Logic and Logical Philosophy
Volume 31 (2022), 261–288
DOI: 10.12775/LLP.2022.010

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A Logic for a Critical Attitude?

Abstract. Individuating the logic of scientific discovery appears a hopeless enterprise. Less hopeless is trying to figure out a logical way to model the epistemic attitude distinguishing the practice of scientists. In this paper, we claim that classical logic cannot play such a descriptive role. We propose, instead, one of the three-valued logics in the Kleene family that is often classified as the less attractive one, namely Hallden's logic. By providing it with an appropriate epistemic interpretation, we can informally model the scientific attitude.

Keywords: scientific attitude; weak Kleene logics; epistemic interpretation

1. Introduction

The descriptive power of logic is traditionally founded on the idea that a logical system suitably describes situations *faithfully representing formal features* of the phenomena under investigation. In other words, logical approaches towards reality usually rely on the ontological assumption that logic can *actually* capture salient features either of the structure of "what is there" (i.e. metaphysics) or of how rational agents reason (or should reason) about (pieces of) reality. It is folklore that classical logic represents a powerful formal model for mathematical reasoning, but it encounters serious difficulties when is applied to modeling certain real-world situations. The simplest, yet most illuminating, objection traces back to Aristotle (*De Interpretatione*, 9), who observed that the truth value of a future sentence such as "there will be a sea-battle tomorrow in front of Athens" can hardly be assigned in the present. This seriously

Special Issue: Logics and their interpretations. I. Guest Editors: H. Antunes and D. Szmuc

Published online February 19, 2022

challenges the principle of bivalence but does not state, in principle, that there is no space for the logical description of statements of this kind. A natural way out from the "sea-battle problem" consists in admitting that statements regarding the future are *neither* true *nor* false at present time: they shall simply be considered as *indeterminate*. A circumstance that can be treated by making use of, for instance, a "third" truth-value: an idea famously discussed, among others, by Łukasiewicz [see, e.g., Żegleń, 1998] and Prior [1953].[1]

Historically, the introduction of (some) non-classical formalisms carried an *ontological* commitment towards the role of logic. This is to say that *indeterminacy* has been assumed to be an ontological property rather than an epistemic incapacity.

Even if we do not want to discuss here any particular positions concerning the (supposed) "rightness" of one logic above the others (e.g. whether classical logic or other formalisms are *the* true logic underlying the metaphysical structure of the real), we think it is relevant to be explicit about the fact that we definitely embrace an epistemic, pluralistic stance, meaning that we evaluate a logic on the basis of the epistemic benefit and adequacy it may grant to the modeller (we will come back on this aspect in Section 5).

Having this said, moving from an ontological concern towards epistemic relevance and wearing the glasses of the philosopher of science, it may turn out that facts regarding science itself, and scientific practice, should be more adequately represented as *indeterminate*, at least for some periods of time (until when further evidence is found to confirm, or reject, hypotheses in a scientific theory). This kind of indeterminacy is, however, of a different type with respect to the above mentioned one, as it refers to the epistemic condition of a subject, such as scientist, a community of scientists, or even just someone exposed to scientific discourse, to establish the "truthfulness" of (part of) a scientific claim (spanning from a not fully structured set of hypotheses to a more refined theory), starting, for instance, from evidence produced.

Taking up a famous argument from [Hacking, 1983, 1992], who claimed that scientific the enterprise involves different forms or styles of

[1] We are not interested here in the fact that a (maybe) more precise and appropriate way to deal with temporal sentences ("it will be the case in the future", "it has always been the case in the past", etc.) has been offered in the form of the so-called *tense logics*, introduced by Prior [1957], in the context of modal logics [see also Øhrstrøm and Hasle, 1995].

reasoning, we wonder whether it is possible to find a logic to model the indeterminacy taking place, within the epistemic approach, that philosophers of science tend to discuss with respect to scientific theories. Our preliminary step is founded on the idea that logic can be used to model or describe commitment with the epistemic confidence towards a scientific theory or even just a set of scientific claims. In other words, when facing the need to express a judgment about a theory or a hypothesis, a logical model can be useful in representing the attitude towards a decision rather than the methods and the criteria of the decision itself (something that pertains, usually, more to rational decision making). Due to the adoption of this conceptual shift, our aim is to provide an epistemic interpretation of (an extension of) the so-called *logic of non-sense* (often referred to as paraconsistent weak Kleene, or Hallden's logic), a formalism arising within the family of Kleene three-valued logics. Subsequently, we will show how this formalism can be adopted to capture a crucial approach to uncertainty, when considering an open-minded epistemic attitude in the evaluation of, partial and temporary approval (and discussion) of scientific theories.

The paper is structured as follows: in Section 2, we recall Kleene three-valued logics, expose the main differences among them and propose our interpretation for paraconsistent weak Kleene. In Section 3, we will briefly examine some philosophical accounts in relation to scientific theories relevant to our discussion. For instance, we recall the well-known Duhem-Quine thesis, and explain the critical approach to scientific theories/claims that we aim to model. In Section 4, we show how the logical formalism introduced insofar can be indeed applied the model our type of uncertainty towards scientific judgement and the attitude towards the acceptance of scientific theories/hypotheses as well. Finally, we close the paper with Section 5, where we summarize and discuss the main ingredients of the proposed approach.

2. Three-valued Kleene logics

Kleene three-valued logics are defined over the same language as that of classical logics, namely $\neg$, $\wedge$, $\vee$; implication can be defined as $A \rightarrow B := \neg A \vee B$ (actually, for instance, only $\neg$, $\wedge$ will be enough as primitives, upon defining $A \vee B := \neg(\neg A \wedge \neg B)$). These logics are characterized by a three-valued semantics. More precisely, in his *Introduction to Meta-*

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\neg$	
1	0
1/2	1/2
0	1

Figure 1. The algebra **SK** (for strong Kleene)

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

$\neg$	
1	0
1/2	1/2
0	1

Figure 2. The algebra **WK** (for weak Kleene)

mathematics, Kleene [1952, § 64] distinguishes between a "strong sense" and a "weak sense" of the propositional connectives in the presence of a third truth-value (according to his aim, meant to model situations where partially defined predicates are present). Each of these meanings is made explicit via a certain algebra of truth-tables. Labelling by $0, 1/2, 1$ the elements of the support set, the two algebras $\mathbf{SK} = \langle\{0, 1, 1/2\}, \wedge, \vee, \neg\rangle$ and $\mathbf{WK} = \langle\{0, 1, 1/2\}, \wedge, \vee, \neg\rangle$ are displayed in Figure 1 and 2, respectively (**SK** stands for *strong* Kleene, and **WK** for *weak* Kleene).

