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Zeno of Sidon *vindicatus*: A Mereological Analysis of the Bisection of the Circle

Abstract. I provide a mereological analysis of Zeno of Sidon's objection that in Euclid's *Elements* we need to supplement the principle that there are no common segments of straight lines and circumferences. The objection is based on the claim that such a principle is presupposed in the proof that the diameter cuts the circle in half. Against Zeno, Posidonius attempts to prove the bisection of the circle without resorting to Zeno's principle. I show that Posidonius' proof is flawed as it fails to account for the case in which one of the two circumferences cut by the diameter is a proper part of the other. When such a case is considered, then either the bisection of the circle is false or it presupposes Zeno's principle, as claimed by Zeno.

Keywords: Euclid's *Elements*; Zeno of Sidon; mereology

1. Introduction

In a long passage of the commentary on Book I of Euclid's *Elements*, Proclus reports a debate between Zeno of Sidon, a leading figure of the late Epicurean school, and the Stoic philosopher Posidonius (*In Primum Euclidis*, 214-218).[1] The debate is centered around Zeno's objection to *Elem.* I.1, where Euclid famously shows how to construct an equilateral

[1] The Greek text is printed in *Procli Diadochi in primum Euclidis Elementorum librum commentarii* ex recognitione Godofredi Friedlein, Teubner, Leipzig, 1873. For the English translation I have consulted Morrow's classical edition (Morrow, 1970) as well as the edition of the fragments of Posidonius (Kidd, 1999, 100-106). Zeno's fragments can be found in (Angeli and Colaizzo, 1979). For the *Elements*, see *Euclidis Elementa*, post J.L. Heiberg edidit E.S. Stamatis, 5 volumes, Teubner, Leipzig, 1969-1977, and in English translation (Heath, 1956).

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triangle on a straight line. Zeno's goal apparently was to refute the whole geometry, but unlike the Epicureans, who rejected the principles of geometry altogether, he was still willing to concede the principles, but denied "that the propositions coming after the principles can be demonstrated unless they grant something that is not contained in the principles" (*In Primum Euclidis*, 199).[2] In particular, he argued that *Elem.* I.1 is not proved unless we assume in addition that "neither circumferences nor straight lines can have a common segment" (*In Primum Euclidis*, 215). Proclus claims that the allegedly missing principle can, in fact, be derived and presents a proof based on the principle that the diameter bisects the circle, a property attributed by Proclus himself to Thales (*In Primum Euclidis*, 157-158). Alas, the bisection of the circle by the diameter, says Zeno, depends on the missing principle, hence the proof is entirely circular. At this point Proclus introduces the Stoic philosopher Posidonius attacking Zeno's objection in the attempt to prove the bisection of the circle without resorting to Zeno's principle.

In this paper I suggest that Posidonius misunderstood Zeno's objection, whether deliberately or inadvertently. I argue that the proof of the bisection of the circle revolves around three cases stemming from the assumption (for *reductio*) that the two circumferences of a circle cut by the diameter do not coincide. Borrowing terminology from modern mereology, I will show that Proclus only deals with the case where the circumferences are disjoint, namely they have no common part. Posidonius, urged by Zeno's objection, reluctantly considers also a second case where they are (properly) overlapping. However, he entirely overlooks the case in which one circumference is a proper part of the other. Not seeing this case led him to mistakenly claim that the circumferences, when not coinciding, must necessarily fall inside one another—if not entirely, at least partly. However, if one circumference is a proper part of the other one, then the two still fail to coincide, but they do not fall—not even partly—inside one another.

I conclude that while Posidonius' reply to Zeno for the case of two (properly) overlapping circumferences is satisfactory (along the way I even conjecture that a detailed proof of this case can be found in a passage by Simplicius), it is not at all clear how the proof can go through

[2] For a complete historical account and a survey of the existing scholarly literature on the criticism of Epicurus and his school towards mathematics, see Verde (2013), especially ch. III.1 "La questione della geometria epicurea", pp. 249-308.

when one of them is a proper part of the other. In fact, under this hypothesis it is easy to see the diameter does *not* cut the circle in half. This is, of course, problematic for Posidonius, whose proof of Zeno's principle rests entirely on the bisection of the circle.

