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# Relating Semantics for Epistemic Logic

**Abstract.** The aim of this paper is to explore the advantages deriving from the application of relating semantics in epistemic logic. As a first step, I will discuss two versions of relating semantics and how they can be differently exploited for studying modal and epistemic operators. Next, I consider several standard frameworks which are suitable for modeling knowledge and related notions, in both their implicit and their explicit form, and present a simple strategy by virtue of which they can be associated with intuitive systems of relating logic. As a final step, I will focus on the logic of knowledge based on justification logic and show how relating semantics helps us to provide an elegant solution to some problems related to the standard interpretation of the explicit epistemic operators.

**Keywords**: relating semantics; epistemic logic; justification logic

## 1. Introduction

This paper is explorative in nature and aims at presenting the advantages deriving from applying relating semantics to epistemic logic in general and to explicit epistemic logic in particular. It is structured as follows. In this introduction relating semantics is put into the context of logical semantics. Section 2 is dedicated to an overview of relating semantics: I distinguish uniform and general relating semantics and derive some simple results about them. Section 3 presents different kinds of semantics for epistemic logic and a general strategy by virtue of which they can be associated with intuitive systems of relating logic. Finally, section 4 is dedicated to the analysis of some aspects of explicit epistemic logic in a relating setting.

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### 1.1. Aspects of propositions

The idea underlying relating semantics is that the truth value of a complex proposition depends not only on its components and the way of composition, but also on the semantic connections between the components, where the semantic connections are based on specific aspects characterizing the propositions expressed by those components [8, 9, 27]. In a very general sense, a logical framework is characterized by the way in which we select the aspects of a proposition that are crucial for identifying its content.[1] So, in a classical setting, we assume that the only aspects of a proposition which matter to logic are

1. its form;
2. its *extension*, i.e. its truth value.

Similarly, in a modal setting, we assume that the only aspects of a proposition which matter to logic are

1. its form;
2. its *intension*, i.e. the set of worlds at which it is true.

The benefit of working under such assumptions is that they allow us to focus on specific classes of operators, the extensional ones in the first case and the intensional ones in the second one, and to develop the corresponding logics. Accordingly, in an extensional framework, we define the truth conditions of a complex proposition in terms of the truth values of its components and, in an intensional framework, we define the truth conditions of a complex proposition in terms of the possible worlds where its components are true. These frameworks, while fruitful in orienting the logical analysis of a vast number of operators, also limit the scope of our study to extensional and intensional contexts, thus preventing us to capture other interesting traits of propositions, specifically the hyperintensional ones.

### 1.2. Aspect based relations

In a general setting, we make the assumption that the aspects of a proposition which matter to logic are

1. its form
2. its $X$, where $X$ is some specific property.

[1] This characterization of a logical framework is exploited in its full generality in [9] and constitutes a key insight in relating semantics.

Since the selection of $X$ is not predetermined, we are in a position to consider any aspect for defining the truth of a proposition. This is one of the starting points of relating semantics: once property $X$ is selected, the truth conditions of a complex proposition are defined in terms of its form and the relations occurring between its components with respect to $X$. Indeed, in a relating setting, the aspects of a proposition which matter to logic are at least

1. its form
2. a selected set of relations between its components.

Hence, in relating semantics the truth conditions of complex propositions involve a crucial reference to those aspects the relations between the component of a proposition are based on.

## 2. Relating semantics

The key idea in relating semantics is that the semantic value of a complex proposition is given by a valuation of propositional variables together with a valuation of the relation between the main components of that proposition. This relation is able to encode both intensional and hyperintensional elements.[2]

### 2.1. Introducing relating semantics

A propositional language is a triple $\mathcal{L} = (Var_{\mathcal{L}}, C_{\mathcal{L}}, ar)$ where $Var_{\mathcal{L}}$ is a set of propositional variables and $(C_{\mathcal{L}}, ar)$, the *type* of $\mathcal{L}$, is such that $C_{\mathcal{L}}$ is a set of logical constants and $ar\colon C_{\mathcal{L}} \to \mathbb{N}$ is a function assigning an *arity* to each constant. The notion of a formula of $\mathcal{L}$ is defined according to the following grammar:

$$\phi := p \mid c(\phi_1, ..., \phi_{ar(c)}), \text{ where } p \in Var_{\mathcal{L}},\ c \in C_{\mathcal{L}}.$$

In interpreting $\mathcal{L}$, we typically work with a set $TV$ of truth values such that $1 \in TV$. A model for $\mathcal{L}$ is a tuple $M = (V, f)$ constituted by a function $V\colon Var_{\mathcal{L}} \to TV$, that assigns a truth value to each variable, and a function $f$, that assigns a truth function $f_c$ to each $c \in C_{\mathcal{L}}$ in such a way that the following condition holds

If $c \in C_{\mathcal{L}}$, then $f_c\colon TV^{ar(c)} \to TV$.

[2] See [16] for an extended introduction to the ideas underlying this kind of semantics.

The notion of truth in a model is then defined so that

$M \models p$ iff $V(p) = 1$, where $p \in Var_{\mathcal{L}}$;
$M \models c(\phi_1, ..., \phi_{ar(c)})$ iff $f_c([\phi_1]^M, ..., [\phi_{ar(c)}]^M) = 1$.

Here $[\phi_i]^M$ is the semantic value of formula $\phi_i$ in $M$, in this case a truth value in $TV$.

A relating semantics is obtained by enriching the notion of a model. If $M$ is a model for $\mathcal{L}$, $M_{\mathcal{R}}$ is the extension of $M$ obtained by introducing a set $\mathcal{R}$ of relations, in principle one for each $c \in C_{\mathcal{L}}$. In this kind of framework the semantic value of a complex formula is determined by its value in $TV$ and the relations in $\mathcal{R}$. Hence, the notion of truth in a model is defined so that

$M \models p$ iff $V(p) = 1$, where $p \in Var_{\mathcal{L}}$;
$M \models c(\phi_1, ..., \phi_{ar(c)})$ iff $f_c([\phi_1]^M, ..., [\phi_{ar(c)}]^M) = 1$
and $R_c(\phi_1, ..., \phi_{ar(c)})$, where $R_c$ is the relation corresponding to $c$.

This definition captures the idea that the semantic value of a complex proposition depends both on the truth values of its components and on a semantic relation between them.

*Example* 1. Let $\mathcal{L}$ be the language of classical propositional logic. Assuming that the implication is introduced by definition, we have $C_{\mathcal{L}} = \{\neg, \wedge, \vee\}$ with $ar(\neg) = 1, ar(\wedge) = ar(\vee) = 2$. A model of $\mathcal{L}$ is a tuple $M = (V, f)$ where $f$ is defined as follows

$f_{\neg} \colon \{0,1\} \to \{0,1\}$ is such that $f_{\neg}(x) = 1 - x$;
$f_{\wedge} \colon \{0,1\}^2 \to \{0,1\}$ is such that $f_{\wedge}(x_1, x_2) = \min(x_1, x_2)$;
$f_{\vee} \colon \{0,1\}^2 \to \{0,1\}$ is such that $f_{\vee}(x_1, x_2) = \max(x_1, x_2)$.

The truth of a complex proposition is determined in terms of the semantic values of its components. E.g.

$M \models \phi_1 \wedge \phi_2$ iff $\min([\phi_1]^M, [\phi_2]^M) = 1$.

The corresponding condition in relating semantics is

$M \models \phi_1 \wedge \phi_2$ iff $\min([\phi_1]^M, [\phi_2]^M) = 1$ and $R_{\wedge}(\phi_1, \phi_2)$.