Each algebra (set of tables) naturally gives rise to two options for defining a three-valued logic, depending on whether only 1 is taken as a designated value, or 1 together with $1/2$.

Thus, the family of Kleene logics consists of the following:[2]

- strong Kleene logic [Kleene, 1952] is induced by the matrix $\langle\mathbf{SK}, \{1\}\rangle$;
- the logic of Paradox LP [Priest, 1979]: induced by $\langle\mathbf{SK}, \{1, 1/2\}\rangle$;
- Bochvar's logic [Bochvar and Bergmann, 1981] B_3 as induced by the matrix $\langle\mathbf{WK}, \{1\}\rangle$;
- paraconsistent weak Kleene PWK, or Hallden's logic [Halldén, 1949; Prior, 1957], as induced by the matrix $\langle\mathbf{WK}, \{1, 1/2\}\rangle$.

Some comments are in order. First, observe that, in the algebra **SK**, $\wedge$ (equivalently, $\vee$) induces a partial (lattice) order defined as usual,

[2] We are exclusively considering logics defined by single matrices.

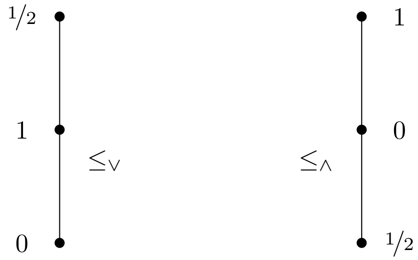

Figure 3. Graphical representation (Hasse diagrams) of the two partial orderings induced over $\{0, 1/2, 1\}$ by $\vee$ (left hand side), and $\wedge$ (right hand side), respectively.

namely: $x \leq y$ iff $x \wedge y = x$ (or, equivalently, $x \vee y = y$). Accordingly, truth-values are ordered as $0 < 1/2 < 1$, suggesting that $1/2$ can be interpreted as a "middle value" (the rational number $1/2$, if $0, 1$ are interpreted as natural numbers). It is easy to check that $\wedge$ and $\vee$ are interpreted, in the strong tables, as the minimum and the maximum, with respect to the order $\leq$. However, the same does not happen to be the case in the algebra **WK**, which is not a lattice (it is an involutive bisemilattice, see [Bonzio et al., 2017]), as it fails to satisfy absorption[3] and, hence, induces no lattice order over the set $\{0, 1, 1/2\}$. As consequence, the two binary operations $(\wedge, \vee)$ induce two different partial orders (sketched in Figure 2), defined as follows

$$a \leq_\wedge b \text{ if and only if } a \wedge b = a,$$
$$a \leq_\vee b \text{ if and only if } a \vee b = b.$$

The connectives $\wedge$ and $\vee$ are not interpreted in **WK** as the minimum or the maximum of any of the two orders. Consequently, $1/2$ cannot be interpreted as an intermediate value, as in the strong algebra. It is worth mentioning that both Strong Kleene and Bochvar are theoremless logics (i.e. they have no tautologies). On the other hand, LP and PWK have the same tautologies of classical (propositional) logic: they are both *paraconsistent* (i.e. they do not satisfy the *absurdo quodlibet*), as (classical) contradictions are satisfiable. Interestingly, both logics fail to satisfy Modus Ponens. In contrast with LP, PWK also fails conjunction

[3] An algebra $\boldsymbol{A} = \langle A, \wedge, \vee \rangle$ satisfies absorption whenever $a \wedge (a \vee b) = a$, for every $a, b \in A$. In our case $1 \wedge (1 \vee 1/2) = 1/2 \neq 1$.

simplification, i.e. $A \wedge B \nvdash_{\mathrm{PWK}} A$ (and also $A \wedge B \nvdash_{\mathrm{PWK}} B$).[4] The two weak logics are dual in the following sense: $A \vdash_{\mathrm{PWK}} B$ if and only if $\neg B \vdash_{B_3} \neg A$.

In the context of weak Kleene logics, Bochvar and Bergmann [1981] advanced the proposal of extending the language with an additional unary connective "t", with the following interpretation.

φ	$\mathrm{t}\,\varphi$
1	1
$1/2$	0
0	0

Intuitively, this allows to distinguish between two types of statements: φ, $\neg\varphi$, $\varphi \wedge \psi$, etc. are statements of the first type, while "φ is true" ($\mathrm{t}\,\varphi$), "φ is false" ($\neg\,\mathrm{t}\,\varphi$), "$\varphi$ and ψ is true" ($\mathrm{t}(\varphi \wedge \psi)$) and so on, are statements of the second type. When the language is extended with t, so that statements of the second type are also allowed, one obtains the so-called *external calculus*, which differs from the *internal calculus*, based on the primitives $\neg$, $\wedge$, $\vee$. The external calculus is introduced by Bochvar and Bergmann [1981] [see also Finn and Grigolia, 1993], upon choosing $\{1\}$ as the truth set, and for PWK in [Segerberg, 1965], choosing $\{1, 1/2\}$ as truth set (this logic is actually referred to as H_0). It is vital to observe that formulas falling under the scope of "t" are evaluated in the two element Boolean algebra (they cannot take the value $1/2$). This means that, in the external calculi, the presence of "t" (and all connectives definable from "t") allows us to control the "infectivity" of $1/2$, which is limited to *open* variables.[5]

The choice of the algebra of the truth-tables together with a specific truth-set suggest different interpretations for the truth value $1/2$. This may lead to different applications, for each logic, in philosophy or other disciplines. In Strong Kleene, $1/2$ is read as "neither true nor false". This logic has applications in artificial intelligence as a model of partial information [Abdallah, 1995] and non-monotonic reasoning [Turner, 1984], and in philosophy as a bedrock logic for Kripke's theory

[4] As suggested by an anonymous reviewer, it is useful to point out that it is not the case that LP is stronger than PWK: the two logics are actually incomparable (as, for instance, $p \wedge \neg p \vdash_{\mathrm{PWK}} (p \wedge \neg p) \wedge q$ but $p \wedge \neg p \nvdash_{\mathrm{LP}} (p \wedge \neg p) \wedge q$).

