A and $\circ$-free formulas formed with A behave classically. Thus, two consequence relations are expressed in the same formal system: preservation of truth and preservation of evidence. The intended meaning of $\circ A$, however, is that the evidence available for A (positive or negative) is conclusive.

A logic of evidence and truth appeared for the first time in [Carnielli and Rodrigues, 2015a], where we introduced the logic LET_K, which is classical positive propositional logic with the following added:

(4) $\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$,
(5) $\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B$,
(6) $\neg(A \rightarrow B) \dashv\vdash A \wedge \neg B$,
(7) $A \dashv\vdash \neg\neg A$,
(8) $\circ A \vdash A \vee \neg A$,
(9) $\circ A, A, \neg A \vdash B$.

Item (9) is called the *principle of gentle explosion* [Carnielli and Coniglio, 2016, p. 33]. Note that

(10) $\vdash A \vee (A \rightarrow B)$

is an axiom of LET_K. The logic LET_J, introduced in [Carnielli and Rodrigues, 2017a], is LET_K without (10). LET_J is an extension of

positive propositional intuitionistic logic, and is better suited to the interpretation in terms of evidence precisely because (10) does not hold. The 'K' and the 'J' mean that these logics extend, respectively, classical and intuitionistic positive propositional logic. The $\circ$-free fragment of LET_J we called BLE, the basic logic of evidence, which is equivalent to Nelson's logic $N4$.[2] In [Rodrigues et al., 2021] we introduced the logic LET_F, which is LET_J without implication and with a non-classicality operator $\bullet$ added. The intended meaning of $\bullet A$ is that there is no conclusive evidence for A. The following hold in LET_F:

(11) $\vdash \circ A \vee \bullet A$,
(12) $\circ A, \bullet A \vdash B$.

The 'F' in LET_F is due to the fact that its fragment without $\circ$ and $\bullet$ is FDE.[3]

[2] Nelson's logic $N4$ is $N3$ without the principle of explosion. A comparison between Nelson's motivations for the logics $N3/N4$ and the interpretation in terms of evidence that motivates BLE can be found in [Carnielli and Rodrigues, 2017a, Sect. 5.3]. In fact, as has been pointed out by a referee, Nelson's interpretation in terms of P- and N-realizability — as well as López-Escobar's interpretation in terms of proofs and refutations proposed in [López-Escobar, 1972] — anticipates not only our proposal in terms of two primitive concepts of positive and negative evidence, but also Fitting's proposal of the corresponding justification logics (see Section 2.4 below).

[3] The investigation of paraconsistency as conflicting evidence began in 2012. The starting point was the intuitive idea that contradictory propositions could be interpreted in terms of non-conclusive evidence, and we tried to combine this idea with an investigation of Logics of Formal Inconsistency (LFIs). The preliminary results were published in [Carnielli and Rodrigues, 2017b], where the possibility of interpreting mbC (an LFI studied in [Carnielli et al., 2007]) in terms of evidence and truth was investigated. The logic $mbCD$, which is mbC without excluded middle plus (8), was introduced in [Carnielli and Rodrigues, 2015b]. LET_K was obtained by enhancing the negation of $mbCD$ with (4), (5), (6), and (7). Although in [Carnielli and Rodrigues, 2017a] LET_J was presented as an extension of BLE, the latter was obtained as a fragment of LET_J. It was Marcelo Coniglio who pointed out to us that BLE, the $\circ$-free fragment of LET_J, coincides with Nelson's $N4$. It is worth noting that in [van Benthem et al., 2014; van Benthem and Pacuit, 2011] we find a proposal of a logic designed to deal with "possibly contradictory evidence". However, the approach is in terms of neighborhood semantics and the result is quite different from the logics proposed in [Carnielli and Rodrigues, 2017a; Rodrigues et al., 2021]. Another interesting proposal of logics designed to deal with possibly conflicting evidence is found in [Carrara et al., 2019], where two pragmatic logics are introduced. To the extent that such logics express propositional attitudes, their approach is also different from ours, although there are some connections between the notion of evidence assumed by them and the one adopted by us.

Logics of evidence and truth have been criticized by (among others) Barrio [2018] and Lo Guercio and Szmuc [2018] (from now on L&S). A central point in the criticisms advanced by them is the notion of evidence.[4] Our aim here is to clarify the notion of evidence, which is indeed the cornerstone of the epistemic approach to paraconsistency proposed by us, and to answer Barrio's and L&S's criticisms. We will try to show that the criticisms of the notion of evidence proposed by us largely depend on the idea that one should not believe, nor accept, what is not justified by strong or conclusive evidence. In this way, the notion of evidence is understood with a bias that assumes that evidence justifies a doxastic attitude toward a proposition that has to comply with the standards put forth by epistemology. As we will see, however, this is not what we mean by evidence. The remainder of this text is structured as follows. Section 2 tries to make more precise the notion of evidence presented in [Carnielli and Rodrigues, 2017a; Rodrigues et al., 2021]. The notion of non-conclusive evidence is explained as *non-factive justifications*, which are 'justifications that might be wrong'[5], and so do not guarantee, nor imply, the truth of the propositions they are supposed to justify. We also argue that such characterization of evidence is in line with how the word 'evidence' is used in philosophy and in real-life reasoning. In sections 3 and 4 we deal, respectively, with Barrio's and L&S's objections.