We are now in a position to introduce different connectives based on the interpretation of $R_{\wedge}$. For instance, interpreting $R_{\wedge}$ as the universal relation enables us to recover the extensional conjunction, representing the fact that $\phi_1$ *and* $\phi_2$, while interpreting $R_{\wedge}$ as the relation obtaining between $\phi_1$ and $\phi_2$ when the proposition expressed by $\phi_1$ describes a state of affairs that precedes the state of affairs described by the proposition expressed by $\phi_2$ enables us to introduce a diachronic conjunction representing the fact that $\phi_1$ *and then* $\phi_2$.[3]

In a more general setting, given a logical constant $c$ with arity $n$, the truth condition for a formula like $c(\phi_1, ..., \phi_n)$ is defined in terms of a set $\boldsymbol{p}$ of parameters. Let $TC(c, \boldsymbol{p})$ be such condition. Then

$$M, \boldsymbol{p} \models c(\phi_1, ..., \phi_n) \text{ iff } TC(c, \boldsymbol{p})$$

and the corresponding truth condition in the relating setting is

$$M_{\mathcal{R}}, \boldsymbol{p} \models c(\phi_1, ..., \phi_n) \text{ iff } TC(c, \boldsymbol{p}) \text{ and } R_c(\phi_1, ..., \phi_n)$$

where it is assumed that the default option for $R_c$ is the universal relation on the set of formulas of $\mathcal{L}$.

*Example* 2. Let $\mathcal{L}$ be the language of classical propositional modal logic, so that $C_{\mathcal{L}} = \{\neg, \wedge, \vee, \Box\}$ and $ar(\Box) = 1$. A model for $\mathcal{L}$ is a triple $(W, R, V)$ such that $W \neq \varnothing$, $R \subseteq W \times W$, $V \colon Var_{\mathcal{L}} \to \wp(W)$. The semantic value of a proposition is the set of worlds in $W$ in which that proposition is true. The function assigned to $\Box$ is

$$f_{\Box} \colon \wp(W) \to \wp(W), \text{ such that}$$

$$f_{\Box}(X) = \{w \in W : \forall x \in W (R(w, x) \Rightarrow x \in X)\}.$$

Again, the semantic value $[\phi]^M$ of a complex proposition can be determined in terms of the semantic values of its components. E.g.

$$M, w \models \Box\phi \text{ iff } w \in f_{\Box}([\phi]^M) \text{ iff } \forall x \in W (R(w, x) \Rightarrow x \in [\phi]^M).$$

In this case, $TC(c, \boldsymbol{p})$ is $w \in f_{\Box}([\phi]^M)$, and the only parameter is $w$. The corresponding condition in relating semantics is

$$M, w \models \Box\phi \text{ iff } w \in f_{\Box}([\phi]^M) \text{ and } R_{\Box}(\phi),$$

where $R_{\Box}$ is the relation corresponding to $\Box$.

In this framework, the possibility is open to incorporate condition $TC(c, \boldsymbol{p})$ into the relating relation. If $TC(c, \boldsymbol{p})$ is incorporated into the relation, we say that the corresponding relating semantics is *deflating*, otherwise we say that it is *inflating*. The main concern here is in inflating relating semantics, since deflating relating semantics turns out to be too general for yielding interesting results.

[3] See [16], where a number of possible interpretations are proposed.

### 2.2. Basic facts in relating semantics

I say that a system of relating semantics is *uniform* when a unique relation $R$ is involved in the truth conditions of *all* the logical operators,[4] whereas a general system allows for associating different relations with different operators. The logic induced by a uniform system of relating semantics is said to be uniform as well. Finally, the notion of logical consequence is defined in the usual manner: w.r.t. $R$ and a class $\boldsymbol{R}$ of relations

(i) $X \Vdash_R \phi$ iff $\forall M, \boldsymbol{p}(M_R, \boldsymbol{p} \models X \Rightarrow M_R, \boldsymbol{p} \models \phi)$;
(ii) $X \Vdash_{\boldsymbol{R}} \phi$ iff $\forall R \in \boldsymbol{R}, M, \boldsymbol{p}(M_R, \boldsymbol{p} \models X \Rightarrow M_R, \boldsymbol{p} \models \phi)$.

Let us now prove some elementary, though general, results on uniform and deflating semantics.[5]

Theorem 2.1. *Suppose $\boldsymbol{R}$ is a class of relations on $\mathcal{L}$ defined by a conditional like $\forall x_1, ..., x_n(R(x_1, ..., x_n) \Rightarrow C)$, where $C$ is any condition. Then the uniform relating logic determined by $\Vdash_{\boldsymbol{R}}$ is empty.*

The idea is to select a suitable $R$.

Proof. Let $c$ be any $n$-ary constant. Define $R$ so that $R$ is empty. Since $M_R, \boldsymbol{p} \models c(\phi_1, ..., \phi_n)$ iff $TC(c, \boldsymbol{p})$ and $R(\phi_1, ..., \phi_n)$, $M_R, \boldsymbol{p} \not\models c(\phi_1, ..., \phi_n)$. Therefore, for any formula $c(\phi_1, ..., \phi_n)$ there is a model in which that formula is not true. ⊣

As a corollary, we get that the logic determined by the class of all relations on $\mathcal{L}$ in a uniform relating semantics is empty. This fact shows that interesting systems of relating logic are either not uniform, like the epistemic systems discussed below, or such that the relations they involve are not defined by a conditional like $\forall x_1, ..., x_n(R(x_1, ..., x_n) \Rightarrow C)$, e.g. systems where the relations are reflexive.[6]

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[4] A uniform relating semantics coincides with a mono-relational semantics for a language with only relating connectives. See [16] for further information on various versions of relational semantics.

[5] A more comprehensive and detailed overview of the semantics of relating logic is proposed in [18, sec. 2].

[6] The systems of relatedness logic introduced in [8] are not touched by Theorem 1: first, those systems are not uniform, since they include classical operators which are such that no relation occurs in their truth conditions; second the relations involved in the truth conditions of the implication are typically reflexive.

A first theorem concerning the deflating version of relating semantics is the following.

Theorem 2.2. *Suppose $c$ is an $n$-ary constant whose* semantic definition *is known. Then, a relating semantics for $c$ can be introduced with respect to which the logic of $c$ is sound and complete.*

Proof. Let $c$ be such an operator and $M, \boldsymbol{p} \models c(\phi_1, \ldots, \phi_n)$ iff $TC(c, \boldsymbol{p})$ be the correlated truth condition. Define $R$ so that

$$R(\phi_1, \ldots, \phi_n) \text{ iff } TC(c, \boldsymbol{p}).$$

Then, the conclusion follows, since:

$$M_R, \boldsymbol{p} \models c(\phi_1, \ldots, \phi_n) \text{ iff } R(\phi_1, \ldots, \phi_n). \qquad \dashv$$

This theorem shows that, in general, introducing a relating semantics whose relating relation incorporates the known truth conditions of an operator is always possible, even though it is not particularly enlightening. In spite of that, this implies that any logic concerning operators that satisfy the condition posed by the theorem can be viewed as a specific relating logic.

I conjecture that a similar theorem holds when the operator is defined in terms of logical axioms.

Conjecture. *Suppose $c$ is an $n$-ary constant whose* axiomatic definition *is known. Then, a relating semantics for $c$ can be introduced with respect to which the logic of $c$ is sound and complete.*

This conjecture is more interesting than the previous theorem.