[5] A propositional variable p is open in a formula φ if there is at least an occurence of p which does not fall under the scope of t (example: p and q are open in $(\mathrm{t}\,p \wedge q) \to p$).

of truth and other related proposals [Field, 2008]. In LP, $1/2$ witnesses the compresence of truth and falsity. This logic has been introduced, and fervently supported, by Graham Priest in the context of a dialetheic approach (in ontology or metaphysics). Roughly speaking, according to Priest, it would be possible to claim that contradictions are real (i.e. reality contains contradictions) and because of that LP turns out to be the "right logic" to describe (or to refer to) reality.

As regards weak logics, the majority of commentators agree with interpreting the third value as *meaningless* (a reconstruction of the logico-philosophical debate about meaningless sentences can be found in [Szmuc and Ferguson, 2021]). This is mostly due to its infectious behavior in the internal calculus (which is the standard way in which weak Kleene logics are mostly intended). Indeed, it is easy to see that any sentence containing a sub-sentence evaluated to $1/2$ is itself evaluated to $1/2$, independently from its complex form. This is usually referred to as the *Principle of Component Homogeneity* or the *Contamination Principle* [see Ciuni and Carrara, 2016]. This interpretation of the third value makes Bochvar's logic valuable for modeling computer programs affected by errors [Ferguson, 2017] or to interpret references to non-existing objects [Prior, 1957]. One of the most recent and more fascinating interpretations, is due to Beall [2016]. Motivated by the choice of the unique value 1 in the truth set, Bochvar's logic accounts not only for truth-preservation but also for *truth and topic preservation*. In detail, truth values in the weak tables shall be interpreted as "true and on-topic" (1), "false and on-topic" (0) and "off-topic" ($1/2$) with no explicit reference to the truth. All the above-mentioned proposals use Bochvar's logic to provide ways to "capture pieces of realities". However, in our view, none of them seems to provide a convincing defense for PWK. We will advance our proposal for a particular epistemic interpretation of weak Kleene logics and show its potential applications.

2.1. Epistemic interpretation of (paraconsistent) weak Kleene

All the interpretations, and consequent applications, examined so far, of Kleene logics, share the idea that these non-classical formalisms can adequately describe some state of affairs: partial information, contradictions, computer errors, etc. Accordingly, we call these approaches *ontological* interpretations. However, the tendency within philosophy of logic to privilege interpretations harboring an explicit ontological commitment, appears problematic if one aims to provide a convincing and feasible interpretation to paraconsistent weak Kleene. Indeed, assuming that $1/2$ is read as "meaningless", how can the choice of the truth set $\{1, 1/2\}$ be rationally justified? In other words, since $1/2$ models meaningless sentences, why should these be preserved through logical inference?

Accordingly, we believe that fruitful answers to the above question can hardly be supported by adopting $1/2$ to account for ontological issues (a choice that would privilege Bochvar's logic over PWK). In our opinion, a feasible way out can be provided by adopting a conceptual shift: indeed logics, in general, can be be applied to capture peculiar *epistemic attitudes or stances* rather than ontological states. Without entering the complex debate (and the connected difficulties see, e.g., [Bonzio et al., 2021]) of how an agent constructs a belief set, we assume the traditional idea that the epistemic attitudes of agents consist in believing a proposition, disbelieving it or suspending judgment. The three possibilities rely on a logical background: a proposition is believed when is acknowledged true and disbelieved when false. In a trivalent perspective, judgment suspension corresponds to evaluating a sentence as having the third value $1/2$. In other words, the third value is used to describe a form of epistemic uncertainty (or the certainty that a sentence can be neither true nor false) affecting agents or groups of agents, like those within a scientific community, or even laypeople dealing with complex information relative to scientific claims (we will come back to the idea of an epistemic attitude in the next section).

It is worth noticing that, in our approach, we are using the term *epistemic* simply in contrast with *ontological*: it goes beyond our purpose to address the existing differences between various forms of epistemic accounts and between the epistemic and the doxastic level.

Although "ontological" interpretations are most common in logic, they are not the only possible ones and logics in the Kleene family have received also epistemic interpretations. Recently, these have involved also PWK in, at least, two cases which are worth mentioning. Szmuc [2019] applies PWK in the context of the formation of group judgments, where agents in a group may express *inconsistent* opinions, namely that a sentence is both true and false. Carrara and Zhu [2021] provide an interesting application of PWK in the context of belief revision theory, where sentences, alongside being true or false, can be considered also as on-topic/off-topic (the reading of the value $1/2$ is line with Beall's interpretation). Although our approach is considerably less detailed than

[Carrara and Zhu, 2021] (which describes precisely what is meant by a "PWK epistemic agent"), it shall be pointed out that our interpretation of epistemic attitudes is in accordance with theirs.

Therefore, after these preliminary remarks, it is now time to better illustrate what do we mean by an *epistemic/critical* interpretation of paraconsistent weak Kleene logic (and H_0). Following Szmuc [2019], this consists of:

1. interpretation of the truth-values;
2. interpretation of the truth tables;
3. interpretation of the logic (meant as a logical consequence).

1. We interpret the truth-values as: true (1), false (0) and uncertain, also dubious, or unknown to an agent ($1/2$). Truth values have to be read in a specific *epistemic sense* (i.e. being critical towards information), namely a statement is evaluated as true by an agent, when they have sufficient reasons (enough scientific evidence, in the common practice of science) to think it as true. This criterion applies to the other truth-values as well. In particular, when coming to $1/2$: this can be then read as not enough evidence has been produced, or contradictory evidence has been produced for accepting an hypothesis, or a part of a scientific theory, as true (or rejecting it as false).

2. The interpretation of the truth-tables in Figure 2 remains the standard one: for each connective, $1/2$ behaves as infectious (except for formulas of H_0, falling under the scope of t).

3. Defending the adoption of PWK (or, its external companion H_0) translates into explaining why epistemic uncertainty spreads into a theory and, more importantly, why should an epistemic status of uncertainty be preserved through logical inferences. We will elaborate and discuss these reasons in Section 4 and provide a convincing justification for the use of H_0 to model the epistemic attitude when reasoning within scientific theories (and also, pseudo-theories).