Can the proof of the bisection of the circle be emended? One way, which Zeno attributes to his adversaries, is to simply assume Zeno's principle. Indeed, if Zeno's principle is assumed, then the bisection of the circle is safe, since the circumferences can neither (properly) overlap nor can one be a proper part of the other—otherwise they would have a common segment. But, then, the bisection of the circle would depend on Zeno's principle and consequently the former could not be used to prove the latter, on the pain of circularity. This, I believe, vindicates Zeno's objection as it shows that the bisection of the circle is either false or presupposes the principle that straight lines and circumference cannot have common segments.

2. Zeno's objection to the equilateral triangle

In *Elem.* I.1 Euclid famously shows how to construct an equilateral triangle on a straight line. The proof uses the first and third postulate ("to draw a straight line from any point to any point" and "describe a circle with any center and distance"), the first common notion ("things which are equal to the same thing are also equal to one another"), as well as the definition of circle ("a circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another"). For convenience, let us recall the proof. Let AB be a straight line, where A and B are the centers of two circles with distance AB (third postulate). From the point C where the circles cut one another, draw the straight lines CA and CB to the points A and B, respectively (first postulate). From the definition of a circle, it follows that the straight lines CA and CB are both equal to AB. Therefore, they are also equal to one another by the first common notion. Thus, the triangle ABC is equilateral.

According to Proclus, Zeno observed that the proof requires the principle, never stated by Euclid, that "neither circumferences nor straight

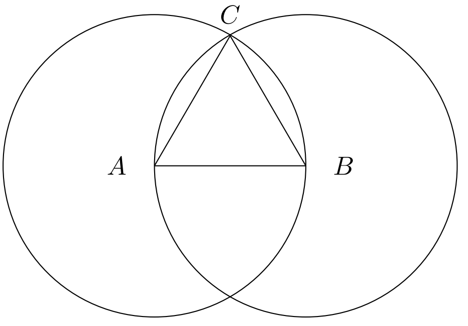

Figure 1. The diagram of *Elem.* I.1

lines can have a common segment" (*In Primum Euclidis*, 215).[3] I shall call this principle *Zeno's principle*.

To show that without Zeno's principle the proof of the first proposition is not valid, Zeno provides a counter-example. He shows how to construct a non-equilateral triangle using the same principles as Euclid, but allowing the two straight lines from A and B to the intersection point C to have the common segment CE (see Figure 2). On the one hand, it still follows from the definition of circle that CEA and CEB are equal to AB (hence they are also equal to one another). However, the triangle so constructed is not equilateral, for the sides EA and EB are less than the base AB.

3. A (circular) proof of Zeno's principle

After presenting Zeno's objection, Proclus tries to dispel the skepticism around *Elem.* I.1 by appealing to the definition of a straight line and the second postulate ("to produce a finite straight line continuously in

[3] τό μήτε περιφερῶν μήτε εὐθειῶν εἶναι τμήματα κοινά. Notice that the first formulation of Zeno's objection reported by Proclus only excludes the common segment of two straight lines (*In Primum Euclidis*, 214). From the ensuing discussion, however, it is clear that Zeno never intended to focus exclusively on straight lines, without mentioning the circumferences. While several early modern scholars, including Gerolamo Saccheri, attempted to prove that two straight lines cannot have a common segment, nobody ever seemed to be interested in establishing a similar result for circumferences. Saccheri proof of Zeno's principle is to be found in Lemma 2, Book One, First Part of his famous *Euclides ab omni naevo vindicatus* (De Risi, 2014, 294-296).

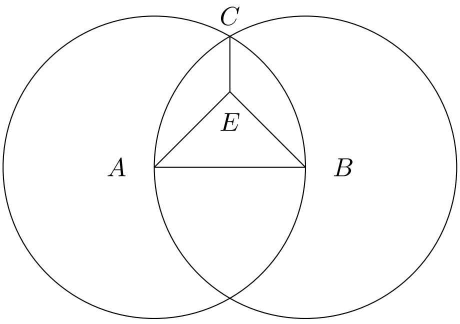

Figure 2. Zeno's counterexample to *Elem.* I.1

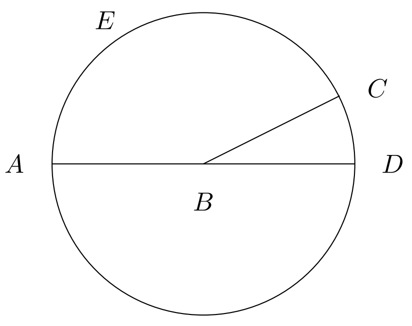

Figure 3. Proclus' diagram against Zeno's counterexample

a straight line"). These principles, he argues, are enough to guarantee that it is impossible for two straight lines to have a common segment. In particular, Proclus specifies that the second postulate is not to be taken as merely asserting the existence of a line extending a given line, but implying that such an extension is also unique.[4]

Much more interesting, however, is the mathematical proof of the principle that he offers shortly after. Proclus does not inform us about his sources and for all we know this part may be his own contribution. Nonetheless, it is more likely that the proof is due to Posidonius.