Proof idea. Let $c$ be such an operator and define its truth conditions so that $M_R, \boldsymbol{p} \models c(\phi_1, ..., \phi_n)$ iff $R(\phi_1, ..., \phi_n)$. If an appropriate translation of the axioms on $c$ in terms of $R$ is available, introduce a condition on the models for each axiom that connects $c$ to a classical connective. The conclusion follows. $\dashv$

Let us exemplify this theorem, and the way the translation could go, by taking the case of a non-monotonic conditional, $\hookrightarrow$, captured by the following axioms in a modal framework where $\to$ is the usual implication:

1. Reflexivity: $\phi \hookrightarrow \phi$;
2. Inclusion: $(\phi \hookrightarrow \psi) \to (\phi \to \psi)$;
3. Cut: $(\phi \hookrightarrow \alpha) \wedge (\phi \wedge \alpha \hookrightarrow \psi) \to (\phi \hookrightarrow \psi)$.

First, set $M, w \models \phi \hookrightarrow \psi$ iff $R_w(\phi, \psi)$, where the models are assumed to satisfy the conditions:

1. $R_w(\phi, \phi)$;
2. if $R_w(\phi, \psi)$ and $M, w \models \phi$, then $M, w \models \psi$;
3. if $R_w(\phi, \alpha)$ and $R_w(\phi \wedge \alpha, \psi)$,then $R_w(\phi, \psi)$.

Next, define a canonical model where $R_w(\phi, \psi)$ iff $\phi \hookrightarrow \psi \in w$. The correspondence between axioms and semantic conditions ensures that the axiomatization is sound and that the canonical model is a model for the logic we are considering. In addition, the canonical definition of $R_w$ ensures that a truth lemma is provable, so that the system is also complete.

Taking stock, Theorem 2.2 and the previous conjecture are interesting is so much as they allow us to appreciate the *generality* of a relating framework, even if the resulting semantics is in a sense parasitic either on an existing semantic specification or on an existing axiomatic specification of the logic, and so provides us with little insight on the sense and properties of the operators we are interested in.

## 3. Epistemic systems

The main point of this paper is the application of relating semantics to epistemic logic. So, in this section I will survey some well-know approaches to epistemic logic and then sketch some connections between these approaches and relating semantics.

The set $Fm(\mathcal{L})$ of formulas of the basic epistemic language $\mathcal{L}$ is defined according to the following grammar, where *Var* is a set of propositional variables and $K$ the epistemic operator representing knowledge:

$$\phi := p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid K\phi$$

while $\rightarrow$ and $\leftrightarrow$ are defined as usual. I work in a framework where epistemic models are based on a set $W \neq \varnothing$ of possible worlds and a valuation function $V\colon Var \rightarrow \wp(W)$ assigning sets of worlds to propositional variables. The truth of a non-modal formula is defined as follows.

1. $M, w \models p$ iff $w \in V(p)$;
2. $M, w \models \top$;
3. $M, w \models \neg\phi$ iff $M, w \not\models \phi$;
4. $M, w \models \phi \wedge \psi$ iff $M, w \models \phi$ and $M, w \models \psi$;
5. $M, w \models \phi \vee \psi$ iff $M, w \models \phi$ or $M, w \models \psi$.

I also assume a functional version of possible worlds semantics, which exploits the correspondence between relations on a set and functions into its power set. Accordingly, models are introduced as tuples involving a function $R\colon W \to \wp(W)$, while the corresponding relation can be defined in such a way that $R(w, v)$ iff $v \in R(w)$, where $R(w)$ is the set of worlds that are accessible from $w$. The notions of logical consequence and validity are defined as usual.

### 3.1. Epistemic semantics

The semantic frameworks for epistemic logic which I am going to focus on are: relational semantics; perspectival semantics; neighborhood semantics; and liberal semantics. These frameworks help us to capture different intuitions on knowledge, and therefore to introduce different principles characterizing $K$. I will assume a fair amount of familiarity with such frameworks and present them from the more specific to the more general, highlighting the basic logical traits of the notion of knowledge they represent.[7] $\Vdash_{\boldsymbol{M}}$ is the notion of logical consequence defined with respect to the class of models introduced from time to time.

#### 3.1.1. Relational semantics

A model is a tuple $M = (W, R, V)$, with $R\colon W \to \wp(W)$ such that $w \in R(w)$ for every $w \in W$. The definition of truth is such that

$M, w \models K\phi$ iff $R(w) \subseteq [\phi]^M$, where $[\phi]^M = \{w : M, w \models \phi\}$.

Intuitively, every world $w \in W$ is associated with a set $R(w)$ of possible worlds, which are the ones that are not excluded by what the agent knows at $w$, and a proposition is known to be true just in case it is true at all these worlds.[8] It is well-known that in this setting we have *full logical omniscience*:

$\Vdash_{\boldsymbol{M}} K\top$;
if $\Vdash_{\boldsymbol{M}} \phi$, then $\Vdash_{\boldsymbol{M}} K\phi$;
if $\Vdash_{\boldsymbol{M}} \phi \to \psi$, then $\Vdash_{\boldsymbol{M}} K\phi \to K\psi$;
if $\Vdash_{\boldsymbol{M}} \phi \leftrightarrow \psi$, then $\Vdash_{\boldsymbol{M}} K\phi \leftrightarrow K\psi$;
$\Vdash_{\boldsymbol{M}} K(\phi \to \psi) \to (K\phi \to K\psi)$.

[7] A classical introduction covering most of the systems we are about to present is given in [10]. For a general introduction to the modal logics underlying the epistemic systems we are going to consider see [5, 7].

[8] See [24, 25] for an overview of this approach.

Therefore, the notion of knowledge we are working with is a notion of either *implicit* or *ideal* knowledge.

#### 3.1.2. Perspectival semantics

A model is a tuple $M = (W, \mathcal{P}, V)$, with $\mathcal{P}\colon W \to \wp\wp(W)$ such that $\mathcal{P}(w) \neq \varnothing$ and $w \in \bigcap \mathcal{P}(w)$ for every $w \in W$. The definition of truth is such that

$M, w \models K\phi$ iff $X \subseteq [\phi]^M$ for some $X \in \mathcal{P}(w)$.

Intuitively, every world $w \in W$ is associated with a set $\mathcal{P}(w)$ of sets of possible worlds. $\mathcal{P}(w)$ represents the strong body of evidence available to the agent, and a proposition is known to be true just in case there is some evidence supporting it.[9] In this setting we have a form of *limited logical omniscience*:

$\Vdash_{\boldsymbol{M}} K\top$;
if $\Vdash_{\boldsymbol{M}} \phi$, then $\Vdash_{\boldsymbol{M}} K\phi$;
if $\Vdash_{\boldsymbol{M}} \phi \to \psi$, then $\Vdash_{\boldsymbol{M}} K\phi \to K\psi$;
if $\Vdash_{\boldsymbol{M}} \phi \leftrightarrow \psi$, then $\Vdash_{\boldsymbol{M}} K\phi \leftrightarrow K\psi$.

In addition, a relational model can be identified with a perspectival model where, for every $w \in W$, $\mathcal{P}(w)$ is closed under taking intersections. In this case $R(w) = \bigcap \mathcal{P}(w)$.

#### 3.1.3. Neighborhood semantics

A model is a tuple $M = (W, \mathcal{N}, V)$, with $\mathcal{N}\colon W \to \wp\wp(W)$ such that $\mathcal{N}(w) \neq \varnothing$ and $w \in \bigcap \mathcal{N}(w)$ for every $w \in W$. The definition of truth is such that

$M, w \models K\phi$ iff $[\phi]^M \in \mathcal{N}(w)$.