3. On modeling critical attitude and our philosophy of logic

Being *critical* is believed to be a key component of a scientific mind. Scientists do not usually refute nor do they welcome new ideas (for the conservative stance of a scientific community see [Kuhn, 1962]). Rather, they usually examine (or should) novelties and new approaches with a critical attitude. In order to be convinced of different perspectives, scientists do not evaluate claims and theoretical stances just on the basis of their likelihood.

Our goal, in this section, is to show how this particular epistemic condition, which concerns the scientific attitude in its almost daily practice, can be logically modeled and fruitfully. By this we mean that it is possible to characterize in a more precise and manageable way some epistemic categories produced by philosophical analysis, through a formal representation. In doing so we will first propose a brief treatment of some of the most relevant positions in the philosophy of science on this problem. Next, we will show how some of the metatheorical conclusions reached in this field can be adequately and operationally represented by the epistemic interpretation we propose about Kleene logics.

Starting from Kuhn's work we know that, in evaluations of theoretical propositions, scientists are guided and influenced by various factors (not all of them specifically epistemic). It is not our purpose to go into this examination in detail. However, it is clear that all these aspects generate an *attitude*, generally fostered and shaped by both scientific education and the research environment, which is aimed at evaluating given theoretical propositions, empirical results, or even raw data (which might be related to the current investigation).

It is our belief that this situation, which corresponds to the phase in which a scientist considers and evaluates the various elements that will inform their judgment (which, as it has been said, also depends on external factors), cannot be adequately represented by a type of reasoning embedding a "bivalent attitude"(meaning that it is based on the logical *principle of bivalence*, according to which any statement is either true or false). As a matter of fact scientists may have legitimate reasons to support a particular theoretical proposition or to positively welcome an experimental result, but at the same time they can be aware that their reasons could present weaknesses or may not not conclusive. Indeed, in this crucial phase of scientific work, a determined judgment has not yet been formed. Thus researchers, although they can definitely express preferences or provide indications, cannot take a clear and determined decision.

This situation, far from being rare, is quite common in daily scientific practice. Scientific judgments, by their nature, are never exact, but rather exhibit differences concerning their *epistemic robustness* (i.e. the capacity to resist being questioned) depending on the moment of their construction (on the issue concerning the robustness of scientific representations; see for instance [Boniolo et al., 2017]). Indeed, for many phases of scientific research, scientists have only more or less grounded "epistemic hopes", since their judgment is yet to be established and it is as if *suspended*. Traditionally, these issues are viewed as logically intractable. We argue that it is definitely true that they seem classically intractable, where, once again, we refer to the difficulty of treating them by recourse to classical logic. However, this does not necessarily mean giving up on logic (something that, surprisingly, many philosophers of science tend to forget). In this move we obviously do not want to argue that our proposal is the most appropriate logic for describing these aspects. Our aim is not to establish that there is a "right logic" (or possibly what it is). Indeed, again, our aim is not ontological or metaphysical. Rather, we adopt a particular philosophy of logic which consists in showing how formal modeling might produce precious conceptual tools. In other words, starting from a real situation, in this case the scientist's attitude in evaluating theoretical assertions and hypotheses (up to scientific theories), we show how, by representing this attitude with a specific formalism, we can derive some consequences that provide an effective (useful) description of the situation. Of course, it will always be possible to disagree and choose a different logic, thus showing what different consequences can be obtained. Indeed, we believe that this approach to the philosophy of logic can be truly useful to the philosophy of science, thus bringing together two fields of research which very often are sadly far from each other. In the following sections, therefore, we will try to show how many judgments concerning theoretical aspects of science are often treated as "neither true nor false". Later we will try to show how this condition can be treated by our trivalent proposal.

3.1. Being critical on scientific theories

The philosophical discussion around scientific theories is a complex and much-debated matter. From the original nucleus of reflections [obviously it is crucial to mention logical empiricism, see, e.g., Creath, 2020], the analysis of the characteristics of scientific knowledge has greatly expanded. When scientists have to express themselves on whether a given scientific claim or hypothesis is either "true" or "false", within a given framework, they first check for evidence. Thus, researchers look for a way to provide a consistent explanation for collected evidence. Still, intuitively, this means that scientists usually need to certify (and this might happen in various ways) what evidence tells in relation to the structure of their theoretical accounts. Accordingly, evidence can *support* a theory or give reasons to reject it.[6] The epistemic problem here precisely refers to the modality according to which evidence may provide its supportive function. In other words, how does evidence bear on the soundness of a particular claim or hypothesis, so that either we accept it or refute it?

Famously, Karl Popper [1959] warned against the perils of confirmation. Indeed, by recognizing an asymmetry between verification and falsification, Popper argued that a high number of confirmations is never sufficient to conclusively verify a universal assertion (i.e. prototype of scientific laws) while a single negative example is sufficient to invalidate it. Accordingly, Popper identified the "falsification criterion" as a *demarcation* criterion.

Popperianism is still quite popular among scientists and in popular or journalistic representations of what makes science different (epistemically) from other human activities. However its acceptance is not theoretically undisputed. Among different criticisms, it is important to recall the famous so called *Duhem-Quine thesis* [see Duhem and Wiener, 1954; Quine, 1951]. In brief, this thesis holds that a single experiment cannot discriminate a theoretical assumption, but only over a bundle of them. In other words a scientist cannot empirically test a well-isolated hypothesis (i.e. the so called *experimentum crucis*), but only the whole theoretical framework. This means that when empirical data do not match with the overarching frame, this implies that, at least one of the hypotheses constituting the framework may not be tenable and needs to be rejected or modified. However, data, as such, do not help in understanding which specific hypothesis should be changed, modified or discarded. As a matter of fact, in actual scientific practice, it is very unlikely that the rejection of a single hypothesis as such necessarily leads scientists to abandon the entire theoretical framework they work with. Indeed it is literally a framework, a complex structure in which different forms of relationships may exist among different hypotheses. Moreover, not all hypotheses have the same "epistemic weight" within a framework. This means that some hypothesis of a theory may be more (or less) im-

[6] Moreover, one must not forget that this impacts also on the fact that evidence *per se* is not enough to determine the choice of a framework over another one. This situation is generally known as *contrastive under-determination of theory* by which we mean that the very same evidence can provide support for diverse theoretical accounts.