2. On the notion of evidence

The notion of evidence for a proposition A has been explained in [Carnielli and Rodrigues, 2017a] as 'reasons for believing in or accepting A'. To summarize: (i) these reasons may be non-conclusive or wrong, (ii) evidence for A does not imply the truth of A, (iii) there may be simultaneous conflicting evidence for a pair of propositions A and $\neg A$, (iv) evidence for A is objective in the sense that it is independent from the belief of an agent in A, and for this reason, a logic of evidence is not a logic of propositional attitudes.[6]

[4] The notion of evidence is also central in the criticisms advanced by Arenhart [2020] and Priest [2019], but we will deal with their objections elsewhere.

[5] We borrowed this terminology from [Fitting, 2016a,b], cf. Sect. 2.4 below.

[6] See [Carnielli and Rodrigues, 2017a, p. 3792 and footnote 5]. Item (iv) indicates that the 'acceptance' of a proposition mentioned above is not supposed to be rational acceptance by an agent (we return to this point in Section 4).

The word 'evidence' does not have a unified meaning, either in philosophy or in the ordinary use of language. In philosophical discussions, it has a connection with justification, but the latter is usually understood with more epistemic weight. Sometimes 'evidence' is used in a sense very close to 'proof', or with the assumption that non-conclusive or wrong evidence cannot be properly called evidence because a rational agent should not accept it as evidence. *LET*s, however, as just mentioned above, were not designed to be logics of propositional attitudes. Although evidence is an epistemic notion closely connected with justification, the intended interpretation of *LET*s does not involve any kind of doxastic attitude of an agent toward a proposition.

In what follows we argue that evidence as 'reasons for believing', and with the four features listed above, is in line with how evidence is used in philosophy (Section 2.1) and in ordinary language (Section 2.2). In addition, we will emphasize the link between evidence and justification, which will be used in Section 2.3 to propose a definition of the notion of evidence. We will also see the basic ideas of the formalization of evidence proposed by Fitting [2016a,b], where we find the concept of a non-factive justification used to explain non-conclusive evidence.

2.1. Evidence, reasons, and justifications

In [Kelly, 2014], besides an explanation of evidence as 'reasons for believing', we find the idea that evidence is what 'confers justification' for a proposition:

> Inasmuch as evidence is the sort of thing which confers justification, the concept of evidence is closely related to other fundamental normative concepts such as the concept of a *reason*. Indeed, it is natural to think that 'reason to believe' and 'evidence' are more or less synonymous.
>
> [Kelly, 2014, Sect. 1]

Indeed, if evidence for A are reasons for believing in A, and if such reasons are conclusive, an agent would be justified in believing in A. But these reasons may be non-conclusive, and, even if they are conclusive, the mere presence of conclusive evidence is not a sufficient condition for the belief of an agent in A.

The idea that evidence may be conclusive or non-conclusive, and that conclusive evidence implies, or is taken to imply, truth, is found in [Achinstein, 2010]:

> I begin with a notion which I shall call potential evidence. [...] First, e can be potential evidence that h even if h is false. Secondly, potential evidence is objective in the sense that whether e is potential evidence that h does not depend upon anyone's beliefs about e or h or their relationship.
> I turn now to a second concept, veridical evidence [...] e is veridical evidence that h only if e is potential evidence that h and h is true. However, this is not yet sufficient. [...] Veridical evidence requires not just that h and e both be true but that e's truth be related in an appropriate manner to h's. [Achinstein, 2010, pp. 4–5]

Veridical and potential evidence correspond respectively to conclusive and non-conclusive evidence, and nothing prevents that there may be circumstances in which there is potential evidence for contradictory propositions. Note also that according to Achinstein evidence is objective in the sense that evidence for A does not depend on the belief of an agent in A. It is implicit in the quotation above that veridical evidence for A is considered as a guarantee of the truth of A, and therefore it is a conclusive (or factive) justification for A.[7]

The connection between evidence and justification is clear in the following quotation from Kim:

> [T]he concept of evidence is inseparable from that of justification. When we talk of 'evidence' in an epistemological sense we are talking about justification: one thing is 'evidence' for another just in case the first tends to enhance the reasonableness or justification of the second. [Kim, 1988, p. 390]

In that paper, Kim is interested in the conditions a belief in A must satisfy if an agent is justified in accepting A as true [Kim, 1988, p. 381]. So, in the passage quoted above, he is thinking of justifications that are conclusive or are accepted as conclusive. Thus he does not use the word 'justification' in the same sense we use it here, which includes also non-factive justifications. But what Kim says about evidence, that it 'tends to enhance the justification' of a proposition, and allows evidence to be non-conclusive and partial, and is in line with the notion of evidence proposed by us.

[7] A similar distinction is found in [Kelly, 2014]: "A given piece of evidence is *defeasible* evidence just in case it is in principle susceptible to being undermined by further evidence in this way; evidence which is not susceptible to such undermining would be *indefeasible* evidence."