Intuitively, every world $w \in W$ is associated with a set $\mathcal{N}(w)$ of sets of possible worlds. $\mathcal{N}(w)$ represents the propositions the agent knows, so that a proposition is known to be true just in case it is one of the members of $\mathcal{N}(w)$.[10] In this setting we have a more interesting form of *limited logical omniscience*:

if $\Vdash_{\boldsymbol{M}} \phi \leftrightarrow \psi$, then $\Vdash_{\boldsymbol{M}} K\phi \leftrightarrow K\psi$.

[9] See [4, 26] for developments of this interesting new approach.

[10] See [10, 21] for an introduction to this approach.

In addition, a perspectival model can be identified with a neighborhood model where, for every $w \in W$, $\mathcal{N}(w)$ is upward closed, while a relational model can be identified with a neighborhood model where, for every $w \in W$, $\mathcal{N}(w)$ is a filter. In this case $R(w) = \bigcap \mathcal{N}(w)$.

Neighborhood models provide the *most general intensional representation* of knowledge. Indeed, it is only assumed that the agent knows the propositions that are logically equivalent to the propositions he knows.

#### 3.1.4. Liberal semantics

A model is a tuple $M = (W, \mathcal{T}, V)$, with $\mathcal{T}\colon W \to \wp(Fm(\mathcal{L}))$. The definition of truth is such that

$M, w \models K\phi$ iff $M, w \models \phi$ and $\phi \in \mathcal{T}(w)$.

Intuitively, every world $w \in W$ is associated with a set $\mathcal{T}(w)$ of formulas. $\mathcal{T}(w)$ represents the propositions the agent is justified to assume, so that a proposition is known to be true just in case it is true and is one of the propositions expressed by a formula in $\mathcal{T}(w)$.[11] In this setting we have no form of *logical omniscience*, but this comes at a cost, since we also have no hint on the logical principles characterizing $K$. Liberal semantics is extremely general: a neighborhood model can be identified with a liberal model where $\mathcal{T}(w)$ is closed under logical equivalence for every $w \in W$. In this case $\mathcal{N}(w) = \{[\phi]^M : \phi \in \mathcal{T}(w)\}$.

Liberal models provide the *most general hyperintensional representation* of knowledge. Indeed, we are free to represent properties of $K$ simply by introducing constraints on formulas in $\mathcal{T}(w)$.

#### 3.1.5. Hybrid liberal semantics

A final framework for representing knowledge is given by a semantics that combines elements from intensional systems and liberal systems [see 10, ch. 9]. In a hybrid liberal semantics a model is a tuple $M = (W, R, \mathcal{A}, V)$, where $R\colon W \to \wp(W)$ is such that $w \in R(w)$ for every $w \in W$ and $\mathcal{A}\colon W \to \wp(Fm(\mathcal{L}))$ is a function that associate a set $\mathcal{A}(w)$ of formulas to each $w \in W$. $\mathcal{A}(w)$ represents the propositions the agent is aware of. Truth is defined so that:

$M, w \models A\phi$ iff $\phi \in \mathcal{A}(w)$;
$M, w \models K\phi$ iff $R(w) \subseteq [\phi]^M$ and $\phi \in \mathcal{A}(w)$.

[11] See [10, 14] for an introduction to this approach.

In this setting, a formula like $K\phi$ states that $\phi$ is explicitly known, since $\phi$ is not only true at every world which is consistent with the agent's knowledge, but it is also a formula the agent is explicitly aware of. So, *explicit* knowledge is defined in terms of *implicit* knowledge and *awareness*.[12] The relationship between the logic of $K$ induced by the basic system of hybrid liberal semantics and the logic induced by a basic system of liberal semantics is the following:

1. every hybrid liberal model can be transformed into a liberal model satisfying the same formulas by defining $\mathcal{T}$ so that $\mathcal{T}(w) = \{\phi : R(w) \subseteq [\phi]^M\} \cap \mathcal{A}(w)$;
2. every liberal model can be transformed into a hybrid liberal model satisfying the same formulas by defining $R$ so that $R(w) = \{w\}$ for every $w \in W$ and $\mathcal{A}$ so that $\mathcal{A} = \mathcal{T}$.

The relationship between hybrid liberal semantics and liberal semantics in general is an open problem.

In conclusion, before moving on to relating semantics for epistemic logic, it is worth noting that liberal and neighborhood semantics are more likely viewed as *ways of representing* knowledge rather than *ways of understanding* knowledge. In fact, in such settings, knowledge has no specific characteristic: if we want to represent a property of knowledge we have to impose it explicitly on $\mathcal{N}$ or $\mathcal{T}$. This gives us a lot of freedom in terms of representation, but it gives us no idea about a principled way to identify a correct logic of knowledge. As we will see, this aspect is in part shared by the relating semantics we are going to consider.

### 3.2. Relating Epistemic logic

Let us now see whether the previous systems can be subsumed under relating semantics and whether this subsumption allows us to better understand the logic of knowledge.[13] I introduce the following relating models, where the relevant relation connects a formula with the agent's *knowledge base* at a world, i.e. the set of propositions which constitute

[12] The epistemic logic proposed in [10, ch. 9] includes an operator representing implicit knowledge, so that the definition of explicit knowledge in terms of implicit knowledge and awareness can be introduced in the language.

[13] For a first integration of relating relations into possible worlds semantics see [17, 19].

the agent's knowledge at $w$. Accordingly, $R_w(\phi)$ states that $\phi$ is related, in a sense to be specified, with the agent's knowledge base.

Definition 3.1. A model for $\mathcal{L}$ is a tuple $(W, R, V)$ where

$R \subseteq W \times Fm(\mathcal{L})$.

The definition of truth is such that

$M, w \models K\phi$ iff $M, w \models \phi$ and $R_w(\phi)$.

This definition is general enough to provide us with a comprehensive scheme to be further specialized with respect to the kind of semantics we want to capture. The idea behind such definition is straightforward: the agent knows that $\phi$ just in case $\phi$ is true and is related, in a sense to be specified, to her knowledge base.

Conditions on $R$ capturing *Relational Logic*:

if $[\phi]^M = W$, then $R_w(\phi)$;
if $R_w(\phi)$ and $R_w(\phi \to \psi)$, then $R_w(\psi)$.

Axiomatization:

axioms and rules for propositional logic;
if $\phi$ is derivable, then $K\phi$ is derivable;
$K(\phi \to \psi) \to (K\phi \to K\psi)$;
$K\phi \to \phi$.

These conditions assume that every validity is related to the knowledge base of the agent and that the consequent of any implication that is related to that knowledge base is also related to the knowledge base, provided its antecedent is so related. They can be exploited to formalize an idea of knowledge according to which the agent knows that $\phi$ just in case $\phi$ follows form his knowledge base. The relation involved is that of consequence: $R_w(\phi)$ states that $\phi$ follows from what the agent knows at $w$. Hence, the notion of knowledge here involved is the notion of implicit knowledge: the agent knows all the logical truths and all the known consequences of what he knows. Furthermore, since $[\phi]^M \subseteq [\psi]^M$ implies $R_w(\phi \to \psi)$, by the conditions on $R$, the agent also knows all the logical consequences of what he knows.

Conditions on $R$ capturing *Perspectival Logic*:

if $[\phi]^M = W$, then $R_w(\phi)$;
if $R_w(\phi)$ and $[\phi]^M \subseteq [\psi]^M$, then $R_w(\psi)$.

Axiomatization:

axioms and rules for propositional logic;
if $\phi$ is derivable, then $K\phi$ is derivable;
if $\phi \rightarrow \psi$ is derivable, then $K\phi \rightarrow K\psi$ is derivable;
$K\phi \rightarrow \phi$.