portant than others within the complex network of the theory (a key idea to which we will come back). This has the relevant implication that the loss or the modification of a specific part may have a different impact on the general structure. Scientific theoretical frameworks show less rigorous consistency criteria than logical systems. Indeed, scientific accounts might easily display "holes" or "gaps" and thus be quasi-coherent (meaning that they do not need to completely avoid contradictions in order to be accepted). Thus, this blurriness does not necessarily involve the rejection of the entire theoretical framework. On the contrary this is quite frequent and it constitutes a common activity in scientific research, i.e. it urges scientists to work on these gaps by updating or modifying those parts that are less in accordance with the main body of the theory. For instance, when the famous hypothesis accounting for the genetic information flow by Francis Crick [1970] has been developed (the famous "Central Dogma of Molecular Biology"), researchers were already aware of phenomena such as viral retro-transcription, which one could consider a possible violation of the account. However the general architecture of the theoretical framework has not been abandoned. Recently some scholars, such as Eugene V. Koonin [2012], have suggested that the case of "prions" might count as a violation of the Central Dogma. Nevertheless, in many research contexts the Central Dogma is still (operatively) accepted and adopted, despite the fact the scientists do not recognize its full validity. From this perspective, the Duhem-Quine thesis reminds that scientific hypotheses are not fully "discrete". Rather, they are *intertwined* with each other in forming the framework. This means that the entire network will be modified by the modification of its constitutive parts. Indeed, unlike formal systems, scientific theories are not thought as starting from "fundamental building blocks", i.e. they are not designed to be constructed axiomatically. Scientific theories (especially those in the life sciences and biomedicine) look like more as the work of a tinkerer, who fixes and adjusts here and there in order to have the whole system functioning, tolerating minor discrepancies among different parts. Because of that, critical attitude should not be seen as the mechanistic application of a protocol. Rather, it precisely changes through time. This means that judging evidence in scientific practice may vary according to particular phases of the scientific enterprise. This is because science is not a static activity. It rather is dynamic and under constant revision. Scientists do change their views and feelings through time, along with the development of the research. Of course such a change is not just a matter of common opinions. The transformation of scientific beliefs and theories is a complex phenomenon involving conceptual and empirical issues as well as the socio-political context. Fundamental, in this sense, is also the work of Norwood Hanson [1958], who is responsible for having brought to the attention of philosophers the importance of context in scientific discovery and for undermining the clear separation between theoretical and observational language. Starting from similar observations, but with different results, it is crucial to mention the great work of Thomas Kuhn [1962] on the nature of scientific revolutions. Famously, Kuhn argued that the development of science is neither linear nor uniform. It should rather be described as divided in several phases affecting in different ways what he calls *paradigm*, i.e. theories, laws, their applications, tools forming an overarching and enough unitary explanatory frame for a particular area of science (e.g. Newton's *Principia Mathematica* in physics). Of particular interest for our analysis is the perspective elaborated by Imre Lakatos [1978]. Roughly speaking, the Hungarian philosopher elaborates a proposal aimed at combining the theses of Popper and Kuhn. For Lakatos, if one wants to capture some of the dynamics of scientific research, it is necessary both to find criteria for demarcation and evaluation of the hypotheses about theories, and to take into account the transformation of science itself as a product of human activity. To this end, Lakatos maintains that the evaluation of theories, and the passage from one theoretical framework in favor to another, can be described as a process governed by methodological rules. Therefore, he elaborates an epistemological category, known as *research program*, in order to account for this scenario.

According to Lakatos, the selection criteria in the endorsement of theoretical assumptions by scientists must not be thought too rigidly. This is also because scientific practice shows how the questioning of a theoretical assumption by experimental results does not imply, *per se*, the abandonment of the theory. It is precisely for this reason, that Lakatos elaborates the notion of research program. Indeed, a research program is a new epistemic category that is composed of a set of theories and hypotheses, which individually can be refuted but between them are supported and united by a *central core* (considered as the non-amendable part), and from a series of *auxiliary hypotheses*.

The central core is constituted by a number of principles on which there is a more or less unanimous consensus of the scientific community (in a given phase). This consensus, which is the expression of a methodological choice guided by various forms of heuristics (and which therefore also involve meta-theoretical aspects), certifies the precise will of the community not to question these principles (we can think of them as accepted as *true* by the scientific community). On the other hand, the auxiliary hypotheses (arranged as to "protect" the central core) are instead subject to falsification and are engaged in a continuous process of adjustment which can also lead to their abandonment. The auxiliary hypotheses therefore serve, in Lakatos's view, to respond to anomalies (in the sense of Kuhn) allowing the research program to update itself, in the light of new empirical evidence, without its central assumptions having to be completely revised.

Furthermore, in Lakatos' idea, the theoretical nature of the central core of the research program is not substantially affected by the experimental results. In other words, when facing experiments that seem to contradict theoretical assumptions, scientists can choose whether to carry out a *peripheral adjustment* and modify the auxiliary hypotheses (which are affected by the experimental results) or whether to alter the fundamental theoretical assumptions (and therefore inaugurate in fact a different research program).[7]

As mentioned, according to Lakatos, the methodological choices underlying a research program are characterized by various forms of heuristics, divided into positive and negative ones. The first establish which research lines are to be favored, identifying the interesting problems for the research program and suggesting what the tools are to be able to face and solve them. The latter, on the other hand, establish which lines of research should instead be abandoned. Lakatos' perspective is interesting for our purposes for at least two reasons. First of all it furnishes a way to qualitatively distinguish hypotheses and theoretical elements within a scientific research program. We believe that, according to our proposal, this qualitative distinction can now be modeled logically. Secondly, Latakos' proposal provides a dynamic image of scientific research, which presents the possibility of changing and updating "truths" but without implying a pure conventionalism or relativism. In this sense, the epistemic solidity of a scientific assertion, with respect to simple views, is saved and indeed reinforced. Because of that, it is our opinion

[7] On this aspect it is appropriate to mention here an advantage constituted by our proposal and discussed in Section 4.1. In fact we will present an extension of PWK, called Halldén external calculus, which allows to model, even at a logical level, the difference between any modifications of the auxiliary hypotheses with respect to a change that occurs in the theoretical core.

that a logical model is now feasible, given that the evaluative uncertainty of hypotheses and theory is no longer confined in a "non-logical space" but rather is precisely incorporated in it.