The proof is very simple. Assume for a contradiction that there exists a common segment AB of two straight lines AC and AD.

[4] On uniqueness, especially with reference to Zeno's argument, see Maffezioli (2022).

With center B and radius AB, describe the circle ACD. Since ABC and ABD are both diameters of the circle and since the diameter cuts the circle in half, the semicircle AEC and ABD are equal to one half of the circle. Therefore AEC and AED are equal to one another by the first common notion. But this is impossible, for the former is contained into the latter.

Thus, a crucial aspect of the proof of Zeno's principle is that it rests on the fact that the diameter cuts the circle in half, a property included by Euclid in the definition of diameter ("a diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle"). It is commonly believed that the reason why Euclid preferred to have the bisection of the circle embedded in the definition of the diameter is that the standard proof is by superimposition, a technique that he apparently deemed to be not rigorous enough. Although the bisection of the circle is granted by the definition of a diameter, we know of previous attempts to prove it as a theorem. Proclus himself attributes such a proof to Thales (*In Primum Euclidis*, 157-158). "Imagine the diameter drawn and one part of the circle fitted (συαρμοζόμενον) upon the other. If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the center to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible. The one part, then, fits (ἐφαρμόζει) the other, so that they are equal, consequently the diameter bisects the circle."(*In Primum Euclidis*, 158). The proof is accompanied by a diagram, drawn at the bottom of f. 86r of the *Monac. gr.* 427, showing the result of putting the two parts of the circle cut by the diameter on top of one another and a straight line from the center cutting the circumferences of the two parts (see Figure 4).

Thus, if it were possible for the diameter to cut two unequal parts of the same circle, then such a circle would have two radii of unequal length, which contradicts the definition of the circle.

Zeno thought that also the proof reported by Proclus of the principle that there are common segments of straight lines and circumferences was to be rejected. For, he argues, such a proof depends on the bisection of the circle and this, in turn, "depends on our previous assumption that two circumferences cannot have a common segment" (*In Primum Euclidis*, 216). In other words, the attempt to prove the principle is

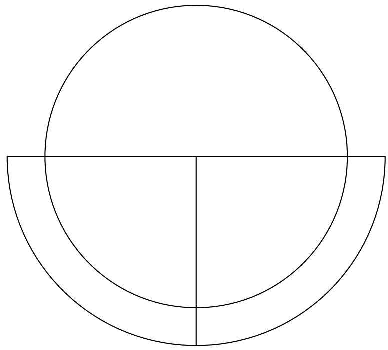

Figure 4. The diagram from *Monac. gr.* 427

entirely circular. Thus, concludes Zeno, "as long as it has not been proved that the diameter bisects its circle, the proposition before us cannot be demonstrated" (*In Primum Euclidis*, 216).

The question naturally arises as to why Zeno demands a proof of the bisection of the circle. As we have seen, this property had already been granted by Euclid in the definition of diameter. Why bother to prove it? A plausible answer is that originally Euclid's definition of diameter did not include the bisection clause. It is even conceivable that such a clause is a later interpolation, inserted precisely to address the issue raised in the debate between Zeno and Posidonius (Netz, 2015, 294). In this case, Zeno's demand for a proof would be entirely legitimate and the proof he most likely had before his eyes was precisely the one attributed by Proclus to Thales (Luria, 1933, 170).

Now, where exactly is the principle that there are no common segments of straight lines and circumferences hidden in the proof of the bisection of the circle? Recall that the proof begins with the assumption that the two parts of the circle cut by the diameter are not equal to one another. This assumption, together with the fourth common notion ("things that coincide with one another are also equal to one another"), entails by *modus tollens* that they do not coincide.