These conditions assume that every validity is related to the knowledge base of the agent and that the every proposition which follows from a proposition related to that knowledge base is also related to the knowledge base. They can be exploited to formalize an idea of knowledge according to which the agent knows that $\phi$ just in case $\phi$ follows from an element of the knowledge base. The relation involved is that of a sort of point-wise consequence: if $\phi$ is related to the knowledge base and $\psi$ is implied by $\phi$, then $\psi$ is also related to the base, but the agent is not allowed to aggregate bits of the knowledge base to derive further consequences. To be sure, it is possible for the agent not to know that $\psi$ even if $\psi$ is implied by known propositions. Hence, the notion of knowledge here involved is a limited version of implicit knowledge: the agent knows all the logical truths and all the point-wise consequences of what he knows. Still, it is not necessary for the agent to know all the known consequences of what she knows, due to the fact that different truths can be known from different perspectives.

Conditions on $R$ capturing *Neighborhood Logic*:

if $R_w(\phi)$ and $[\phi]^M = [\psi]^M$, then $R_w(\psi)$.

Axiomatization:

axioms and rules for propositional logic;
if $\phi \leftrightarrow \psi$ is derivable, then $K\phi \leftrightarrow K\psi$ is derivable;
$K\phi \rightarrow \phi$.

In this case every proposition which is intensionally equivalent to a proposition related to the agent's knowledge base is also related to that knowledge base. The relation involved is simply the relation of intensional equivalence: we abstract from the way in which a proposition represents a set of possible worlds and assume that every proposition representing a certain set of possible worlds is related to the knowledge base provided that one of such propositions is so related. The notion of knowledge here involved is a basic version of implicit knowledge: the agent knows all the propositions that are logically equivalent to what he knows. In this case, nothing is assumed about knowledge of either logical truths or logical consequences of what is known. The idea is just that the content of knowledge is independent of the way of presentation.

Conditions on $R$ capturing *Liberal Logic*:

no condition.

Axiomatization:

axioms and rules for propositional logic;
$K\phi \to \phi$.

The notion of knowledge here involved is the liberal one: no assumption is at work. When considered from the point of view of relating semantics, it is plain that a liberal logic is able to represent both rational and irrational epistemic agents. Indeed, since no condition on the propositions that are related to the knowledge base is assumed, there is no constraint on what the agent is supposed to know given his knowledge base.

**Soundness and completeness.** With respect to the previous systems, soundness is straightforward, while completeness is proved as follows. Let $M$ is the canonical model defined so that $R_w(\phi)$ iff $K\phi \in w$.

LEMMA 3.1 (Truth Lemma). *$M, w \models \phi$ iff $\phi \in w$.*

PROOF. $M, w \models K\phi$ iff $M, w \models \phi$ and $R_w(\phi)$, by the definition of truth, iff $\phi \in w$ and $K\phi \in w$, by the induction hypothesis and the definition of $R_w$, iff $K\phi \in w$, since $K\phi \to \phi \in w$. ⊣

LEMMA 3.2 (Canonicity Lemma). *$M$ is a model of $\mathcal{L}$.*

PROOF. Straightforward, since there is no condition on $M$. ⊣

Finally, the proof that the conditions on $R$ are satisfied in the canonical models for relational, perspectival, neighborhood, and liberal logic follows from the definition of $R$ and the axioms on $K$ that characterize the corresponding logics.

### 3.3. Discussion

Is there any advantage in adopting a relating semantics in studying the logical structure of knowledge? As far as I can see, there are two possible benefits in introducing this kind of semantics.

First, in a sense, the connection between algebraic and possible world semantics becomes more transparent. In fact, viewing $R_w(\phi)$ as the semantic content of $K\phi$, the conditions on the algebraic operations that correspond to the intensional operators can be obtained immediately by substituting algebraic operations for operations and relations on suitable sets. In more detail, we get the following correspondences.

Basic normal modal systems:

| | |
|---|---|
| if $a = 1$, then $\Box a = 1$ | if $[\phi]^M = W$, then $R_w(\phi)$ |
| if $\Box a \wedge \Box(a \to b) \leq \Box b$ | if $R_w(\phi)$ and $R_w(\phi \to \psi)$, then $R_w(\psi)$. |

Basic monotonic modal systems:

| | |
|---|---|
| if $a = 1$, then $\Box a = 1$ | if $[\phi]^M = W$, then $R_w(\phi)$ |
| if $\Box a$ and $a \leq b$, then $\Box b$ | if $R_w(\phi)$ and $[\phi]^M \subseteq [\psi]^M$, then $R_w(\psi)$. |

Basic classical modal systems:

| | |
|---|---|
| if $\Box a$ and $a = b$, then $\Box b$ | if $R_w(\phi)$ and $[\phi]^M = [\psi]^M$, then $R_w(\psi)$. |

Similarly, the connection between axioms on $K$ and conditions on the accessibility relation $R$ in relational semantics becomes more transparent. Let us see why. As said before $R_w(\phi)$ states that $\phi$ is related with the agent's knowledge base and that the precise sense of this relation is to be further specified. A natural suggestion is that $R_w(\phi)$ holds when $\phi$ follows from that knowledge base. This interpretation allows us to better understand the accessibility relation $R$ involved in relational epistemic models. Indeed, we are typically invited to think of $R$ as a possibility relation: it defines what worlds an agent considers possible in any particular world $w \in W$, where a world is possible in $w$ provided that it is consistent with the agent's knowledge base. Interestingly, we are now in a position to make this intuition precise. Since $R_w(\phi)$ holds when $\phi$ follows from the agent's knowledge base, we can assume that $v$ is considered possible in $w$ just in case no sentence which follows from the agent's knowledge base is false at $v$. Therefore, we can assume this

*Equivalence Thesis*: $v \in R(w)$ iff $\forall\phi\ (R_w(\phi) \Rightarrow M, v \models \phi)$.

On top of that, we are in a position to derive the most used conditions on $R$ directly from the corresponding axioms about $K$.

*Example* 3. Suppose we trust the $KK$-thesis and, accordingly, we characterize $K$ in terms of

Axiom 4: $K\phi \to KK\phi$.

The condition on $R$ matching this axiom in relating semantics is:

$R4$: for all $w$ and $\phi$, if $R_w(\phi)$, then $R_w(K\phi)$.

The corresponding condition in standard relational epistemic models can be derived as follows. If $v \in R(w)$ and $R_w(\phi)$, then $R_w(K\phi)$, by $R4$. So, $M, v \models K\phi$, by the equivalence thesis, and therefore $R_v(\phi)$. Hence, if $v \in R(w)$ and $R_w(\phi)$, then $R_v(\phi)$, which implies that all the worlds that are considered possible in $v$ are also considered possible in $w$. In conclusion, if $v \in R(w)$, then $R(v) \subseteq R(w)$, which amounts to the fact that $R$ is transitive.

A similar reasoning allows us to derive that $R$ is to be

1. reflexive, since our models are suitable;
2. symmetric, in case $\neg\phi \to K\neg K\phi$ is assumed;
3. Euclidean, in case $\neg K\phi \to K\neg K\phi$ is assumed;
4. convergent, in case $\neg K\neg K\phi \to K\neg K\neg\phi$ is assumed.

The second benefit stems form the fact that in devising a relating epistemic logic we are forced to identify and justify the assumptions we use to characterize the notion of knowledge. Thus, while in standard epistemic semantics we start figuring out a set of appropriate conditions on the accessibility relation and then we try to understand and justify the properties induced by such conditions on $K$, in relating epistemic semantics we are forced to start with conditions on $K$, and so to support these conditions based on an explicit epistemological theory. I think this is an advantage relative to standard epistemic semantics, since it seems to me that our judgments concerning how $K$ behaves are more secure than our judgments concerning how epistemically possible worlds are related. To be sure, I tend to be of the opinion that, for instance, it is not because we include the actual world in the set of worlds that are epistemically accessible that we assume that knowledge implies truth, but it is because we assume that knowledge implies truth that we include the actual world in the set of worlds that are epistemically accessible. Similarly, I think that it is not because we assume that the accessibility relation is transitive that we trust in the $KK$-thesis, if we trust in it, but it is because we trust in the $KK$-thesis that we assume that the accessibility relation is transitive.