Once the characterization of evidence as reasons is accepted, the connection between evidence and justification is also found in [Pollock, 1974]:

> Justification proceeds in terms of *reasons*. When one belief justifies another, then the former is said to be a reason for the latter. [...]
> Let us say that a *good* reason is one that is sufficient to justify the belief for which it is a reason. We often have reasons both for believing something and for disbelieving it. These are all reasons, both pro and con, but they do not all justify what they are reasons for. Each one by itself, in the absence of any competing reasons to the contrary, would be a good reason for believing that for which it is a reason, but taken together with the other reasons it may no longer be a good reason. Thus reasons need not always be good reasons. [Pollock, 1974, pp. 33–34]

We cannot go into a detailed analysis of Pollock's interesting book here but, as far as we can see, he is interested in answering skeptical arguments, and so he is also not talking about non-factive justifications in the sense explained above. But his talking about *reasons pro and con* for a proposition A that do not provide justifications (in the strict sense) fits well with the idea of non-conclusive conflicting evidence that does not guarantee the truth nor the falsity of A.

Finally, it is worth looking at how *The Cambridge Dictionary of Philosophy* [Audi, 1999, p. 293] explains evidence:

> [i.] In philosophical discussions, a person's evidence is generally taken to be all the information a person has, positive or negative, relevant to a proposition. [...]
> [ii.] According to a traditional and widely held view, one has knowledge only when one has a true belief based on very strong evidence. Rational belief is belief based on adequate evidence, even if that evidence falls short of what is needed for knowledge. [...]
> [iii.] The evidence one has for a belief may be conclusive or inconclusive. Conclusive evidence is so strong as to rule out all possibility of error.

So, evidence is positive and negative information (i) which may be conclusive or non-conclusive (iii), and evidence may not imply knowledge, nor imply rational belief (ii). Also clear in the quotation above is the connection between evidence and justification. A central point of our characterization of evidence is that the expressions 'x is evidence for A' and 'x justifies A' may be taken as synonymous in several contexts, and this can be extended to non-conclusive evidence and non-factive justification.

From the aforementioned quotations, although there is no well-defined philosophical concept of evidence, it should be clear that the notion of evidence as reasons for believing proposed by us is in line with the use of 'evidence' and 'reasons' in epistemology. The connection between evidence and justification is also clear, although the standard understanding of the latter in epistemology may not include non-factive justifications.

2.2. Evidence in natural language

Now we turn to the notion of evidence in ordinary language. As a general rule, the usage of a word in natural language is ambiguous, and the word 'evidence' is no exception. The point is not to show that the concept of evidence, as it has been discussed above and will be better characterized in Section 2.3 below, does correspond to the meaning of 'evidence' in natural language. Rather, the point is to show that the concept of evidence proposed by us is indeed found in the ordinary usage of language — and, we would say, is extensively found.

If we take a look at how the word 'evidence' is actually used, we find that in a number of circumstances evidence can be non-conclusive, objective, and independent of belief and truth. In *The Concise Oxford English Dictionary* [2004] we read that evidence is "*information indicating* whether a belief or proposition is true or valid" (our emphasis). *The Cambridge Dictionary Online* [2020] explains evidence as "one or more *reasons for believing* that something is or is not true" (our emphasis). And more importantly, if one does a Google search for 'conflicting evidence', 'lack of evidence', 'conclusive evidence', 'non-conclusive evidence', 'inconclusive evidence', 'partial evidence', 'false evidence', or 'misleading evidence', restricted to reliable sources of English language usage like 'nytimes.com', 'bbc.com', and '.edu', one will find hundreds, even thousands, of collocations with 'evidence' in line with the notion of evidence as characterized by us.[8]

[8] Regarding partial and non-conclusive evidence, it is also worth mentioning that evidence may come in different weights or grades, and so may be more or less conclusive. See, for example, how evidence in legal reasoning is dealt with in [Gordon and Walton, 2009].

2.3. A definition of evidence

The notion of evidence can be made more precise based on the definition of information as *meaningful data* and its connection with the definition of knowledge as justified true belief. Dunn [2008] explains information as

> what is left from knowledge when you subtract justification, truth, belief, and any other ingredients such as reliability that relate to justification. [Dunn, 2008, p. 581]

This idea of information as a pure semantic content that may be false and does not depend on the belief of an agent is a propositional version of the 'standard account' of information, as we read in [Fetzer, 2004]:

> [T]he "standard account" of information would define it simply as meaningful data, which might or might not be true. [Fetzer, 2004, p. 224]

The definition of *data* of course has its own difficulties. Although this characterization of information differs from Floridi's claim that information has to be veridical, in [Floridi, 2011, pp. 85–86] we find an illuminating discussion of the characterization of data as *difference*, or *lack of uniformity*. So, things like a drop of blood on the floor and a scratch on the skin qualify as meaningful data, and such things may also constitute evidence for some proposition. Information as meaningful data does not need to be linguistic, but linguistic information, in its most basic form (without truth, justification, belief, reliability), is just a semantic content, that is, a *proposition*.

Evidence may appear in the form of linguistic or non-linguistic information. It may be conveyed by propositions as well as by things like blood spots, details in a photograph, fossil records, fingerprints on a gun, documents, etc. These 'pieces of information' are *justifications* which may be fallible, partial, wrong — and so non-factive — or even in some cases conclusive. Now, we define *evidence for a proposition* A:

$$\text{Evidence for } A \stackrel{\text{def}}{=} \langle \Theta, A \rangle,$$

where Θ contains whatever is taken as evidence, linguistic or non-linguistic, for A.

