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Revisiting the Conditional Construal of Conditional Probability

Abstract. We show how to extend any finite probability space into another finite one which satisfies the conditional construal of conditional probability for the original propositions, given some maximal allowed degree of nesting of the conditional. This mitigates the force of the well-known triviality results.

Keywords: conditional probability; probabilities of conditionals; Lewis' triviality results

1. Introduction

The *conditional construal of conditional probability*, that is, the CCCP, is — roughly — the idea that the probability P of the indicative conditional $A \to B$ is equal to $P(B|A)$. The idea, still intuitive for many,[1] met with a series of "triviality results", which show the limits of the idea in its various rigorous formulations [see, e.g., Hájek, 2015; Hájek & Hall, 1994; Hall, 1994; Lewis, 1976].

Suppose you have a field of propositions $\mathcal{F}$ on which a probability function P is defined, forming a probability space. Assume you would like the P to be defined on conditionals $A \to B$, for $A, B \in \mathcal{F}$, so that the CCCP is satisfied. Are these conditionals already in your space?

[1] For example, [Dorst, 2020, p. 595] refers to "the fact that natural language expresses conditional probabilities as probabilities of (indicative) conditionals". This is not an isolated incident; however, since this is a technical note, we have decided not to report any more sources of similar sentiments.

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Sometimes it is evident they are *not*, even before we start asking more difficult questions about the nesting of $\rightarrow$ and whatever conditions we might reasonably expect to be satisfied by it (e.g., should $A \rightarrow A$ equal $\mathbf{1}_{\mathcal{F}}$?). Consider, for example, the space $(W = \{a, b, c\}, \mathcal{P}(W), P)$ with P uniform. In such a case $P(\{a\}|\{a, b\}) = 0.5$; however, there is no proposition in the domain of P with that probability, so the CCCP cannot be satisfied (i.e., the space contains no candidate for the proposition $\{a, b\} \rightarrow \{a\}$).

It follows from the results of [Hall, 1994; van Fraassen, 1976] that the only probability spaces[2] for which the CCCP is satisfied (and in which $\rightarrow$ meets some sensible minimal conditions) are *full*; in other words, they are *atomless*, that is, for any proposition A such that $P(A) > 0$, for any $0 < x < P(A)$, there is a B such that $B \subseteq A$ and $P(B) = x$. From some perspective, these are *huge* spaces. Suppose you are only interested in finitely many propositions. It might be surprising, and unwanted, if your goal is "just" to contemplate conditionals constructed from those propositions and their Boolean combinations, to be forced to use an uncountably infinite probability space. We will show here that, for finite probability spaces, if you specify the maximal degree of nesting of $\rightarrow$, it is always possible to extend your space to a *finite* one, so that the CCCP holds for the conditionals constructed from the propositions from the original space, up to the specified nesting degree.

We shall make all this precise now.

2. The definitions

Definition 2.1 (Extension of a probability space). A probability space $(W', \mathcal{F}', P')$ is an *extension* of the probability space $(W, \mathcal{F}, P)$ *by means of* h if h is a Boolean algebra embedding of $\mathcal{F}$ into $\mathcal{F}'$ such that for any $A \in \mathcal{F}$, $P'(h(A)) = P(A)$.

That is, an extension of a probability space is a "new" space which preserves all the "old" probabilities, possibly involving more propositions. In the context of probabilities of conditionals, extensions have been used, e.g., by van Fraassen [1976]. More generally, in formal philosophy it is quite natural to use extensions when investigating the capturing of the same phenomena in various contexts which differ in how

[2] Of cardinality higher than 2.

many other aspects of the relevant situations are taken into account. One may in such cases speak of "the same" propositions even if the various objects in question belong to the event algebras of different spaces; the crucial thing is the measure-preserving nature of the embedding which defines the extension. For example, given a probability space, it might be asked whether all pairs of propositions possess a "statistical common cause", while the affirmative answer might be given by a procedure for producing a desired extension, in which such a "common cause" exists for each pair of the *images* of the original propositions [Marczyk & Wroński, 2014]. If the elements of W are taken to be epistemically possible worlds, and if P is some agent's credence function, one can contemplate extensions in which W' consists of suitably "fine-grained" worlds from W, corresponding to the fact that the agent started taking more factors into account.

Following [Hall, 1994], we will use the term "model" to refer to a probability space enjoined with an interpretation of $\to$:

Definition 2.2 (Model). A quadruple $(W, \mathcal{F}, P, \to)$ is a *model* iff $(W, \mathcal{F}, P)$ is a probability space and $\to$ is a total function from $\mathcal{F} \times \mathcal{F}$ into $\mathcal{F}$.