As highlighted before, the aforementioned aspects can be regarded as an advantage only from the point of view of those who want to *represent* knowledge, not from the point of view of those who exploit semantic considerations in order to *understand* knowledge. In fact, the main disadvantage of this framework seems to be related to the limits in the explicative power of the resulting semantic system: we are able to extract from it only what we already put into it in terms of properties of $R$, so that it will be hardly the case that we discover something unexpected on knowledge based on such a system.

## 4. Explicit epistemic logic

Let us now move to the relational interpretation of explicit epistemic semantics. First of all, explicit epistemic logics are a class of logics aimed at modeling knowledge and other epistemic operators in light of the explicit sources supporting the epistemic states. Thus, formulas like $K\phi$, stating that the agent knows $\phi$, are replaced by formulas like $t : \phi$, stating that the agent knows $\phi$ on the basis of $t$, where $t$ is a suitable source of knowledge. In this context sources of knowledge can be analyzed both from an externalist and from an internalist perspective:[14] from an externalist point of view, $t : \phi$ is interpreted as stating that $t$ is sufficient for $\phi$ to be justified; from an internalist point of view, $t : \phi$ is interpreted as stating that $t$ is available to the agent as a justification of $\phi$, in case we assume that $t$ is a potential justification, or that $t$ is accepted by the agent as a justification of $\phi$, in case we assume that $t$ is an actual justification. Since justifications are key elements, the language $\mathcal{L}$ of such epistemic logics contains terms explicitly referring to them:

$$t := c \mid x \mid t \cdot s \mid t + s, \text{ where } c \in C(\mathcal{L}), x \in V(\mathcal{L});$$

$$\phi := p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid t : \phi, \text{ where } t \in Tm(\mathcal{L}).$$

Here, $C(\mathcal{L})$ and $V(\mathcal{L})$ are sets of constants and variables for various epistemic sources and $Tm(\mathcal{L})$ is the set of terms for justifications. Finally, $Fm(\mathcal{L})$ is the set of formulas of $\mathcal{L}$. A term can stand for a justification that results from composing other justifications. In particular, $t \cdot s$ results from $t$ and $s$ is such a way that it justifies all the propositions which follow by modus ponens from implications justified by $t$ with an antecedent justified by $s$, while $t + s$ results from $t$ and $s$ is such a way that it justifies all the propositions which are justified by either $t$ or $s$.

[14] See [12, chs. 5–6] and [22, ch. 5] for a basic introduction to such topics.

Definition 4.1. A model for $\mathcal{L}$ is a tuple $M = (W, R, \mathcal{E}, V)$ where

1. $R\colon W \to \wp(W)$, is such that $w \in R(w)$ for each $w \in W$;
2. $\mathcal{E}\colon W \times Tm(\mathcal{L}) \to \wp(Fm(\mathcal{L}))$.

The definition of truth is such that

$M, w \models t : \phi$ iff $R(w) \subseteq [\phi]^M$ and $\phi \in \mathcal{E}(w, t)$.

Intuitively, every pair constituted by a world $w \in W$ and a term $t \in Tm(\mathcal{L})$ is associated with a set $\mathcal{E}(w, t)$ of formulas representing the propositions the agent knows at $w$ based on $t$. Thus, the basic idea is that $\phi$ is known by virtue of $t$ provided it is one of the formulas that are justified by $t$ at $w$.[15]

**Problems and desiderata.** Logics of this kind are extremely interesting due to their expressive power, but are subject to some significant problems. On the one hand, it is difficult to capture the distinction between *actual* and *potential* justifications, i.e., justifications that are available to the agent because the agent possesses them and justifications that are available to the agent just because they exist and the agent is in a position to possess them.[16] On the other hand, in a general setting, the distinction between operations $\cdot$ and $+$ is not transparent.[17] As to the desiderata, given our intuitive judgments about the structure of justification, we would like to be able to say that operation $+$ is idempotent, commutative and associative, but these properties are not satisfied if their satisfaction is not explicitly imposed. As we will see shortly, the development of a relating semantics for this kind of systems provides a nice solution to these problems and a costless satisfaction of the desiderata.

### 4.1. Axiomatization

Let **EK** be the basic logic of explicit knowledge we want to discuss. This logic is axiomatically defined as follows:

1. axioms and rules for propositional logic;
2. $t\colon (\phi \to \psi) \to (s\colon \phi \to t \cdot s : \psi)$;

[15] See [1, 2, 3, 11] for an introduction to the logic of justification, which is one of the principal systems of explicit epistemic logic, and its possible worlds semantics.

[16] See [13] for a throughout discussion.

[17] This distinction is perfectly fine in a specific setting like the one provided by the original logic of proofs, since in that setting $\cdot$ and $+$ receive an intuitive interpretation in terms of different constructions of complex proofs.

3. $t : \phi \vee s : \phi \rightarrow t + s : \phi$;
4. $t : \phi \rightarrow \phi$:
5. $c_1 : ... : c_n : \phi$, for $c_1, ..., c_n \in C(\mathcal{L})$, provided $\phi$ is an axiom from 1–4.

The sole rule is modus ponens. Schemas 2, 3, and 4 are introduced to capture the intended interpretation of $\cdot$ and $+$ and to restrict the class of justifications in such a way that all the justifications we consider are justifications supporting knowledge, i.e., are sufficient for the justified proposition to be true. The final schema ensures that all axioms are justified.[18] The corresponding conditions on $\mathcal{E}$ are the following:

if $\phi \rightarrow \psi \in \mathcal{E}(w, t)$ and $\phi \in \mathcal{E}(w, s)$, then $\psi \in \mathcal{E}(w, t \cdot s)$
(matching axiom 2);

$\mathcal{E}(w, t) \cup \mathcal{E}(w, s) \subseteq \mathcal{E}(w, t + s)$
(matching axiom 3);

if $\phi$ is an axiom, then $c_1 : \ldots : c_n : \phi \in \mathcal{E}(w, c)$
(matching axiom 5).

It is then not difficult to see that the system is sound.

### 4.2. Completeness

The canonical model is defined so that

$W$ is the set of **EK**-complete sets of formulas;
$R$ is such that $v \in R(w)$ iff $\{\psi : \exists t(t : \psi \in w)\} \subseteq v$;
$\mathcal{E}$ is such that $\phi \in \mathcal{E}(w, t)$ iff $t : \phi \in w$;
$V$ is such that $v \in V(p)$ iff $p \in v$.

Completeness is then proved based on the following lemmas.

Lemma 4.1 (Truth Lemma). *$M, w \models t : \phi$ iff $R(w) \subseteq [\phi]^M$ and $\phi \in \mathcal{E}(w, t)$ iff $\phi \in \mathcal{E}(w, t)$ iff $t : \phi \in w$.*

Lemma 4.2 (Canonicity Lemma). *$M$ is a model of $\mathcal{L}$.*

Conditions on $\mathcal{E}$ follow from axioms 2, 3, 5 [see 1, 11].

[18] Here we work with an elementary system in which all the validities are justified by all the constants in $C(\mathcal{L})$; more general systems can be constructed by introducing a constant specification. See [3, ch. 2] for further details.