An important point about the relation between Θ and A is that it is not to be thought of as a relation of logical consequence, that is, A does not follow logically from Θ through some notion of consequence. In fact, it is not clear that a general account of the relation between

Θ and A could even be spelled out. It involves what could be called a *multi-rational argumentative context* that includes, besides different accounts of logical consequence, probability, generalizations (induction), analogies, protocols, abduction, causality, and more. (A precise account of the relation between Θ and A would presumably shed light on the role of justification in the definition of knowledge as justified true belief.) The proposal of a logic of evidence is to model reasoning with evidence, but *it is not* to give an account of what constitutes evidence for a given proposition.

Science denialism illustrates what we mean by conflicting evidence based on non-factive justifications. Let us take as an example the creationist claim that God created the Earth less than ten thousand years ago. The scientific consensus estimates the age of the Earth at about 4.5 billion years. On the Web we find both propositions, together with a number of justifications. The same applies to other examples of denialism: HIV is just a passenger virus and is not the cause of AIDS, vaccination is not safe, there is no anthropogenic climate change, tobacco does not cause cancer, etc.[9] Justifications for these propositions, as nonsensical as they might be, are out there, objectively available, sometimes without even identifying who made them available.

Less worrying examples of non-factive justifications are found in day-by-day scenarios. Consider the propositions:

(13) Goneril just checked the Internet Banking and his bank balance is \$1,234.56.

(14) It is likely that it will rain next weekend in Belo Horizonte because Climatempo said so.

The justification in (13) is considered factive because bank access over the Web is supposed to be secure and reliable. On the other hand, the justification in (14) is non-factive because of the uncertainty of weather forecast.

Given the account of evidence proposed above and the concept of information as meaningful data, let us to return to the definition of knowledge as justified true belief. As already said, if the evidence available for A is conclusive, then an agent would be justified in believing in A, but the mere presence of conclusive evidence is not a sufficient

[9] On scientific denialism and the (non-factive) justifications that come together with it, see, e.g., [Oreskes, 2019].

condition for the belief of an agent in A. What distinguishes reliable from unreliable information is the justification that comes together and can be part of the information being conveyed. If we take it that reliable information is information for which there is conclusive evidence, the difference between reliable information and knowledge is that reliable information is objective and does not require the belief of an agent.

2.4. On non-factive justifications

As already stated, we borrowed the expression 'non-factive justification' from [Fitting, 2016a]. In this paper, Fitting proposes a formalization of the notion of evidence presented by us in [Carnielli and Rodrigues, 2017a], "making use of well-developed ideas coming from justification logic" [Fitting, 2016a, p. 1149]. He provides an embedding of *BLE* into the modal logic *KX4*, and then an embedding of *KX4* into the justification logic *JX4*, analogous to the embedding of intuitionistic logic into *S4* and the latter into the justification logic *LP* (the *logic of proofs* [Artemov and Fitting, 2020], not to be confused with the *logic of paradox* [Priest, 1979]). Although Fitting's results are relevant for the discussion carried out here, we will not go into an analysis of them. We want just to point out to how Fitting understands the justifications formalized by *S4/LP* and *KX4/JX4*.

The modal logic *KX4* is obtained by dropping the factivity axiom

(T) $\quad \Box A \rightarrow A$

from *S4* and adding

(X) $\quad \Box\Box A \rightarrow \Box A$.

Let $\Box A$ be read as asserting that there is evidence for A. The validity of (T) expresses that the evidence "is certain and never mistaken", but in order to "weaken *S4* to a logic of evidence that may be erroneous" [Fitting, 2016a, p. 1152] (T) has to be dropped. In *KX4*, according to the reading proposed by Fitting, $\Box A$ means that the evidence for A

> is not necessarily correct, that is, factual. We can understand *KX4* as being a logic of implicit uncertain evidence.
>
> [Fitting, 2016a, p. 1150]

> [The idea is to] think of *KX4* as an implicit logic of non-factive evidence, in the same way that Gödel thought of *S4* as a logic of provability".
>
> [Fitting, 2016a, p. 1152]

The modal logic *S4* embeds into the justification logic *LP*. The latter is equipped with terms that represent the strongest kind of evidence, proofs [Fitting, 2016a, p. 1150]. *KX4*, in its turn, embeds into the logic *JX4*, which is equipped with terms that represent

> specific items of uncertain evidence and there are operations on these pieces of evidence. We can think of *JX4* as a logic of explicit evidence. [Fitting, 2016a, p. 1150]

The evidence is explicit because *JX4* (as well as *LP*) is equipped with 'explicit justifications terms' that represent 'explicit reasons'. In *LP* these reasons are proofs, in *JX4* non-factive justifications [Fitting, 2016a, p. 1156]. So, while the evidence represented by intuitionistic logic, *S4*, and *LP* is factive and implies truth, the evidence of *BLE*, *KX4*, and *JX4* is non-factive, in the sense of the notion of evidence explained above.