The models we will be interested in are those in which the probabilities of conditionals equal the corresponding conditional probabilities, and the $\to$ satisfies the conditions of the "minimal logic of conditionals" [van Fraassen, 1976, p. 277].

Definition 2.3 (Satisfying the CCCP). A model $(W, \mathcal{F}, P, \to)$ *satisfies the CCCP with regard to Prop* $\subseteq \mathcal{F}$ iff for any $A, B, C \in \mathit{Prop}$:

(∗) $P(A \to B) = P(B|A)$;
(I) $(A \to A) = W$;
(II) $A \cap (A \to B) = A \cap B$;
(III) $(A \to C) \cap (A \to B) = A \to (C \cap B)$;
(IV) $(A \to C) \cup (A \to B) = A \to (C \cup B)$.

3. The result

Here is the main result of this note:

Theorem 3.1. *Suppose a finite probability space* $(W, \mathcal{F}, P)$ *is given. There exists a finite model* $(W', \mathcal{F}', P', \to)$ *such that*:

- $(W', \mathcal{F}', P')$ *is an extension of* $(W, \mathcal{F}, P)$ *by means of some Boolean algebra embedding* h;
- $(W', \mathcal{F}', P', \rightarrow)$ *satisfies the CCCP with regard to the image of* h.

Before we embark on the proof, a few introductory remarks. We will be extending a probability space with $W = \{w_1, \ldots, w_n\}$ and measure P on $\mathcal{F}$ to a new one with $W' = \{w_{1,1}, w_{1,2}, \ldots, w_{1,k}, \ldots, w_{n,k}\}$. That is, each of the original w_i's will effectively be "split" into k elements; the extension homomorphism will take each w_i into the set $\{w_{i,1}, \ldots, w_{i,k}\}$. The k number is constant for all w_i's; we need to make sure it is big enough that all the required conditional probabilities can be made to "fit inside" the new space. We calculate the k as follows: first, label each $P(\{w_i\})$ in some unique way; second, define $\mathcal{F}_n$ as the set of all subsets of W of cardinality at least 2 which do not include w_n; third, fully expand the product $\prod_{S \in \mathcal{F}_n} P(S)$ so you end up with a sum of values of expressions involving your previously introduced labels; fourth, set k to be the number of such expressions occurring in that sum (even though in some cases the values of these expressions might be identical).

For example, set $n = 4$, $W = \{w_1, w_2, w_3, w_4\}$ and $P(\{w_1\}) = a$, $P(\{w_2\}) = b$, $P(\{w_3\}) = c$, $P(\{w_4\}) = d$. Then we expand the product $\prod_{S \in \mathcal{F}_4} P(S)$ as $(a+b+c)(a+b)(a+c)(b+c) = a^3b + a^3c + a^2b^2 + a^2b^2 + a^2bc + a^2bc + a^2bc + a^2bc + a^2c^2 + a^2c^2 + ab^3 + ab^2c + ab^2c + ab^2c + ab^2c + abc^2 + abc^2 + abc^2 + abc^2 + ac^3 + b^3c + b^2c^2 + b^2c^2 + bc^3$, therefore $k = 24$. And so our extension will include a W' consisting of 96 elements.

Were we only interested in the cardinality of the extension, we could just give the appropriate formula. However, in the proof below we will actually refer to the expressions inside the sums $\prod_{S \in \mathcal{F}_i} P(S)$ for various i's. Given some canonical labelling of the probabilities of singleton propositions, and the above definition of $\mathcal{F}_i$, we hereby define $\mathbf{Sum}(i)$ as the sum being the expansion of $\prod_{S \in \mathcal{F}_i} P(S)$. Assume the expansion proceeds via the same algorithm for each i (below we will use expressions like "the jth element of $\mathbf{Sum}(i)$"). Continuing the previous example, "a^2c^2" is an element of $\mathbf{Sum}(4)$, as we have seen, but it is *not* an element of e.g. $\mathbf{Sum}(1)$. If $a = 0.75$, $b = c = 0.1$, and $d = 0.05$, then 0.000025 is the value of some expression from $\mathbf{Sum}(1)$ (of c^2d^2, for example)[3], but it is not the value of any expression from $\mathbf{Sum}(4)$.

We will sometimes *mention* $\mathbf{Sum}(i)$ below (when referring, e.g., to some expression in $\mathbf{Sum}(i)$), but also sometimes *use* it: if we divide by

[3] We will omit quotes from now on.

Sum(i), we of course divide by the value of the sum. Context, as usual, will disambiguate.