Subsequently, the proof proceeds by inspecting the cases where the circumferences of the two parts of the circle cut by the diameter fail to coincide. In fact, in the proof transmitted by Proclus only one case is given. "For we presupposed that one of the two circumferences would coincide with the other or else, not coinciding, fall either inside or out-

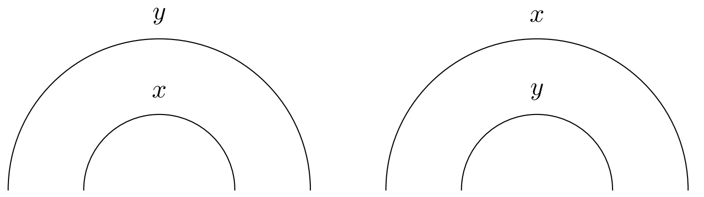

Figure 5. The case in which the circumferences are disjoint

side it" (*In Primum Euclidis*, 216). The case can be represented as in Figure 5.

However, argues Zeno, if the (circumferences of the) two parts of the circle cut by the diameter were to share a segment, namely an arc, they would still fail to coincide, but they would not fall inside one another; on the common segment they would coincide. Thus, in order to fall inside one another, it should be additionally assumed, beside them being not coinciding, that they do not have common segments. Consequently, the bisection of the circle *does* presupposes Zeno's principle and the attempt, reported by Proclus, to prove such a principle using the bisection of the circle ends up being entirely circular.[5]

To express the idea that the two parts of the circle cut by the diameter have a common segment, Zeno says that their circumferences coincide partly (ἐφαρμόζειν κατά τι μέρος). Thus, when one part does not coincide with the other one, "there is nothing, he says, to prevent its failing to coincide as a whole but coinciding in part" (*In Primum Euclidis*, 216). The distinction between total and partial coincidence is overtly ungrammatical. In geometry the verb 'to coincide' used in the active form (ἐφαρμόζειν) is an intransitive verb with the meaning of 'to fit exactly' or 'to coincide entirely with'. Thus, the idea of the two circumferences coinciding partly is an oxymoron, just like the idea of them being completely coinciding a pleonasm. On the other hand, when the verb is used in the medio-passive form ἐφαρμόζεσθαι it generally means 'to be applied to', "without any implication that the applied figure will exactly fit, or coincide with, the figure to which it is applied" (Heath, 1956, 224-

[5] To be sure, there is nothing wrong *per se* in assuming Zeno's principle to prove the bisection of the circle. The problem is, of course, that one cannot later use the bisection of the circle to prove Zeno's principle.

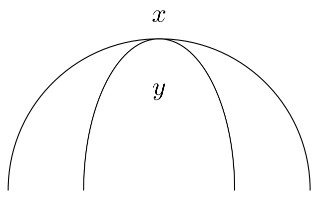

Figure 6. Two cases in which the circumferences overlap

225). In fact, Proclus does use the medio-passive form, too.[6] However, he only intends to refer to the superposition technique. Zeno, instead, uses explicitly the expression 'coinciding in part' (ἐφαρμόζειν κατά τι μέρος), thus leaving no doubt that, however ungrammatical might have sounded, he considered the possibility that the two circumferences are only partly coinciding at least mathematically plausible.

4. Posidonius and Simplicius on the bisection of the circle

Posidonius reacts to Zeno's objection of circularity. While conceding that the proof of the bisection of the circle needs to be supplemented with the extra case where the circumferences of the two parts of the circle cut by the diameter have a common segment, Posidonius firmly claims that proof of this case does not substantially differ from the one where they fall inside one another.

Although no diagram has come down to us, I believe that Figure 6 captures the spirit, if not the letter, of Posidonius' understanding of Zeno's objection.

According to Posidonius, we can still reach a contradiction as we did in other case—only now we need to make sure that the straight line from the center does *not* end on the common segment. After all, this always possible, since even if the two circumferences have a common segment, they are still partly inside one another! Thus, "laughing at the shrewd Epicurus", Posidonius concludes that "at the part where they [*scil.* the circumferences] do not coincide one circumference is inside, the other outside, and the same absurd consequences result when we draw a straight line from the center to the outer circumference" (*In Primum*

6 Cf. the use of the present participle συαρμοζόμενον.

Euclidis, 217). Therefore, there is no need to assume Zeno's principle to prove the bisection of the circle: the standard proof may still go through with minor adjustments and the the bisection of the circle can be used to prove Zeno's principle without circularity.