3. On Barrio's objections

The criticisms advanced by Barrio [2018] have two main points. The first concerns intended interpretations of formal systems. Barrio claims that a formal system has no canonical interpretation [Barrio, 2018, pp. 88, 101, 108] and argues that both *BLE* and LET_J can not only be interpreted without the notion of evidence, but also are amenable even to a dialetheist interpretation [Barrio, 2018, p. 101ff.]. The second point is that truth and conclusive evidence are different concepts, which (according to him) are mistakenly collapsed by the logic LET_J [Barrio, 2018, p. 109]. In what follows, we answer these two objections.

3.1. On interpretations of logics

According to Barrio,

> [the background position] is to reject the one canonical *interpretation thesis*: the idea according to which a logical system has one standard interpretation [Barrio, 2018, p. 88].
> Sometimes is really important to have in mind some informal reading of the inferential rules or models of a pure system to propose a logic. I am not rejecting this idea at all. Simply, I am saying that the question of giving a reading of a pure system of logic is not a natural consequence of this pure system. [...] But one has to have in mind that pure logics usually have multiple interpretations. [Barrio, 2018, p. 96]

> *BLE* and LET_J are compatible with dialetheia. This means that these systems can have different interpretations as pure logics, even an interpretation that is committed with true contradictions.
>
> [Barrio, 2018, p. 101]

A canonical interpretation would be an interpretation induced by, and so appropriated for, a given formal system. Barrio claims (i) that the interpretation of the logics *BLE* in terms of evidence and LET_J in terms of evidence and truth should not be taken as canonical and (ii) that our claim that the logic LET_J is anti-dialetheist is mistaken because LET_J is amenable to a dialetheist interpretation. Barrio provides an interesting analysis of semantic aspects of *BLE* and LET_J, and suggests an interpretation of LET_J according to which $A \wedge \neg A$ means that A is both true and false and $\circ A$ means 'A is not a dialetheia'. Triviality follows only if a formula A is and is not a dialetheia [Barrio, 2018, p. 103].

We agree with Barrio, and maybe his objections arose because we emphasized an anti-dialetheist view of paraconsistency but were not sufficiently as clear as we should have been about the intended interpretations of *BLE* and LET_J. Let us try to state our position more clearly.

First, we also think that formal systems do not have unique interpretations and agree that LET_J may be interpreted dialetheistically. In fact, there is some textual evidence for that. We say in [Carnielli and Rodrigues, 2017a] that our purpose is to present a paraconsistent formal system *and* a corresponding interpretation according to which true contradictions are not tolerated [Carnielli and Rodrigues, 2017a, p. 3789]. So it is not the formal system alone that is incompatible with dialetheism. When we present the inference rules of *BLE* we say that positive intuitionistic logic, a fragment of *BLE*, is well suited to express both preservation of evidence and the positive *BHK*-interpretation [Carnielli and Rodrigues, 2017a, p. 3795].

Second, concerning a dialetheist reading of LET_J, we remark that nothing prevents LET_J from being interpreted in a dialetheist way [Carnielli and Rodrigues, 2017a, p. 3811], and in [Carnielli and Rodrigues, 2017a, p. 3810] we even suggest a dialetheist interpretation of LET_J in terms of the concepts 'just true' and 'just false'. The sense in which LET_J *together with the intended interpretation* is anti-dialetheist is that the simultaneous truth and falsity of A is expressed by $A \wedge \neg A \wedge \circ A$, and the latter yields triviality; cf. [Carnielli and Rodrigues, 2017a, p. 3807] and (9) above.

3.1.1. On philosophical interpretations

Now we turn to the topic of philosophical interpretations of formal systems, discussed by Barrio [2018] and also by Barrio and Da Ré [2018] (from now on B&D). A tripartite distinction between pure logics, applied logics, and philosophical interpretations of logics is proposed by B&D. The distinction between pure and applied logics is well-known: the former is a relation of logical consequence, syntactic or semantic, mathematically defined; the latter is a logic applied to some purpose, e.g., to model electric circuits or fragments of real-life reasoning [cf. Barrio and Da Ré, 2018, pp. 157ff.]. To these two concepts B&D add the concept of philosophical interpretation of a logic.

> It is clear that one can use philosophical interpretations to try to obtain additional understanding of certain pure logical theories (from a proof-theoretical or model-theoretical point of view). It is also true that at least in some cases pure logics are developed to find an answer to a philosophical interpretation. For example, pure modal logics have been motivated by our interest in explaining what is metaphysically necessary or possible. [Barrio and Da Ré, 2018, p. 159]

As we understand it, a philosophical interpretation of a logic is the intended meaning attributed to its expressions motivated by, or connected to, philosophical concepts. In order to be an applied logic, a formal system has to have intended meanings attributed to its expressions. These intended meanings, at first sight, may come without a philosophical interpretation, but it is reasonable to suppose that any applied logic is amenable to a conceptual — and so philosophical — discussion, given that in formalizing a domain a logic says things about that domain. Depending on how we understand the concept of an applied logic, it may be that a formal system has a philosophical interpretation but no application, the latter understood in a strict sense. But in a broad sense, a philosophical discussion about some fragment of natural language formalized by a logic can be regarded an application of a logic. In any case, although there is no clear dividing line between applications and philosophical interpretations, we think that having these two concepts at hand, as B&D claim, provides a better understanding of formal systems.[10]

[10] Arenhart [2020], in opposition to B&D, argues that the notion of philosophical interpretation of a logic is not really necessary, and that the distinction between pure and applied logics would be enough for the philosophical study of formal systems. A more detailed analysis of Arenhart's paper will be done elsewhere.