The proof below assumes that the $\mathcal{F}$ in Theorem 3.1 is the full powerset of the W; the supposition is harmless, since the spaces in question are finite, and is made in the interest of set-theoretical clarity.

PROOF OF THEOREM 3.1. Let $\Omega = (W, \mathcal{P}(W), P)$, where $W = \{w_1, w_2, \ldots, w_n\}$ (that is, $n > 2$: in smaller spaces the issue of the CCCP is trivial) and $P(\{w_i\}) = x_i$. Let $\mathcal{F}_i = \{S \subset W : w_i \notin S \wedge \#S \geq 2\}$; of course $\#\mathcal{F}_i = \text{const.}$ Let $\Omega' = (W', \mathcal{P}(W'), P')$ and $W' = \{w_{1,1}, w_{1,2}, \ldots, w_{1,k}, \ldots, w_{n,k}\}$, where k is the number of the elements of the sum **Sum**(i) defined as the expansion of the product $\prod_{S \in \mathcal{F}_i} P(S)$, using the labelling of the probabilities $P(\{w_i\})$, for various i, by x_i.

For $x_{i,j}$ being the j-th element of **Sum**(i), set $P'(\{w_{i,j}\}) := \frac{x_i \cdot x_{i,j}}{\mathbf{Sum}(i)}$. Define $A_i := \{w_{i,j} : 1 \leq j \leq k\}$. Notice that

$$P'(A_i) = \sum_{j=1}^{k} P'(\{w_{i,j}\}) = \frac{x_i \cdot \sum_{j=1}^{k} x_{i,j}}{\mathbf{Sum}(i)} = x_i = P(\{w_i\}).$$

Therefore, if we define the homomorphism h by setting $h(\{w_i\}) := A_i$ (with all the other cases handled recursively), we see that Ω' is an extension of Ω by means of h. We now need to add to it the function $\rightarrow$ so that it becomes a model satisfying the CCCP with regard to the image of h.

For any $X \subseteq W$, let I_X be its set of indices, i.e., $I_X := \{i : w_i \in X\}$. Of course, both $I_{X \cup Y} = I_X \cup I_Y$ and $I_{X \cap Y} = I_X \cap I_Y$ hold.

The reasoning to come will make use of the fact that each A_j can be partitioned into sets $S_j^{i,A} \subseteq A_j$ such that for $i \in I_A, j \notin I_A$ we have:

$$P'(S_j^{i,A}) = \frac{x_i \cdot x_j}{P(A)}.$$

Let us argue that such sets $S_j^{i,A}$ exist. Here goes: since $j \notin I_A$, then $P(A)$ is a value of some expression in **Sum**(j). Therefore **Sum**$(j) = P(A) \cdot c = c \cdot (\sum_{k \in I_A} x_k)$, but since $i \in I_A$, then $\sum_{k \in I_A} x_k = x_i + s$, and so **Sum**$(j) = cx_i + cs$. Let us, then, put the following definition:

$$S_j^{i,A} := \{w_{j,y} : \sum P'(\{w_{j,y}\}) = cx_i\}.$$

That such $w_{j,y}$'s exist follows from the definition of P'. Note, then, that

$$P'(S_j^{i,A}) = \sum_{x \in S_j^{i,A}} P'(\{x\}) = \frac{x_j \cdot cx_i}{\mathbf{Sum}(j)} = \frac{x_j \cdot cx_i}{c \cdot P(A)} = \frac{x_i \cdot x_j}{P(A)}$$

as required.

It is easy to see that

$$\bigcup_{i\in I_A} S_j^{i,A} = A_j.$$

We now have everything in place to define $\to$. Let us, then, for any $A, B \in \mathcal{P}(W)$, put

$$h(A) \to h(B) := \bigcup_{i\in I_A\cap I_B}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}),$$

defining $\to$ in some arbitrary way for inputs outside the image of h (recall that the definition of a model requires $\to$ to be a total function from $\mathcal{P}(W')$).

Notice that $(*)$ from Definition 2.3 holds for the h-images of the original propositions:

$$\begin{aligned}
P'(h(A) \to h(B)) &= \sum_{i\in I_A\cap I_B}\Big(P'(A_i) + \sum_{j\notin I_A} P'(S_j^{i,A})\Big) = \\
&= \sum_{i\in I_A\cap I_B} x_i + \sum_{i\in I_A\cap I_B}\sum_{j\notin I_A} \tfrac{x_j x_i}{\sum_{k\in I_A} x_k} = \\
&= \sum_{i\in I_A\cap I_B} x_i + \sum_{i\in I_A\cap I_B} \tfrac{(1-\sum_{k\in I_A} x_k)\cdot x_i}{\sum_{k\in I_A} x_k} = \\
&= \frac{\sum_{i\in I_A\cap I_B} x_i}{\sum_{k\in I_A} x_k} = \tfrac{P'(h(A)\cap h(B))}{P'(h(A))} = \\
&= P'(h(B) \mid h(A)) = P(B \mid A).
\end{aligned}$$

What remains now is to show that conditions (I)–(IV) of Definition 2.3 are satisfied in the same domain.