4. Modeling the critical attitude

Our aim here starts from the consideration that the history of science, especially from the point of view of its practice, can be labelled as a particular "history of being critical", in the sense of rational uncertainty. By that we mean that science may be seen as the great attempt to reconcile a general condition of irreducible, incomplete knowledge with the the rejection of radical scepticism. In other words, science represents (currently) the best way to deal with our uncertainty without surrendering to the claim that if knowledge is unsure, nothing can be known. In our view, this is not just a pleasant philosophical stance. It rather mirrors a quite common attitude of everyday scientific work. Our idea is that this particular attitude, which we examined in the previous section, can be grasped, at a more formal level, by a three-valued logic in the weak Kleene family. In the following subsections we will rather discuss some examples of this situation. It is important to emphasize that the type of uncertainty involved here is not probabilistic uncertainty. By that we mean that a scientist does not evaluate or judge the reliability of a theoretical claim by weighing it in probabilistic terms but rather on the basis of its "accordance" (i.e. the claim is "embedded within" the paradigm, in Kuhnian terms) with the theoretical framework in which they operate. In other words, it is a kind of uncertainty related to a sort of suspension of judgment on the part of the scientist who is (objectively) unable, given evidence, to evaluate a scientific hypothesis (or a part of a scientific theory) as either true or false. Moreover, it is important to underline that the character of our proposal is *descriptive* and not *normative*. By that we mean that our model should not be intended as outlining any attempt to deal with these kinds of epistemic uncertainties but rather as a fruitful way to formally describe something that, at first sight, might look formally intractable.

4.1. Scientific theories

For our purposes, the most important lesson from the Duhem-Quine thesis (see Subsection 3.1) is that a scientific theory is a *complex net-*

work: a net of intertwined hypotheses. According to this view, it is not implausible to think that a scientific theory, thought as "a whole" , in the form of a more or less structured collection of hypotheses, is interpreted as uncertain if it contains at least a part that is reckoned as uncertain (we may think of $1/2$ as logically modelling such uncertainty). Even though the network of a scientific theory is difficult to describe, we implicitly accept the assumption, according to which different hypotheses, or parts of a scientific theoretical framework, can be thought of as logically connected. In other words, we are translating the Principle of Component Homogeneity, characterizing weak Kleene logics, into the following:

Scientific Component Homogeneity: when scientists (have reasons to) believe that a part of a theory is uncertain, then it is reasonable to believe that the whole theory is uncertain.

It is worthy briefly specifying here what we mean by "uncertain". Generally speaking, uncertainty refers to an epistemic situation in which available information is partial or imperfect. Uncertainty is a central topic of many philosophical investigations. It is, moreover, a key notion in several areas of scientific inquiry: certainly probability, but also game theory, physics, economics and cognitive science. Thus, the notion of being uncertain may be variously defined, according to the context. Our situation clearly expresses a form of uncertainty which is not *within* the area of investigation but *concerning* the area of investigation. In other words, it is a type of uncertainty that does not refer to something in the theory but about the theory itself. Because of that, in our view, being uncertain here roughly means that, at given time and data, one is not able to discriminate among diverse theoretical alternatives. Thus, one is dubious. That is, one does not necessarily consider something either true or false. To put it differently, that one's judgment, one's evaluation, is as if it is postponed until a situation arises in which some elements can finally make one lean towards one alternative rather than another. As we have already specified in Section 3, this situation is not unusual but rather quite common in scientific practice. Indeed, it constitutes the daily process by which ordinary science proceeds in its details. Let us go back to the principle of Scientific Component Homogeneity. In the form presented, it definitely risks of sounding too strong due to its implicit universal quantification. Indeed, one could ask if it makes sense to evaluate a scientific theory as uncertain in the case whether only a secondary hypothesis (of the theory) is uncertain. This suggests a return to the idea (outlined in the previous section) that parts and hypotheses within a scientific theory may have different "epistemic weights". Accordingly, we can think to some hypotheses as forming the *core* of a theory, while others to be somehow secondary, or less important with respect to the former. We have previously explained that, from a philosophical point of view, this difference traces back to the work of Lakatos [1978]. However, we are taking up the spirit of this distinction outlined in Lakatos, while we are not complying with (exactly) the same meaning of the terms. If follows that in the context of scientific theories we can rephrase the principle of Scientific Component Homogeneity by limiting it to the *core* facts, or hypothesis, of a scientific theory. In order to provide a rough idea of what the uncertainty of a core hypothesis may look like, let us consider an extremely relevant scientific theory: Quantum mechanics (QM). This theory is famously characterized by the phenomenon of *entanglement*. From the theoretical point of view, it is a simple consequence of Von Neumann's axiomatization of the theory [see, e.g, Dirac, 1981; von Neumann et al., 2018], which states that the Hilbert space of a composite system is the tensor product of the single spaces of the components. From this, it easily follows the existence of the so-called non-factorized states, as the tensor product of two (or more) Hilbert spaces contains states (vectors) that cannot be decomposed as tensor product of two (or more) basis vectors (a peculiarity that does not happen to be the case if different constructions, such as the direct product, were chosen for describing composite systems). Non-factorized states are called "entangled" and give rise to non-local phenomena, such as the measurement of an observable (as the spin of an electron) relative to a particle. This is entangled with another which can be very far in the space, which causes the collapse of the wave function of both particles. The theoretical prediction of non-local phenomena has been judged as problematic from some commentators, not least Albert Einstein himself [see the famous EPR argument Einstein et al., 1935]. Indeed, the effective *reality*[8] of non-local phenomena, predicted by the theory (elaborated between the 1920s and the 1930s), was not determined for many years, essentially because entangled particles were not (experimentally) observable until the design of specific experiments conducted by Alain Aspect and his group in [1982]. More precisely, in these experiments,

[8] By the term *reality*, is meant here the adherence/correspondence between the theory and the empirical observations of the theoretical predictions.

the failure of the Bell [1964] inequality was detected for the first time, confirming the effective reality of entangled particles. Concerning our proposal, the relevant point is the following: QM, as a physical theory, has been developed and applied in many directions since its formalization (and even before) to our days, over the decades in which its mirroring of certain aspects of reality was not yet clear, i.e. non-local phenomena were predicted by the theory but not yet observed. It is reasonable to say that the theory has contained for years a core part[9] which has been disputed, from the perspective of the philosopher of science interested into the foundation of a theory. A certain *epistemic uncertainty* has surrounded the theory; nevertheless it has not been a sufficiently good reason to reject it (along with its application), due to its explanatory robustness and also on the pragmatical level.