Let us start by taking a look at some logics and their interpretations. Classical propositional logic (*CPL*) is perfectly suited to express preservation of a realist and non-dialetheist notion of truth, but *CPL* can also be interpreted in terms of electrical circuits. Intuitionistic logic has been motivated by Brouwer's views on mathematics and its standard interpretation is given by the *BHK*-interpretation [see, e.g., Heyting, 1956; Troelstra and van Dalen, 1988]. The latter can be traced back to Kolmogorov's reading of Heyting's formalization of intuitionistic logic as solutions of problems [Kolmogorov, 1932]. According to a common reading, the modal logic *S5* expresses Leibniz' notion of necessity [e.g. Chellas, 1980, p. 3], and there are a plenty of modal logics (alethic and non-alethic) that are philosophically motivated. Belnap-Dunn 4-valued logic expresses the deductive behavior of information in an inconsistent and incomplete database [see, e.g., Belnap, 1977]. Priest's logic of paradox *LP* [Priest, 1979] not only fits the claim that there are true contradictions but also has been motivated by a metaphysical thesis about the nature of truth. And last but not least, the intended interpretation of LET_J in terms of evidence and truth has been motivated by a pluralist view of logic which fits Belnap's idea of a 'local logic' to be used in contradictory contexts without rejecting classical logic [Belnap, 1977, p. 30]. Are all these logics applied logics? It depends on what we understand by the application of a logic. Let us take a closer look at *CPL* and modal logics.

At first sight, it may seem that there is no genuinely philosophical motivation behind *CPL* applied to electric circuits, but that would be a hasty conclusion. Shannon, in his 1937 Master's thesis [Shannon, 1938], formally proved that the behavior of certain types of electrical circuits is logically equivalent to Boolean algebras, and indeed such applications of *CPL* are due to the fact that *CPL* and Boolean algebras are just 'two sides of the same coin'. As Tarski proved in 1935 [see Font et al., 2003], and as already foreseen by Boole in the 19th century, Boolean algebras arise in logic in an intrinsic form: to any consistent (or non-contradictory) theory T of *CPL*, there is a corresponding Boolean algebra formed by equivalent classes of sentences, called the Lindenbaum-Tarski algebra of T, with operations given by the connectives of the language of T. So Tarski's proof, based on Lindenbaum's proposal, basically says that consistent theories of *CPL* have as algebraic counterparts Boolean algebras of sets. On the other hand, the well-known representation theorem of Stone shows that any Boolean algebra is reducible to a concrete algebra of sets [see Halmos, 1963; Stone, 1936]. Therefore the algebraic

side of *CPL* consists of canonical Boolean algebras, and *CPL* has an intrinsic Boolean-algebraic nature.[11]

Concerning modal logics, there are a lot of philosophical discussions about alethic, epistemic, deontic, and temporal logics. Although there are interactions between modal logics and computer science [see, e.g., Garson, 2018, Sect. 13, and Goranko and Rumberg, 2020, Sect. 11], it is not the case that every philosophical discussion of modal concepts is connected to an 'effective application' — i.e. an application not restricted to a conceptual discussion. It is to be noted, however, that although conceptual discussions, inside and outside philosophy, are not always connected with applications in the strict sense, such discussions may be interesting and relevant in themselves. Of course, an 'effective application' is not a necessary condition for a conceptual discussion to be worth pursuing. In any case, even without a clear notion of an applied logic, it is certain that philosophical interpretations, in B&D's sense, are far from being unnecessary additions to the distinction between pure and applied logics.

3.1.2. Interpretations and logical pluralism

The topic of interpretations of logics, which is the framework of Barrio's criticisms, is central in the approach to logical pluralism defended by us. The idea is that classical logic and some paraconsistent and paracomplete logics 'talk about different things', and that what a logic 'talks about' is the property of propositions being preserved by its inferences. Such a property may be given by the intuitive meaning of the value 1 in a bi-valued semantics that may be non-deterministic. More precisely, if L is a non-classical logic in a language with the standard propositional connectives and L is a fragment of classical logic, the property preserved by L (if any) is the intuitive meaning of the semantic value 1 in a non-deterministic valuation semantics[12]. And a philosophical interpretation

[11] This also suggests that the interpretation of classical logic as preserving a realist and non-dialetheist notion of truth might well be taken as the canonical interpretation of classical logic. In this case, *CPL* would be a counterexample to Barrio's claim that no formal system has a canonical interpretation.