Although Posidonius's reasoning is substantially convincing, there remains a noticeable absence of details for this case. In fact, Posidonius does not even seem to be willing to provide a sketch of the proof (Acerbi, 2007, 403).[7] Fortunately, there exists another version of the proof due to Simplicius and reported by al-Nayrizi. Interestingly, Simplicius' proof, unlike the one reported by Proclus and attibuted to Thales, does consider the case where the circumferences do not entirely fall inside one another, but rather they intersect in a point. In light of the striking similarities between Posidonius' attempt to fix the bisection of the circle and Posidonius proof, I shall give a detailed review of the latter.

Let $ABGD$ be a circle with center E and diameter BD. To show that the semicircles BGD and BAD are equal to one another, Simplicius assumes for a contradiction that they are not. Therefore, BGD is either bigger or smaller than BAD. In the first case, from the center E draw the straight line EG to the (circumference of) BGD, however it may fall upon it. Then, let BGD be fitted upon BAD. Clearly, BGD will exceed BAD, because it is bigger by hypothesis. Now, let BZD be the part of BGD that exceeds BAD (see Figure 7).

Now, EG coincides with EZ as the result of the superimposition of the two semicircles. Therefore, the two are equal by the fourth common notion ("things which coincide with one another are equal to one another"). But EG is also equal to EA, since they are radii of $ABGD$. Therefore, EZ is equal to EA by the first common notion, which is clearly impossible. Hence, BGD is not greater than BAD. Similarly Simplicius proves that BGD is not smaller than BAD, but the proof is entirely symmetrical and presents no substantial novelty. In fact, he avoids drawing a second diagram with only denotative letters changed by assuming that the given circle is $BZDG$.

The importance of Simplicius' proof is that he additionally considers a third case in which BGD does not entirely fall inside BAD, nor does it entirely fall outside of it, but rather it intersects it at some point A. We also have a diagram accompanying the proof of the third case (see Figure 8).

[7] See fn. 44.

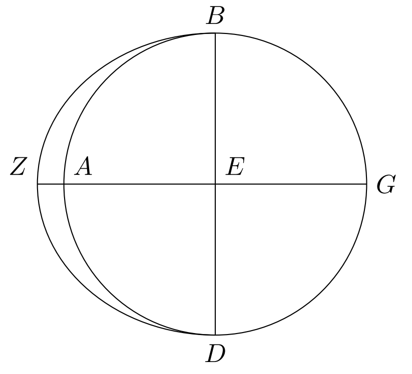

Figure 7. The first case considered by al-Nayrizi

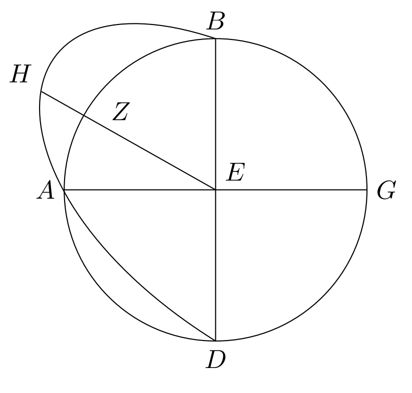

Figure 8. The second case considered by al-Nayrizi

Clearly, the reasoning of the first two cases does not apply here to reach a contradiction. For EG is still equal to EA as the result of BGD be fitted upon BAD, but since EG falls neither outside nor inside BAD it is not possible to draw a contraction. That does not mean, says Simplicius, that a contradiction cannot be reached. In fact, by drawing a straight line EH from the center E to any point H distinct from A on the the circumference BHD and letting Z be the point at which EH cuts the semicircle BAD, we can see, on the one hand, that EZ is equal to EA since they are radii of the semicircle BAD. On the other hand, EA is also equal to EH as a consequence of the construction. Hence, by transitivity of equality, we draw the absurdity that EZ is equal to EH.

Posidonius' reply and the third case in Simplicius' proof are strikingly similar. Firstly, both authors recognize that the standard proof of the bisection needs to supplemented with the case where the circumferences of the two parts of the circle cut by the diameter can fail to coincide without falling inside one another. Secondly, both seem offer the same solution.

These similarities suggest that the source of Simplicius' text in this part is the debate between Zeno and Posidonius reported by Proclus. That Simplicius has a source is confirmed by his vague, albeit suggestive, reference to a potential interlocutor in introducing the third case of the proof. While the rest of the proof is expressed in the typically neutral mathematical jargon, in introducing the third case Simplicius refers, if only incidentally, to someone ("if someone should say"). In the Latin translation of al-Nayrizi's commentary by Gerard of Cremona we read: "Quod si quis dixerit, quod medietas, que est BGD, cum supraponitur alie medietati circuli, que est BAD, non cadit tota intra, nec tota extra, sed secat eam in puncto a, sicut in alia figura signatum". Certainly we can only speculate who hides behind the *quis*. However, the similarities of the two proofs as well as the fact that Simplicius had a source make at least plausible thinking that he might be Zeno.

