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Defusing a Paradox to a Hypodox

Abstract. One way of resolving a paradox is to defuse it to a hypodox. This way is relatively unknown though. The goal of this paper is to explain this way with varied examples. The hypodoxes are themselves a broad class: both the Truth-teller and the 21st birthday of someone born on 29th February can be construed as hypodoxes. The most familiar kind of relation between paradoxes and hypodoxes is exemplified by the relation between the Liar and the Truth-teller. This article concerns a second kind where a paradox is defused to a hypodox by restricting or rejecting some granted principles. The Liar paradox has this second kind of relation to a Liar hypodox, which will be introduced. In some cases, defusing a paradox to a hypodox is only a partial resolution, as the hypodox itself may then need resolving. Even so, such a partial resolution decomposes a complex problem into more easily understood problems. Moreover, I compare the result of defusing a paradox to a hypodox with the results of resolving paradoxes in other ways. I give four examples. The first is mainly pedagogic, concerning a birthday. The second is a lightweight legal case, presenting a parking voucher paradox. The third is a formal system in which a Liar and Liar-like sentences are hypodoxical. The fourth is a philosophical critique of ways of solving Bertrand's chord paradox.

Keywords: Hypodox; Paradox; Bertrand's Chord Paradox; Liar Paradox; Liar Hypodox; Curry's Paradox; Yablo's Paradox, Epimenides Paradox, Truth-teller; Parking Voucher Paradox; Paradoxical dilemma; Dilemma over a hypodox; Ship of Theseus; Water-and-wine Problem

1. Introduction

One way of resolving a paradox is to defuse it to a hypodox. This way is relatively unknown though, and consequently is overlooked where it could be useful. In this article, I explain this way, while providing varied

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examples. Moreover, I compare the result of resolving a paradox in this way with the results of resolving it in other ways. Where a paradox *can* be defused to a hypodox, I will talk about *defusion* as a relation from that paradox to that hypodox, even where that relation extends across systems. Just so, a legal antinomy in one legal system might relate to a legal hypodox in another. In some cases, such as the Liar paradox or Bertrand's chord paradox, defusing a paradox to a hypodox is only a partial resolution, as the resultant hypodox itself then needs resolving. Nevertheless, this intermediary step towards a full solution is a way of dividing a paradox into more easily understood problems.

A Truth-teller is a self-referential sentence such as 'This sentence itself is true' that says of itself that it is true. But is it? Construed as a hypodox, it is true xor not true, yet there is a lack of a granted principle determining whether it is true. (Let 'xor' extend English with an explicitly exclusive disjunction.) The semantic value of the Truth-teller is underdetermined. Indeed, there are a number of hypodoxes of self-reference. An 'autological' adjective is true of itself. Is 'autological' autological? Construed as a hypodox, either it is xor it is not, yet there is a lack of a granted principle determining which.

Construed this way, the Truth-teller is a paradigm hypodox.

> Put simply, a hypodox 'might consistently take either truth-value but we have no basis for determining which'.
>
> (Eldridge-Smith, 2007, p. 178)

This generalizes (e.g., for a multi-valued logic, and for other semantic and modal values, such as whether something is necessary). Moreover, whether some things are hypodoxical may depend on contingent circumstances. That is, in certain circumstances, the matter is underdetermined, whereas it is determined (or even incompatibly overdetermined) in others. Furthermore, hypodoxes are not paradoxes, as per the following contradistinction.

> Whereas a paradox incompatibly overdetermines something by granted principles, in some cases given apparently possible circumstances, a hypodox is compatible with granted principles yet underdetermined for lack of a granted principle that determines the matter, in some cases given apparently possible circumstances.
>
> (Eldridge-Smith, 2007, 2008, 2022, 2023, endnote 3)

Granted Principles are granted mechanisms or bases for reasoning. They vary from system to system, but in natural language, the principles one person grants may differ from another. Principles include inference rules, axioms, distinctions or criteria, conceptual principles, and even conventions. Rules (including formation, structural and inference rules) and axioms of a formal system are among its principles. An example of a conceptual principle is the comprehension schema in naive set theory, loosely speaking, that for any condition, there is a set of things that satisfy it. Another example is the Principle of Indifference, loosely speaking, that unless there are relevant differences, possible outcomes are equally probable, which is used in some theories of probability. *Circumstances* include contingencies, stipulative definitions, mere regularities, and received or common opinions. Circumstances may be assumed or given as premises.

Paradoxes and hypodoxes occur in natural language reasoning, in some theories and some systems. Some things that are paradoxical under some circumstances may be hypodoxical under another; and, some things that a paradoxical under one set of granted principles may be hypodoxical under a different set of granted principles (cf. Da Ré et al., 2020, Introduction).