With that in mind, in our approach, the value $1/2$ is meant for interpreting scientific assertions for which there is reasonable evidence to consider them as uncertain, or not enough evidence to consider them either (epistemically) true or false. Should the scientific community (or even a group of scientists working on a problem) reject a whole theory/hypotheses when there is enough evidence to doubt about the truthfulness of a core part of a theory? A positive answer would demand the rejection of many theories but this is not what happens in current scientific practice. Going back to the above example, quantum mechanics should have been rejected soon after its axiomatization by John von Neumann, due to the fervent debate about the reality of non-local phenomena, such as entanglement. However, this is not the way science usually proceeds: scientists keep working within theories which have doubtful or uncertain (core) parts. The reason is that these theories appears convincing, nevertheless they include uncertain parts. In logical terms, they still aim at *reasoning* within the uncertainty of the whole theory and trying their best to solve uncertainty.

From, this perspective the question of whether there are ways to model logically the right epistemic attitude towards theories becomes clearer. The logical systems to be adopted should have the presence of (at least) a non-classical truth-value, to capture epistemic uncertainty. Moreover, they should respect the principle of Scientific Component Ho-

[9] It should be clear that the possibility of non-local phenomena is central in QM, thus part of the core of the theory, as, from the theoretical point of view, is a direct consequence of the one of the axioms of the theory.

mogeneity, at least in its restricted form. Last but not least, logical consequence should preserve the uncertain truth-value: a requirement justified by the necessity of not rejecting a theory which is uncertainty due to the uncertainty of some parts of it.

The above considerations indicate, in our opinion, PWK as a natural candidate for modeling this epistemic attitude towards scientific theories, due to presence of the value $1/2$, which can represent epistemic uncertainty (according to the interpretation in Subsection 2.1), the principle of SCH, and the choice of $\{1, 1/2\}$ as truth-set. However, the standard version of the SCH appears too strong and the "infectivity" of uncertain parts of theory (provoking the uncertainty of the whole theory) should be limited to core facts or hypotheses. To this end, the logic H_0 (or, Halldén external calculus) should be preferred to PWK. Indeed, extending the language of PWK by introducing t allows us to define also further connectives, which are interpreted "classically", namely into the two-element Boolean subalgebra of **WK** ($\{0, 1\}$) [see Bonzio et al., 2017]. These connectives include (but are not limited to) $+\varphi := \neg\,\mathrm{t}\,\neg\varphi \to \mathrm{t}\,\varphi$, $-A := \neg(+A)$ and $\star\,A := tA \vee -A$, whose interpretation is given by the following tables.

φ	$+\varphi$
1	1
$1/2$	0
0	1

φ	$-\varphi$
1	0
$1/2$	1
0	0

φ	$\star\,\varphi$
1	1
$1/2$	1
0	0

According to Bochvar's interpretation of the external calculus, $\mathrm{t}\,\varphi$ is interpreted as "φ is true". Consequently, the readings of $+\varphi$, $-\varphi$ and $\star\,\varphi$ are "φ is not uncertain" (it is either true or false), "φ is uncertain" and "φ is not false" (either true or uncertain), respectively. The presence of these auxiliary connectives allows us to provide a description of the attitude with which scientific theories are approached. Indeed, a theory can be schematized as a (long) conjunction of formulas (in the language of the logic H_0) $A_1 \wedge \cdots \wedge A_n \wedge \star\,A_{n+1} \wedge \cdots \wedge \star\,A_m$, where the subsets $\{A_1, \ldots, A_n\}$ and $\{A_{n+1}, \ldots, A_m\}$ consists of the core and "secondary" facts or hypothesis of the theory, respectively. The rationale behind is that every scientific fact can be interpreted as true, false or uncertain, depending on the evidence at one's disposal. However, when one among the core facts in uncertain, then the whole conjunction modeling the theory becomes automatically uncertain. On the other hand, if a secondary fact is uncertain, this does not affect the truthfulness of the whole theory.

Observe that it is easily checked that the statement $\star A \wedge \star B$ is not equivalent to $\star(A \wedge B)$. This (partially) motivates the choice of the connective $\star$ for modeling secondary facts of scientific theories. Indeed, the conjunction of two secondary statements does not turn into a statement which is still secondary. Plausibly, it can turn the conjunction $A \wedge B$ into a core part of a theory. Let us make an example taken from the history of molecular biology. The standard framework accounting for transcriptional processes in molecular biology did not contain post-transcriptional regulators [such as microRNAs Bartel, 2018] in its original formulation because scientists were not aware of the complexity of gene regulation. Moreover, the existence of varieties of non-coding RNAs was also a known fact but it was not initially associated with the regulation of gene expression which is now widely recognized [Frías-Lasserre and Villagra, 2017]. Roughly speaking, these two "parts" of the framework were peripheral but when, due to the increase of experimental evidence, good reasons were brought to consider these aspects jointly, they radically changed the "core" of the current understanding of gene expression.

There are also other peculiarities of the logic H_0 which are well suited in the application proposed above for modeling the attitude towards scientific theories: indeed, this interpretation provides good reasons for defending some features of the logic, such as the satisfaction of classical contradictions, tautologies and some notable deductive failures. Our epistemic interpretation is prompted by the fact that, within scientific practice it is not uncommon to deal with confirmations of contradictory hypotheses. Upon interpreting them as logical negations, in a broad sense, it seems natural to need a logic which allows for the validation of contradictions, such as $A \wedge \neg A$. Indeed, one such contradiction is actually validated only in case a statement A takes value $1/2$, i.e. A is an uncertain (scientific) fact. Then, there are good reasons to think that $\neg A$ is also uncertain (differently, a scientist could express himself/herself about the truthfulness of A). We might, for instance, think about two different experiments, one *confirming* an hypothesis and another one *falsifying* the very same hypothesis: this creates an epistemic contradiction. A cautious scientist, facing contradictions of this kind, is brought to assume the hypothesis to be uncertain, at least for a reasonable period of time. We emphasize that we are not expressing the idea that contradictions are real (a claim supported, for instance, by Graham Priest); rather we are simply admitting the possibility of contradictions as an epistemic condition, namely the plausibility of the epistemic coexistence of A and $\neg A$.