The substantial difference between the two proofs is, of course, that in Simplicius there is absolutely no mention of the common segment—he only concedes an intersection point of the two circumferences. However, if intersecting at a point can count as partial coincidence at all, then the two circumferences may clearly coincide partly without necessarily having a common segment. While having a common segment is sufficient for the two circumferences to coincide partly, it is by no means necessary. Thus, we should definitely consider Zeno's demand for a third case, i.e., the partial coincidence of the two circumferences, but there is no need for us to leave the *terra firma* of Euclid postulates. This is exactly what Simplicius seems to be doing: he acknowledges an additional case, but he does not commit himself to the existence of a common segment of the two circumferences.

5. A mereological analysis of Zeno's objection

Posidonius' reply to Zeno seems convincing. Once it is granted that the two circumferences cut by the diameter fall inside one another—if not entirely, at least partly—a contradiction easily follows, as illustrated by Simplicius. How could it be that Zeno, who was so subtle that it took Posidonius an entire book to show that his reasoning was rotten (*In Primum Euclidis*, 200), did not see this? In other words, Posidonius' reply arises the suspicion that he misunderstood, if only unwittingly, Zeno's objection.

In this section I argue that Zeno's objection is more serious than Posidonius had thought. By resorting to a mereological terminology I provide an alternative reading of Zeno's notion of partial coincidence that cannot be accommodated using Posidonius's strategy. In particular, I submit that Posidonius interpreted Zeno's objection exclusively in terms of mereological overlap. According to Posidonius' interpretation of Zeno's objection, Zeno thought that while in the standard proof of the bisection of the circle the two circumferences cut by the diameter are taken to be disjoint, we should also consider the case in which they overlap. If interpreted in terms of mereological overlap, the issue raised Zeno can easily be addressed. But I believe that Posidonius' interpretation does not tell the whole story about Zeno's objection. In particular, I think that Zeno suggested that we should consider an additional case, where one of the two circumferences is a proper part of the other. Interestingly, in this case Posidonius' attempt to fix the standard proof of the bisection of the circle fails, for the simple reason that under the hypothesis that one of the two circumferences is a proper part of the other, the diameter does *not* cut the circle in half.

To make this more precise I provide two semi-formal reconstructions of the proof of the bisection of the circle. The first reconstruction corresponds to the proof attributed by Proclus to Thales, whereas the second, also reported by Proclus, represents Posidonius' understanding of Zeno's challenge. Let x and y be the two circumferences cut by the diameter of the circle. Both reconstructions tacitly employ the principle that if x and y coincide, then they are equal, which follows immediately from Euclid's fourth common notion ("things which coincide with one another are equal to one another"). The first reconstruction rests additionally on the principle, labeled here 'Thales' principle', according to which if x and y do not coincide, then they are mereologically disjoint, namely they have no common part. Thales has shown this case to be incompatible with Euclid's definition of circle. (The details of the inference from x and y being disjoint to a contradiction are not relevant for the reconstruction and I shall refer to it as 'Thales' lemma'). Hence, the assumption that x and y are not equal is untenable and $x = y$ is concluded.

1	$x \neq y$	assumption for *reductio*
2	if x and y coincide, then $x = y$	from CN4
3	x and y do not coincide	from 1, 2 by *modus tollens*
4	x and y are disjoint	from 3 by Thales' principle
5	contradiction	from 4 by Thales' lemma
6	$x = y$	from 1, 5 by *reductio*

In the second reconstruction, Thales' principle is rejected and replaced by what we may call 'Posidonius' principle' for which if x and y do not coincide, then they either are disjoint or they properly overlap. Following the standard usage of the term in mereology I say that two objects properly overlap when the have a common part but neither is part of the other—like two intersecting roads, which overlap at their junction but they are not part of one another (Simons, 1987, 12). For simplicity, I will often omit the epithet 'proper' and use 'proper overlap' and 'overlap' interchangeably.[8] Posidonius' reconstruction is identical with Thales' except that a contradiction needs to be derived not just from the assumption that x and y are disjoint, but from the assumption that they overlap as well. While Posidonius accepts Thales' lemma for the case in which x and y are disjoint, he provides the missing details for the case in which they overlap (Posidonius' lemma).