One kind of relation between paradoxes and hypodoxes is exemplified by the relation between the Liar, ‘This sentence itself is false’, and the Truth-teller. Also, Russell’s paradox has that kind of relation to whether the set of all self-membered sets is self-membered. That kind of relation between paradoxes and hypodoxes is symmetric. Indeed, many pairs of a paradox and a hypodox are related as duals in that way (Mackie, 1973, p, 298; Eldridge-Smith, 2007, 2008, 2012, 2022, 2023).

This article concerns a relation of a another kind between paradoxes and hypodoxes, which is not a symmetric. Some paradoxes can be *defused* to hypodoxes. Since a paradox incompatibly overdetermines something by granted principles, it may be appropriate to question those principles. One may reject, restrict, revise or replace some principle (be that a rule of inference, axiom schema, conceptual principle, or etc) to determine the matter. A *correct* solution must also justify the particular rejection, restriction, revision or replacement. But my point here is that the result of such a rejection, restriction, revision or replacement *sometimes* is that the matter becomes underdetermined by the residual granted principles (Eldridge-Smith, 2012). When this happens, a paradox is *defused* to a hypodox.

Conversely, granting additional principles or removing a restrictive condition on some granted principle may make something that was hypodoxical become paradoxical (Eldridge-Smith, 2012). When this happens, a hypodox is *enfused* to a paradox. Of course a goal of adding principles, if they can be justified, is to determine such hypodoxical matters; nevertheless, granting additional principles sometimes overachieves this goal and incompatibly overdetermines the matter.[1] So, *defusion* maps paradoxes to hypodoxes (and, in some cases, to other things); *enfusion* maps hypodoxes to paradoxes (and, in some cases, to other things).

I explicate the defusion relation with reference to sets that respectively represent a *paradoxical dilemma* and a *dilemma over a hypodox*. Section 2 supplies definitions of these. In section 3, I compare the result of defusing a paradox to a hypodox with the results of other ways of resolving a paradox. This method of comparison will be referenced in subsequent sections, especially the last. In section 4, I discuss a pedagogical example involving birthdays. In section 5, I defuse a lightweight legal paradox to a hypodox. In section 6, I give a technical example of defusing a Liar Paradox to a hypodox. I conclude with a philosophical example, discussing how these considerations apply to Bertrand's chord paradox in section 7. In that section, I also briefly discuss von Mises' water into wine problem.

2. Some definitions

Let me define a few things.

2.1. A paradox results in one or more paradoxical dilemmas

Some things of various kinds are said to be paradoxes. These include conclusions (Quine, 1976; Sainsbury, 2009), riddles (Sorensen, 2003), arguments (Mackie, 1973), and inconsistent sets (Lycan, 2010; Rescher, 2001; Schiffer, 2003). According to my definition, some things of each of these kinds are paradoxical:

> A *paradox* is something that incompatibly overdetermines something by granted principles, in some cases, given apparently acceptable circumstances.

[1] One example of this in the literature is Mortensen & Priest's *Truth Teller Paradox*. By adding a principle that a Truth-teller sentence is neither true nor false, Mortensen & Priest (1981) derive a contradiction. In this way, in my terminology, they *enfuse* a hypodox to a paradox.

Among the things a paradox incompatibly overdetermines are one or more sentences (or statements or propositions, as the reader pleases). Such sentences may be indicative or modal, declarative or imperative. (For modal examples, please see Eldridge-Smith (2022)). By *incompatibly overdetermining* a sentence, a paradox *results* in a *paradoxical dilemma*. To define these, I will make use of an exclusive 'or' connective.

> For any sentences A and B, 'A xor B' takes a designated value just if exactly one of {A, B} takes a designated value. Let a sentence of the form 'A xor B' be called an *alternation* in which A and B are *alternatives*.

Some logics have just one designated value, namely, true. The definition allows for logics with more than one designated value.

> A paradox *results* in a sentence A iff either it is given in the circumstances, or the paradox's circumstances are conjointly possible and A follows by granted principles assuming those circumstances. It may *follow* using a system of deduction, semantic reasoning, metatheoretical reasoning, a calculus (such as a probability calculus) or reasoning in natural language. The circumstances may be a null set; some paradoxes do not depend on any particular circumstances.
> A paradox *incompatibly overdetermines* a sentence A iff there is a sentence B such that each of {A xor B, A, B} results from the paradox.

A paradox incompatibly overdetermines something (by definition) and what it overdetermines can be represented or expressed by a sentence. Therefore, for each paradox, there are some sentences A and B, such that the paradox results in each of {A xor B, A, B}.