(I): $h(A) \to h(A) =$

$$\begin{aligned}
&= \bigcup_{i\in I_A}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}) = \bigcup_{i\in I_A} A_i \cup \bigcup_{i\in I_A}\bigcup_{j\notin I_A} S_j^{i,A} = \\
&= \bigcup_{i\in I_A} A_i \cup \bigcup_{j\notin I_A}\bigcup_{i\in I_A} S_j^{i,A} = \bigcup_{i\in I_A} A_i \cup \bigcup_{j\notin I_A} A_j = W'.
\end{aligned}$$

(II): $h(A) \cap (h(A) \to h(B)) =$

$$\begin{aligned}
&= \bigcup_{i\in I_A} A_i \cap \bigcup_{i\in I_A\cap I_B}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}) = \\
&= \bigcup_{i\in I_A\cap I_B} A_i = \bigcup_{i\in I_A} A_i \cap \bigcup_{i\in I_B} A_i = h(A) \cap h(B).
\end{aligned}$$

(III): $(h(A) \to h(C)) \cap (h(A) \to h(B)) =$

$$\begin{aligned}
&= \bigcup_{i\in I_A\cap I_C}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}) \cap \bigcup_{i\in I_A\cap I_B}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}) = \\
&= \bigcup_{i\in I_A\cap I_B\cap I_C}(A_i \cup \bigcup_{j\notin I_A} S_j^{i,A}) = (h(A) \to (h(C) \cap h(B))).
\end{aligned}$$

$$\begin{aligned}\text{(IV)}:\quad & (h(A) \to h(C)) \cup (h(A) \to h(B)) = \\ &= \bigcup_{i \in I_A \cap I_C} (A_i \cup \bigcup_{j \notin I_A} S_j^{i,A}) \cup \bigcup_{i \in I_A \cap I_B} (A_i \cup \bigcup_{j \notin I_A} S_j^{i,A}) = \\ &\quad = \bigcup_{i \in I_A \cap (I_B \cup I_C)} (A_i \cup \bigcup_{j \notin I_A} S_j^{i,A}) = (h(A) \to (h(C) \cup h(B))).\end{aligned}$$

Thus we were able to construct the required extension. ⊣

Due to the fact that the composition of two embeddings is an embedding, iterating the construction used in the proof allows us to achieve an arbitrary level of nesting of $\to$ for the (images of the) original propositions and still satisfy the CCCP. In the parlance of [Hájek & Hall, 1994], this is an argument for a weakened variant of the "Universal tailoring version" of the "hypothesis" underlying the CCCP: "for each P there is some $\to$ such that the CCCP holds" [Hájek & Hall, 1994, p. 76]. Once we specify the maximal degree of nesting of $\to$, then indeed, there *is*; it's just that we may need to move to a different — larger, but still finite — space with a measure P', which agrees with P on all the propositions on which P is defined.

It might be observed that satisfying the CCCP in the sense used throughout this note is not straightforwardly preserved by conditionalisation. Suppose P_E is the measure with the same domain as P, defined in the usual way, that is — under the assumption that $P(E) > 0$ — as $P_E(\cdot) := P(\cdot|E)$. Suppose a finite model $(W, \mathcal{F}, P, \to)$ is given which satisfies the CCCP with regard to some *Prop* $\subseteq \mathcal{F}$. Then, typically — that is, for most choices of E such that $P(E) > 0$, and for most choices of *Prop* — the model $(W, \mathcal{F}, P_E, \to)$ will *not* satisfy the CCCP with regard to *Prop*. However, it is simply a matter of running the above construction again to achieve the needed model. The belief update of conditionalisation, then, if one wants to preserve satisfying the CCCP with regard to some class of propositions, becomes a two-step operation: first, we change the measure; second, we extend the probability space.

A loose end: the reader will surely have noticed that while we have fine-tuned $\to$ to behave properly with respect to the h-images of the propositions from the original space, we have allowed its operation on "new" propositions to be arbitrary, also in the case of pairs where only one element is "new" (i.e., which does not belong to the image of h). It remains to be seen whether (while keeping the space finite) $\to$ can be made to function in a way resembling the conditions of minimal logic in such cases, too; and if so, then to what extent.

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