[8] The terminological clarification is nevertheless important since according to the standard meaning of the term 'overlap' the conclusion of Posidonius' principle would just be an instance of the law of excluded middle.

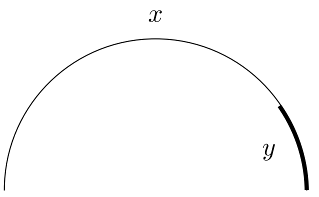

Figure 9. The case in which one circumference is a proper part of the other

1	$x \neq y$	assumption for *reductio*
2	if x and y coincide, then $x = y$	from CN4
3	x and y do not coincide	from 1, 2 by *modus tollens*
4	x and y are disjoint or they overlap	from 3 by Posidonius' princ.
5	x and y are disjoint	assumption
6	contradiction	from 5 by Thales' lemma
7	x and y overlap	assumption
8	contradiction	from 7 by Posidonius' lemma
9	contradiction	from 4, 6, 8 by disj. elim.
10	$x = y$	from 1, 9 by *reductio*

We now proceed to illustrate what I take Zeno's position to be. From a mereological perspective the two alternatives considered by Posidonius do not exhaust all the possible cases in which x and y may fail to coincide. Crucially, x and y may fail to coincide when y is a proper part of x or *vice versa*. For example, the reason I do not coincide with my own hand is neither because my hand and me are disjoint nor because we (properly) overlap. In fact, I do not coincide with my own hand simply because my hand is one of my proper parts. Consequently it is perfectly consistent to think of x and y as not coinciding without thereby being disjoint or partly overlapping. This can be visualized as follows.

In Proclus' terminology, in the diagram above x and y fail to coincide as a whole but coincide in part. The only difference is that while in Posidonius' interpretation the part in which they coincide is neither x nor y, here it is y itself. Thus, Figure 9 is perfectly consistent with Zeno's words: it describes his notion of partial coincidence just as well as Posidonius' case of overlap. Thus, Posidonius' principle needs to be strengthened so as to account also for the case in which one circumference is a proper part of the other: if x and y do not coincide, then they either are disjoint or they overlap or else one is a proper part of the other. I shall refer to this principle as 'strong Posidonius' principle'.

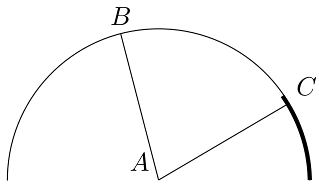

Figure 10. A counterexample to the bisection of the circle

Is strong Posidonius's principle incompatible with Euclid's principles? In the proper part case it is clear that the two circumferences do not fall inside one another—not even partly. Hence, it is not entirely clear how to apply the usual construction (labeled earlier Thales' and Posidonius' lemmas) to draw a straight line from the center to one of the two circumferences in order to obtain a contradiction. Whether such a line falls on the common segment, like AC, or outside it, like AB, it does not seem obvious that producing it allows us to obtain a contradiction.

The burden of the proof is on Posidonius but, as far as we know from Proclus, he focused exclusively on the overlap case, overlooking entirely the proper part case. We will never know whether the omission was deliberate. However, if the proper part case can count at all as a genuine case of failure of coincidence, then there is little hope of drawing a contradiction from the assumption that one circumference is a proper part of the other. For in this case the bisection of the circle simply fails: if one circumference is a proper part of the other, then the diameter does not bisect the circle. In other words, the proper part case is a counterexample to the bisection of the circle.

At this point the role of Zeno's principle become clear. Indeed, Zeno's principle excludes both the overlap and the proper part case. Whether the circumferences overlap or are part of one another, they would have a common segment, which is precisely what Zeno's principle excludes. Thus, all possible cases in which the circumference may fail to coincide lead to a contradiction. If they are disjoint, then a contradiction is concluded by Thales' lemma, whereas if they overlap or are part of one another, then a contradiction is reached by assuming Zeno's principle. This suggests a third, final reconstruction of the proof of the bisection of the circle, based on Zeno's principle:

1	$x \neq y$	assumption for *reductio*
2	if x and y coincide, then $x = y$	from CN4
3	x and y do not coincide	from 1, 2 by *modus tollens*
4	x and y are disjoint or they overlap or one is proper part of the other	from 3 by strong p.p.
5	x and y are disjoint	assumption
6	contradiction	from 5 by Thales' lemma
7	x and y overlap	assumption
8	contradiction	from 7 by Zeno's principle
9	one is proper part of the other	assumption
10	contradiction	from 9 by Zeno's principle
11	contradiction	from 4, 6, 8, 10 by disj. elim.
12	$x = y$	from 1, 11 by *reductio*

The mereological analysis of Zeno's objection thus reveals an important aspect of Zeno's principle: the principle guarantees that in the proof of the bisection of the circle the overlap case as well as the proper part case never arise. And since the latter makes the bisection of the circle fail, Zeno's principle guarantees that the diameter bisects the circle.