### 4.3. Interpretation in relating semantics

Systems of explicit epistemic logic can be interpreted from the point of view of relating semantics without difficulty. Such an interpretation also allows us to simplify the systems by avoiding the introduction of terms. Let us start by discussing the second point. We saw that a formula like $t : \phi$ is interpreted as stating that $t$ is a justification of $\phi$. We also saw that the relation of being a justification is intended in different ways from different perspectives: so, from an externalist point of view, $t : \phi$ is interpreted as stating that $t$ is sufficient for $\phi$ to be justified, whereas, from an internalist point of view, $t : \phi$ is interpreted as stating that $t$ is a reason, either available or potentially accessible, which provides a sufficient justification of $\phi$. In both cases, $t$ can be viewed as a condition that is sufficient for $\phi$ to be justified: a certain fact, for the externalist, or the fact that a certain reason is available to the agent, for the internalist. Therefore, we are allowed to read $t : \phi$ as stating that the fact described by $t$ provides a reason for $\phi$.[19] This leads us to the assumption that a term like $t$ refers to a fact, and therefore that $t$ can be interpreted as expressing a proposition, thus abolishing the distinction between terms and formulas.[20]

Since we are using constants as terms referring to justifications for logical axioms, all the constants can be interpreted as representing a unique fact, that is the fact that justifies logical validities, which we identify with the logical truth $\top$.[21] As a first result, the set $Fm(\mathcal{L})$ of formulas of the language $\mathcal{L}$ of explicit logic can be defined according to this grammar:

$$\phi := p \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \tau : \phi.$$

[19] The idea that reasons are fact-like entities is discussed in [6, 15, 23].

[20] Alternatively, we could maintain the view that a term like $t$ refers to a bit of evidence, and focus on the internalist idea that a bit of evidence becomes a justification of a proposition only when it is acknowledged as such by the agent. In this case, a formula like $t : \phi$ should be interpreted as stating that the fact that $t$ is possessed by the agent constitutes a justification of $\phi$. In turn this idea could be formalized by means of a formula like $\tau_t : \phi$, where $\tau_t$ refers to the fact that the agent possesses $t$. In what follows, for the sake of simplicity, I will stick to the view that terms refers to facts, but all I say can be translated without difficulty into the view just sketched.

[21] This is the simplest assumption we can make in reference to the facts justifying logical principles. It goes without saying that more fine-grained solutions can be adopted in representing reasons for logical validities, but since we are not focusing on this issue, the solution proposed here is sufficient for the present purposes.

Here, $\tau$ is simply a formula and $\tau : \phi$ is to be read as stating that the fact described by $\tau$ is a justification of $\phi$. This change in the language induces a change in the definition of a model. We saw that the main element of a model for a system of explicit epistemic logic is $\mathcal{E}$, i.e., the function which assigns to any pair constituted by a world $w \in W$ and a term $t \in Tm(\mathcal{L})$ the set of formulas representing the propositions that are justified by $t$ at $w$. In the present framework, such function is smoothly replaced by a relation between formulas parameterized by worlds. Hence, a model is a tuple $M = (W, R, R, V)$, with $R\colon W \to \wp(Fm(\mathcal{L}) \times Fm(\mathcal{L}))$, and $R_w(\phi, \tau)$ is assumed to hold precisely when the fact described by $\tau$ counts as a justification of $\phi$ at $w$.

The definition of truth is such that

$$M, w \models \tau : \phi \text{ iff } R(w) \subseteq [\phi]^M \text{ and } R_w(\phi, \tau).$$

Thus, we assume that $\phi$ is known by virtue of $\tau$ provided it is one of the formulas that are justified by the fact described by $\tau$ at $w$. Here, the parameterization with respect to worlds captures the idea that a fact that counts as a justification of $\phi$ at $w$ may not count as a justification of $\phi$ in a different context: e.g., the fact that the agent thinks he sees a cat on a tree may not count as a justification of the proposition that there is a cat on a tree in a context where the agent is in the dark. In view of the changes highlighted before, we modify the axiomatic system as follows.

1. axioms and rules for propositional logic;
2. $\tau : (\phi \to \psi) \to (\sigma : \phi \to \tau \wedge \sigma : \psi)$;
3. $\tau : \phi \vee \sigma : \phi \to (\tau \wedge \sigma) : \phi$;
4. $\tau : \phi \to \phi$;
5. $\top : \phi$, provided $\phi$ is an axiom from 1–4.

The corresponding conditions on $R_w$ are:

if $R_w(\phi \to \psi, \tau)$ and $R_w(\phi, \sigma)$, then $R_w(\psi, \tau \wedge \sigma)$
(matching axiom 2);

if $R_w(\phi, \tau)$ or $R_w(\phi, \sigma)$, then $R_w(\phi, \tau \wedge \sigma)$
(matching axiom 3);

if $\phi$ is an axiom, then $R_w(\phi, \top)$
(matching axiom 5).

It is not difficult to see that the system is sound and complete with respect to its intended semantics. In fact, proving soundness is straightforward, while the proof of completeness is based on the construction of the following canonical model:

$W$ is the set of **EK**-complete sets of formulas;
$R$ is such that $v \in R(w)$ iff $\{\psi : \exists\tau(\tau : \psi \in w)\} \subseteq v$;
$R$ is such that $R_w(\phi, \tau)$ iff $\tau : \phi \in w$;
$V$ is such that $v \in V(p)$ iff $p \in v$.

Completeness is then proved as usual.

Lemma 4.3 (Truth Lemma). *$M, w \models \tau : \phi$ iff $R(w) \subseteq [\phi]^M$ and $R_w(\phi, \tau)$ iff $R_w(\phi, \tau)$ iff $\tau : \phi \in w$.*

Lemma 4.4 (Canonicity Lemma). *$M$ is a model of $\mathcal{L}$.*

Conditions on $R$ follow from axioms 2, 3, 5.

**Problems and desiderata.** The distinction between potential and actual justification is now definable, provided we work in an enriched modal framework where an operator is available representing the fact that the agent has an access to justifications, that is the fact that he possess a justification of $\phi$. Let $E$ be such an operator.[22] A proposition like $E(\tau)$ states that the agent has access to the fact that $\tau$, while a proposition like $\Diamond E(\tau)$ states that she has a potential access to the fact that $\tau$. The notions of potential and actual justification can be now defined and contrasted. In more detail, we are able to distinguish several important conditions.

(i) $\tau : \phi$ ($\tau$ is an objective justification of $\phi$; this notion is neutral with respect to the accessibility of $\tau$ and can be adopted to describe an externalist perspective on justification);

(ii) $E(\tau) \wedge \tau : \phi$ ($\tau$ is an actual justification of $\phi$; this notion encodes the fact that the agent possesses a justification and can be adopted to describe an internalist perspective on justification);

(iii) $\Diamond E(\tau) \wedge \tau : \phi$ ($\tau$ is a potential justification of $\phi$; this notion encodes the fact that the agent is in a position to possess a justification and can be adopted to describe a wider internalist perspective on justification);

[22] The logic of $E$ can be formulated in terms of a simple liberal semantics.

(iv) $E(\tau) : \phi$ ($E(\tau)$ is an actual justification of $\phi$; this notion encodes a more specific internalist intuition, according to which it is precisely the fact that the agent has an access to $\tau$ that constitutes a justification of $\phi$).