[12] We say that a semantics is non-deterministic when the semantic value of molecular formulas is not functionally determined by the values of atomic formulas. On non-deterministic valuation semantics, see [Loparic, 1986, 2010]. Valuation semantics for *LET*s are presented in [Carnielli and Rodrigues, 2015a, 2017a; Rodrigues et al., 2021] and discussed in [Carnielli and Rodrigues, 2019, pp. 181ff.].

of a logic is a conceptual framework that motivates and justifies the meaning attributed to the semantic value 1. In classical logic such a meaning is the classical notions of truth and falsity and, of course, the semantics is deterministic. In intuitionistic logic it is the notion of constructive proof, and in *BLE* it is non-conclusive evidence. This idea also applies, for example, to Nelson's *N3*, which expresses the deductive behavior of constructive truth and constructive falsity, and to *FDE* interpreted as preserving evidence [cf. Rodrigues et al., 2021, Sect. 2.2.1], as well as to a notion of information stripped of ontological and epistemic ingredients as suggested by Odintsov and Wansing [2016].

3.1.3. A brief digression: more on interpretations of logics

In what follows we make some remarks about three more points related to interpretations of logics.

Although it is true that a formal system may have more than one intended interpretation, it must be noted that a formal system can exclude particular interpretations. Classical logic excludes the interpretation in terms of constructive proofs because excluded middle holds, and both classical and intuitionistic logic excludes the interpretation in terms of information because explosion holds. The claim that a formal system allows different intended interpretations is thus significantly constrained by the formal system in question.

Another point related to philosophical interpretations is that a unique interpretation, in principle, should not be compatible with two different formal systems. Consider, for example, two different logics that have been proposed to express constructive proofs, Johansson logic [Johansson, 1937] and Heyting logic [Heyting, 1930, 1956]. Given that explosion holds in Heyting logic (H), but does not hold in Johansson logic (J), for some A and Γ, it may be that $\Gamma \vdash_H A$ but $\Gamma \nvdash_J A$. Therefore, the notion of constructive proof (the philosophical interpretation) cannot be the same in both logics. By a similar argument, the notion of truth preserved by *LP* (Priest's logic of paradox) and *CPL* cannot be the same notion of truth.

And what about logics without interpretations? Indeed, there are several pure logics not amenable to an intuitive interpretation, and without any clear application — for example, da Costa's C_n hierarchy. C_1 is a very weak logic with some counterintuitive properties, e.g. $\neg(A \wedge \neg A)$ and $\neg(\neg A \wedge A)$ are not equivalent. This does not mean, however, that logics without intuitive interpretations are not worth studying. Firstly,

the investigations of pure logics provide insights into the properties of formal systems that are interesting in themselves. Secondly, pure logics may give rise to logics with applications and philosophical interpretations, and C_n is a nice example. The investigation of *C-systems* and *dC-systems* [Carnielli and Marcos, 2002] gave rise to *LFI*s [Carnielli et al., 2007], and the latter to the logics of evidence and truth.

3.2. Truth versus conclusive evidence

The second of Barrio's objections concerns the notion of conclusive evidence, treated in LET_J as truth and subjected to classical logic. Barrio claims that recovering classical logic in LET_J for propositions with conclusive evidence undermines the epistemic approach, because conclusive evidence and truth are different notions.

> [Classical logic] can be interpreted in different ways and one of them could be using strong evidence. But this view seems to reinforce my point: *the strongest kind of evidence* and truth are different philosophical notions. Recovering *CL* inside LET_J produce a deviation in interpretations: truth is transformed into conclusive evidence.
>
> [Barrio, 2018, p. 109]

This is a sensible objection. However, we do not think that conclusive evidence is the same as truth in a metaphysical sense, and when we take conclusive evidence as subjected to classical logic we do not mean that it has to be *identified* with truth. Conclusive evidence is treated as truth because people reason classically about conclusive evidence in the sense that people reason *as if* conclusive evidence were tantamount to truth.

Although in [Carnielli and Rodrigues, 2017a] we have not addressed this point clearly, we did say there that in some contexts we deal simultaneously with evidence and truth, that is, with propositions that *we take* as conclusively established as true or false, as well as others for which only non-conclusive evidence is available. We also said that classical logic is restored for propositions *we want to declare* as either true or false [Carnielli and Rodrigues, 2017a, p. 3803]. We did not, therefore, identify conclusive evidence with metaphysical-transcendent truth.

Finally, it is worth mentioning [Carnielli and Rodrigues, 2021, Section 3.2.2], where we illustrate how *BLE* works with three scenarios: (i) Newtonian mechanics, subjected to classical logic; (ii) Newtonian mechanics plus the theory of electromagnetic fields, a contradictory scenario that would be modeled by *BLE*; and (iii) relativity theory, again

subjected to classical logic. The underlying logic of scenario (ii) has to be non-explosive, and our point is that *BLE* is appropriate to describe the inferences allowed in scenario (ii), where conflicting evidence is available. Classical logic was the underlying logic of Newtonian mechanics not because its propositions were true in a metaphysical sense, but because the evidence available at that time for its propositions was considered conclusive. The same applies to Einstein's theory of relativity, even though it may be revised and adjusted, for example, due to new evidence. Indeed, we do not stop using classical logic in well-established scientific theories just because their propositions in principle can be revised and, therefore, may not be true in the metaphysical-transcendent sense. In fact, it might even be objected that, with just a few exceptions, no proposition of empirical sciences is true in such a transcendent sense, but this does not prevent classical logic from being the underlying logic of such theories.