5. Conclusion

We have proposed a non-classical (three-valued) logic for modeling the critical attitude that scientists and philosophers of science could show towards the acceptance (and rejection) of scientific theories. This consists of an innovative attempt as, to the best of our knowledge, logic – as a formal tool – has not been used in trying to capture peculiar reasoning and attitudes proper to the philosophy of scientific practice.

The choice of a logic in the weak Kleene family, featuring an infectious truth-value is not dictated by chance. Firstly, it relies on the idea that the third truth-value can be provided with an epistemic interpretation and, consequently, represents a specific epistemic uncertainty. Secondly, the type of uncertainty discussed here regards the truthfulness of hypotheses contained in, or parts of, scientific theories and can be "infectious", namely it can shade uncertainty over the whole theory. This idea is grounded on the spirit of the Duhem-Quine thesis, according to which a scientific theory can be seen as a network of intertwined hypotheses, which we rendered as a conjunction of formulas, taking into account the core part of theories and their secondary assumptions (following an idea by Lakatos). A collateral effect of our choice which may attract potential criticisms is that a theory turns out to be epistemically uncertain, instead of false, even when it contains false parts. However, one could defend the idea that this could model, for instance, the functioning principles of pseudo-scientific theories (and, perhaps, might be applied to model so-called "fake-news" as well) in which, roughly speaking, elements of truth coexist with false claims.[10]

We are convinced that we have sufficiently motivated the choice of weak Kleene. Nevertheless, one might object that other many-valued formalisms could be better suited, pointing, for instance, at fuzzy logics. However, in this respect, we believe that *fuzziness* itself looks more like an ontological property than an epistemic capacity. In other words,

[10] With respect to this issue, following the advice of an anonymous reviewer, we have agreed not to develop these aspects here and we will probably reserve a detailed analysis of these problems in the future, in a dedicated publication.

fuzziness deals with the fact that some properties (like "being tall", "being good at",...) do not possess precise demarcation conditions and thus are not well handled by binary evaluations: for instance it is true that Federico is very tall, though we could reconsider this evaluation when he is standing close to the players of the Italian basketball team. In the realm of fuzzy logics, understood here in the rough sense of the logics of continuous t-norms interpreted over $[0, 1]$ [see, e.g., Hájek, 2001], the middle value $1/2$ is traditionally read as "half true, half false" (something in the middle between "begin tall" and "being not tall"). Such an interpretation could unlikely stand for epistemic uncertainty. Yet, one could say that some non-classical formalisms are well-suited to describe pieces of reality and, perhaps, could turn out to be useful for describing some aspects of scientific theorizing. Interestingly enough, logics featuring infectious truth-values (in their algebraic counterparts) outnumber weak Kleene logics: examples include $\mathbf{FDE}_{\varphi}$ (introduced by Priest [2010] and, independently, by [Daniels, 1990]), $\mathbf{S}_{\text{fde}}$ (introduced by Deutsch [1977]), obtained by adding infectious truth-values to Belnap-Dunn logic (also known as *First Degree Entailment*) and the Logic of Paradox, respectively. To the best of our knowledge, specific infinite-valued logics featuring the presence of infectious truth-values have not be studied. Nevertheless, the general approach to the "logics of variable inclusion" introduced in [Bonzio et al., 2021; Bonzio and Pra Baldial., 2021a,b] (see also the monograph Bonzio et al., 2022) shows that one can associate with every logic an "infectious companion". As a consequence, the critical attitude proposed in the present work could be made more general, assuming that other formalisms could be involved in the description of scientific theories, and the consequent critical attitude by the infectious companion of the target logic. It is useful to recall that the modeling of the critical attitude proposed in the present work reserves a *descriptive* role (not a normative one) to logical formalism.

Having specified these aspects it also important to recall that our choice to deal with an epistemic interpretation should also clarify our meta-theoretic position concerning the debate on the nature of logics, i.e. whether a "right" logic actually exists (or whether it should) and why. In this respect, without entering too much into details, our stance is definitely pluralistic and towards positions advocating anti-exceptionalism over logic [see, e.g., Hjortland, 2017]. Because of that, our methodological and theoretical choice has two main consequences. First, it means that, in our opinion, (some) weak Kleene logics (H_0 or PWK) offer an adequate (in the sense also of "operationally effective") way to model the phenomena of interest to us, namely the epistemic condition of scientific judgement in everyday scientific enterprise. Second, the choice of H_0 is justified and motivated here, precisely because of the epistemic advantage it grants us in modelling the phenomenon of our interest. However, this does not mean, *per se*, that H_0 (or PWK) has to be *the* logic to model this kind of phenomena. In other words, weak Kleene logics work effectively, from an epistemic point of view, but that is not a reason to claim that they constitute the only logical tool suitable for this type of analysis.

In conclusion, the present work addresses a double objective. On the one hand, it proposes an application for (the external calculus of) paraconsistent weak Kleene logic. In particular, thanks to the epistemic interpretation of the logic, our approach gives a reason for the choice of designating (preserving it through logical inference) the third value, a principle that, to the best of our knowledge, has not been convincingly discussed elsewhere. On the other hand, it offers the philosopher of science a logical formalism useful for the description of the attitude towards the critical acceptance of scientific theories and hypotheses.

Acknowledgments. Both authors wish to thank Vincenzo Fano, Pierluigi Graziani, Stefano Calboli and the whole group Synergia of the University of Urbino, for their observations and comments on an earlier draft of this work. For useful comments on a preliminary presentation of this work, the authors thank Jan Sprenger, Vincenzo Crupi, Michele Lubrano, Matteo Plebani and Fabrizio Calzavarini of the Department of Philosophy and Educational Sciences of the University of Turin.

F.B. acknowledges the support of the postdoctoral fellowship of the Department of Literature and Philosophy of University of Florence, funded by the Department itself.

S.B. acknowledges the support of the postdoctoral fellowship Beatriu de Pinos, funded by the Secretary of Universities and Research (Government of Catalonia) and by the Horizon 2020 program of research and innovation of the European Union under the Marie Skłodowska-Curie grant agreement No. 801370. He acknowledges the support of the PRIN 2017 Project "From models to decisions" funded by the Italian Ministry for University, Education and Research (MIUR) and also gratefully acknowledges the INDAM GNSAGA, Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.

We finally thank two anonymous reviewers for their fruitful comments on a previous version of the paper.

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