6. Zeno on the proper part case

The question now naturally arises as to whether and to what extent the proper part case can genuinely be ascribed to Zeno. I believe that far from being just an artificial construction, inspired by the much more modern mereological theorizing, the proper part case is, in fact, a better interpretation of Zeno's notion of partial coincidence.

In the proof of Zeno's principle given by Proclus, when the lower part of the circle cut by the diameter ABC is superimposed on the upper part, we end up in a situation which is much more similar to the one described in the proper part case than the the one described in overlap case. Indeed, after the superimposition technique is applied, the semicircle AEC is a proper part of the semicircle AED. Importantly, neither falls inside the other, although the two do share a segment, AEC. In a version of the proof transmitted in a scholium it is also explicitly stated that (the circumference of) the semicircle AEC is smaller than (the circumference of) the semicircle AED, whereas the two should be equal to one another (ἴσον ἄρα ἐστί [...] τό ἔλαττον τῷ μεἰζονι).[9]

[9] The scholium is found in the manuscripts of the so-called Theonine family, in margin of *Elem.* XI.1. The passage is edited by Heiberg and Menge (17th scholium

One may object that the AEC is proper part of AED only because we mistakenly allow ABC to be a diameter of the circle. Although ABC is clearly not a straight line in the ordinary, pre-theoretical sense of the term, it remains to decide whether this can be established by the principles. I think that the motivations for not considering ABC a diameter are not convincing, both mathematically and textually. For a start, we cannot certainly argue that ABC is not a diameter because it does cut the circle in half. That, of course, would amount to presupposing the bisection of the circle. Neither can we argue that ABC is not diameter because is not a straight line. Once again, ABC is unquestionably not a straight line in the ordinary, pre-theoretical sense of the term, but does this follows from the principles? The only genuinely argument in the *Elements* against ABC being a straight line is *Elem.* XI.1, where Euclid shows that straight line cannot be partly on a plane of reference and partly on a more elevated plane. Alas, the proof rests Zeno's principle (in fact, it is the only proposition in the *Elements* in which Zeno's principle is applied).[10] We know for sure that Zeno firmly rejected the practice of proving his principle by using propositions that are established after *Elem.* I.1 (*In Primum Euclidis*, 217).

I conclude that there is quite enough convincing evidence that we should interpret Zeno's partial coincidence not in terms of overlap, as Posidonius did, but in terms of proper parts. This does not mean that the overlap case should should be discharged altogether. Posidonius was certainly right in adding such a case in the proof of the bisection of the circle. And was he also right in claiming that such a case presents no substantial novelty with respect to the first case, hence it can be safely ignored without loss of generality. He was wrong, however, in focusing exclusively on this case.

To summarize: Proclus reports a proof, most likely due to Posidonius, that the bisection of the circle entails (together with Euclid's principles) Zeno's allegedly missing principle that there are no common segments of straight lines and circumferences. Zeno objects that such a principle, in turn, entails (together with Euclid's principles) the bisection of the circle.

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of the Book XI) in *Euclidis Opera Omnia*, ediderunt J.L. Heiberg et H. Menge, vol. V, Teubner, Leipzig, 1888.

[10] Alternatively, one can follow Proclus and simply say that ABC does not satisfy the definition of straight line. The argument, however, is quite obscure (Acerbi, 2007, 112) and it is conceivable that he merely intended to remind his readership of the principles of geometry (Netz, 2015, 289).

Therefore, the bisection of the circle and Zeno's principle are equivalent; assuming the latter is *ipso facto* assuming the former. Posidonius claims the bisection of the circle follows from Euclid's principles without Zeno's principle (a detailed account can be found in a passage by Simplicius). Using mereology, I have argued that Posidonius's strategy to prove the bisection of the circle without Zeno's principle is not satisfactory. This vindicates Zeno's objection that his principle is required (together with Euclid's principles) in the proof of the bisection of the circle.

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