4. Lo Guercio and Szmuc's objections

4.1. On rational acceptance

The crucial point of the objections advanced by Lo Guercio and Szmuc [2018] is that in our characterization of evidence presented in [Carnielli and Rodrigues, 2017a] the acceptance of contradictions must be understood as rational acceptance. Their argument goes as follows.

1. Carnielli & Rodrigues say that there are some contexts in which *contradictions are accepted* [Lo Guercio and Szmuc, 2018, p. 154].
2. They are thinking of *epistemically rational acceptance*, which means that it is rational for an agent S to accept A iff A fits the evidence available to S [Lo Guercio and Szmuc, 2018, pp. 155, 156].
3. So, there are cases in which it is *epistemically rational to accept a contradiction* [Lo Guercio and Szmuc, 2018, p. 156].
4. As a consequence, Carnielli & Rodrigues' intended interpretation of *BLE* entails *a stronger form of extreme permissivism*, which should be rejected [Lo Guercio and Szmuc, 2018, p. 158].

Extreme permissivism is characterized by L&S as the thesis that

> there are possible cases in which not only the evidence permits different doxastic attitudes toward a proposition but it allows radically incom-

> patible ones. [...] Extreme Permissivists may agree that some body of evidence makes it rational either to accept A or to accept $\neg A$, but refrain from admitting that it is rational to accept A and to accept $\neg A$, *at the same time.* [Lo Guercio and Szmuc, 2018, pp. 157–158]

The view L&S attribute to us is indeed a stronger form of extreme permissivism because it

> entails that there are possible cases in which not only the evidence makes it rational either to accept A or to accept $\neg A$, for some proposition A, but also to accept $A \wedge \neg A$, that is, to accept A and to accept $\neg A$ at the same time. [Lo Guercio and Szmuc, 2018, p. 158]

The crucial step in L&S's argument is the second premise. Indeed, if the presence of conflicting evidence for A were enough for rational acceptance of A and $\neg A$, and given the quite implausible strong form of permissivism, the interpretation in terms of evidence could not be maintained. However, we did not mean that the presence of conflicting evidence for a proposition A justifies the rational acceptance of both A and $\neg A$. The idea was that a contradiction is accepted in a context of reasoning[13] in the sense that it is admitted as a premise, is the result of one or more scientific theories in some critical circumstances, or is stored in a database. In fact, the word 'admitted' instead of 'accepted' would have been a better choice since it does not have the epistemic ingredient usually attributed to the latter in epistemology.

It is rational to accept contradictions only in the sense that we (and some machines too) *have to* reason in the presence of contradictions without triviality. It is in this weak sense that a contradiction can be accepted. Thus, what is rational is not to accept contradictions in the sense of 'epistemic rational acceptance', but to deal with contradictions without accepting them as true, and this is precisely the point of the anti-dialetheist reading of LET_J.

4.2. On suspending judgment

In addition to their objection to the notion of evidence based on the idea that evidence implies rational acceptance, L&S also claim that suspend-

[13] By a context of reasoning we understand a scenario in which there is a set of propositions subject to an underlying logic and closed under logical consequence, but not necessarily with the presence of agents. Thus, theories and databases (structured or not), and of course the Web, are contexts of reasoning.

ing judgment is the best alternative in the face of conflicting evidence for A and $\neg A$:

> in these cases the total evidence supports suspending judgment about whether A is the case. If this is on the right track, either the total evidence supports A, or $\neg A$, or it supports *suspending judgment* regarding the matter. [Lo Guercio and Szmuc, 2018, p. 163]

However, the fact that propositions without conclusive evidence are not subjected to classical logic may be understood precisely as a suspension of judgment with respect to the *truth-value* of these propositions. Accepting A and $\neg A$ and their respective justifications as non-factive, without declaring either one as true, is just to suspend the judgment about them.

5. Final remarks

We have tried to make more precise here the notion of evidence presented in [Carnielli and Rodrigues, 2017a; Rodrigues et al., 2021] and have argued that our use of 'evidence' is perfectly in line with how this word is effectively used in philosophy and ordinary language. We have also emphasized the connection between 'evidence' and 'justification', although the latter sometimes has more epistemic weight than the former. We have argued that the notion of a non-factive justification, without that epistemic weight, not only fits the idea of a justification in several scenarios but also, and more importantly, is an indispensable concept for analyzing contradictory contexts where alleged justifications are presented for contradictory propositions, the Web being the paradigmatic example. Indeed, the crucial point in understanding the underlying idea of paraconsistency as conflicting evidence is to extend the notion of justification to include non-factive justifications.

As we said in the beginning of this text, a central point in our replies to Barrio, Lo Guercio, and Szmuc is that the criticisms of the notion of evidence, particularly the notion of non-conclusive evidence (which could be extended to the notion of non-factive justification) stick to the idea that people should not believe in what is not justified by strong or conclusive evidence. In this context, the notion of evidence is read with a bias that assumes that evidence justifies a doxastic attitude toward a proposition that has to comply with the standards put forth

by epistemological discussions about knowledge, justification, epistemic acceptance, etc. But the logics of evidence and truth, although related to that discussion, have a different purpose, viz., to provide a formal system capable of formalizing positive and negative, conclusive and non-conclusive evidence, as explained in sections 1 and 2 above.

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