The distinction between $\cdot$ and $+$ is eliminated: both operations are rendered in terms of a conjunction. This is desirable at the level of abstraction we are considering here. Furthermore, assuming that reasons are not affected by the way their are presented, at least at a propositional level, the fact that operation $+$ is idempotent, commutative and associative is derivable. Indeed, the assumption that reasons are not affected by the way their are presented is captured by a rule like:

*Equivalence*: if $\vdash \tau \leftrightarrow \sigma$, then $\vdash \tau : \phi \leftrightarrow \sigma : \phi$.

whose semantic counterpart is a condition ensuring that $R_w(\phi, \tau)$ is closed under substitution of equivalents in the second argument.

This rule, together with the standard properties of conjunction, suffices for obtaining the features ascribed to $+$.

Furthermore, having construed epistemic reasons as facts allows us to examine some intuitive developments of the basic system proposed above. As a first idea, we can consider particular classes of reasons, e.g. the class of reasons that satisfy monotonicity. I have in mind cases where a proposition about a physical system is justified based on a mathematical model of that system. In these cases, if two models $m_\tau$ and $m_\sigma$ are such that $m_\tau$ is more accurate than $m_\sigma$, then $m_\tau$ enables us to justify more propositions than $m_\sigma$, provided that $m_\tau$ is compatible with $m_\sigma$. So, if $\tau$ is the proposition stating that $m_\tau$ is at our disposal and $\sigma$ is the proposition stating that $m_\sigma$ is at our disposal, we get that $\tau \to \sigma$ and that all propositions justified based on $m_\sigma$ are also justified based on $m_\tau$. Such a class is defined without difficulty by a rule like: if $\vdash \tau \to \sigma$, then $\vdash \sigma : \phi \to \tau : \phi$, which generalizes *Equivalence*.

### 4.4. Developments

Before closing this section, let me hint a second interesting development of justification logic within relating semantics. As in the standard setting, $C(\mathcal{L})$ and $V(\mathcal{L})$ are sets of constants and variables for epistemic sources, $Fm(\mathcal{L})$ is the set of formulas of $\mathcal{L}$, and the set $Tm(\mathcal{L})$ of terms is obtained by combining constants and variables by means of $\cdot$ and $+$.

But this time, terms are thought of as referring to justifications involving both propositions and conditions. A basic epistemic formula of the form $t : (\alpha, \phi)$ is now construed as stating that $t$ justifies $\phi$ provided that $\alpha$ is the case. Hence, $\alpha$ describes a condition that must be satisfied in order for $t$ to justify $\phi$. For instance, if $t$ is a bit of evidence for $\phi$ obtained by visual perception, $\alpha$ can refer to the condition that the relevant visual system is working correctly or, if $t$ is a bit of evidence for $\phi$ obtained by constructing a proof of $\phi$, $\alpha$ can refer to the condition that the proof is actually constructed in a correct way.

Definition 4.2. A model for $\mathcal{L}$ is a tuple $M = (W, R, \mathsf{R}, V)$, where

1. $R\colon W \to \wp(W)$, is such that $w \in R(w)$ for each $w \in W$;
2. $\mathsf{R}\colon W \times Tm(\mathcal{L}) \to \wp(Fm(\mathcal{L}) \times Fm(\mathcal{L}))$.

The definition of truth is such that

$M, w \models t : (\alpha, \phi)$ iff $R(w) \cap [\alpha]^M \subseteq [\phi]^M$ and $\mathsf{R}^t_w(\alpha, \phi)$.

Intuitively, every pair constituted by a world $w \in W$ and a term $t \in Tm(\mathcal{L})$ is associated with a relation $\mathsf{R}^t_w$ such that $\mathsf{R}^t_w(\alpha, \phi)$ holds just in case $\alpha$ stands for the condition to be satisfied and $\phi$ stands for the proposition the agent knows at $w$ based on $t$ under the assumption that such condition is indeed satisfied. The logic of this kind of knowledge, call it **CEK**, for conditional explicit knowledge, can be axiomatically defined as follows:

1. axioms and rules for propositional logic;
2. $t : (\alpha, \phi \to \psi) \to (s : (\beta, \phi) \to t \cdot s : (\alpha \wedge \beta, \psi))$;
3. $t : (\alpha, \phi) \vee s : (\beta, \phi) \to t + s : (\alpha \vee \beta, \phi)$;
4. $t : (\alpha, \phi) \wedge \alpha \to \phi$;
5. $c : (\top, \phi)$, for $c \in C(\mathcal{L})$, provided $\phi$ is an axiom from 1–4.

The sole rule is modus ponens. The corresponding conditions on $\mathcal{E}$ are the following:

if $\mathsf{R}^t_w(\alpha, \phi \to \psi)$ and $\mathsf{R}^s_w(\beta, \phi)$, then $\mathsf{R}^{t \cdot s}_w(\alpha \wedge \beta, \psi)$
(matching axiom 2);

if $\mathsf{R}^t_w(\alpha, \phi)$ or $\mathsf{R}^s_w(\beta, \phi)$, then $\mathsf{R}^{t+s}_w(\alpha \vee \beta, \phi)$
(matching axiom 3);

if $\phi$ is an axiom, then $\mathsf{R}^c_w(\top, \phi)$
(matching axiom 5).

It is not difficult to see that the system is sound. As to completeness, consider the following canonical model:

$W$ is the set of **CEK**-complete sets of formulas;
$R$ is such that $v \in R(w)$ iff $\{\alpha \to \phi : t : (\alpha, \phi) \in w\} \subseteq v$;
$R$ is such that$R^t_w(\alpha, \phi)$ iff $t : (\alpha, \phi) \in w$;
$V$ is such that $v \in V(p)$ iff $p \in v$.

The Truth Lemma is proved as follows.

LEMMA 4.5. *Truth lemma.*

$M, w \models t : (\alpha, \phi)$ *iff*
$R(w) \cap [\alpha]^M \subseteq [\phi]^M$ *and* $R^t_w(\alpha, \phi)$ *iff* $R^t_w(\alpha, \phi)$ *iff* $t : (\alpha, \phi) \in w$.

As to the last equivalence, note that

if $t : (\alpha, \phi) \in w$, then $R(w) \cap [\alpha]^M \subseteq [\phi]^M$.

Indeed, suppose $t : (\alpha, \phi) \in w$ and $v \in R(w) \cap [\alpha]^M$. Then, by the induction hypothesis, $\alpha \in v$ and, by the definition of $R$, $\alpha \to \phi \in v$. Therefore, $\phi \in v$, and we get the conclusion.

LEMMA 4.6 (Canonicity lemma). *$M$ is a model of $\mathcal{L}$.*

Conditions on $R$ follow from axioms 2, 3, 5.

The opportunity of applying the present framework to problems in epistemology seems to be promising. In particular, we can use a formula like $t : (\alpha, \phi)$ to model the idea that a justification of $\phi$ is required to satisfy a certain set of standards, as in the tradition of epistemic contextualism, or the idea that the application of a given procedure yields a justification of $\phi$ only if that procedure is correctly carried out. This line of research is left for further work.

## 5. Conclusion

We saw that relating semantics provides us with an intuitive framework for studying epistemic logic in general and explicit epistemic logic in particular. The relations used in the representation of knowledge are extremely flexible and enable us to deal with epistemic operators from both an intensional and a hyperintensional point of view. Furthermore, the semantics gives us a lot of suggestions both on how to solve some problems related to explicit epistemic logic and on how to further develop systems in the light of new ideas.

In summary, relating semantics is an intriguing tool. On the one hand, the main advantage in using such framework is constituted by the fact that the properties of the epistemic operators, as identified in different epistemological theories, can be incorporated in a semantics in a very natural way, thus allowing us to derive the logic of those operators without difficulty. On the other hand, the main challenge to be addressed concerns how to exploit this framework so as to get not only a precise characterization of the operators we want to analyze, but also a basis for understanding why such operators are to be characterized in the way they are.

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