> Thus, the paradox results in a *paradoxical dilemma* represented by the result set {A xor B, A, B}. (N.b. These are results that hold conjointly, unlike multiple conclusions in a sequent calculus).

I add the following notes:

1. Not every sentence *qua* sentence that is incompatibly overdetermined by a paradox is itself a *paradoxical sentence*. In a paradoxical dilemma sentences A or B may or may not be paradoxical. For example, some paradoxes (and given circumstances) result in this paradoxical dilemma: {Athens is the capital of France xor Athens is not the capital of France, Athens is the capital of France, Athens is not the capital of France}. But none of these sentences is paradoxical. (Even though at least one of these results *qua* conclusion is a *paradoxical conclusion*.)

2. According to some consequence relations, incompatibly overdetermining a sentence will be sufficient for inconsistently overdetermining many sentences, even every sentence in the language.

3. The above definitions do not assume that the consequence relation is classical, although many paradoxes are presented as though classical reasoning is granted.

4. I make many claims about results, particularly those about paradoxical dilemmas, in a meta-language. My meta-language is natural language extended with some formal devices. The consequence relation in the metalanguage I am using is one that preserves designated values. (Its sentential and quantificational logic is that of 1PLITCHv4, as specified in section 6.)

2.2. Hypodoxes

Some things of various kinds are hypodoxical. The Truth-teller is such that one could add a principle such that it is true and one could instead add a principle that it is false. One could add a principle that sentences like a Truth-teller are false, since they have no Truth-maker.[2] Instead, one could add a principle that sentences like a Truth-teller are true, since they have no 'False-maker'. (Were one to resolve the Truth-teller in either of these ways, one would need to justify one of these ways while showing that the other is untenable or at least unjustifiable.) This suggests the following general account.

> Something, X, is a *hypodox* iff it is underdetermined by granted principles, in given circumstances in some cases, and these circumstances (if any) and principles are consistent with adding a principle so that it would result that X is the case and consistent with instead adding another principle so that it would result that X is not the case; but no such principle has been granted, usually because it has not been justified.

Were the granting of just one such additional principle justified, the matter would be determined and the hypodox resolved. Alternatively, one might resolve the hypodox by justifying that it has some sort of gap, especially a truth-value gap, or some kind of glut, especially a truth-value glut. And there are other ways, a fair number of which will be implicit in the next section about different results of resolving a paradox in different ways.

[2] This will determine the semantic status of a Truth-teller but not a Liar sentence, since if a Liar sentence is false, it is consequently true.

Let 'U' stand for the operator 'It is underdetermined for lack of a granted principle whether ...'. Then a sentence A is hypodoxical iff U(A).

> If a paradox is defused to a hypodox, then for some sentence A that was incompatibly overdetermined by the paradox, U(A). Moreover, there is some sentence B, incompatible with A (usually, ∼A) such that the paradox resulted in the paradoxical dilemma {A xor B, A, B}. But then U(B) results from A xor B and U(A). Thus, defusing a paradox to a hypodox results in a *dilemma over a hypodox* {A xor B, U(A), U(B)}

I add the following notes.

1. It is not necessary for the overdetermination or underdetermination of A to be expressible in the object language; so long as it is the case and the dilemma over a hypodox can be expressed in the meta-language.

2. In a logical system with a strong negation operator, ∼, that takes a designated value to an undesignated value, and *vice versa*, then A is paradoxical iff ∼A is paradoxical, and A is hypodoxical iff ∼A is hypodoxical.

3. for the same A and B, a paradoxical dilemma and a dilemma over a hypodox are contraries; this is because *a fortiori* of being overdetermined, A and B are both determined if the paradoxical dilemma obtains; whereas A and B are both underdetermined if the hypodox obtains. This accords with what was said in the previous section, hypodoxes are not paradoxes.

3. Results of resolving a paradox in a Hendecagon of Opposition

There are different results from resolving a paradox in different ways. In this section, these results are compared with respect to the general form of a paradoxical dilemma. Defusing a paradox to a hypodox results in a dilemma over a hypodox. This is distinguishable from nine other results of resolving a paradox in different ways. These together with accepting the paradoxical dilemma itself make up eleven results of resolving a paradox in different ways. These results are related as contraries.

Consider the Ship of Theseus paradox.

> Over a period of years, in the course of maintenance a ship has its planks replaced one by one — call this ship A. However, the old planks are retained and themselves reconstituted into a ship — call this ship B. At the end of this process there are two ships. Which one is the original ship of Theseus? (Clark, 2012, p. 230)

Notice that this paradox contains no apparent use of negation. Nevertheless, it results in a paradoxical dilemma (PD), {either Ship A xor Ship B is the ship of Theseus, Ship A is the ship of Theseus, Ship B is the ship of Theseus}. It is a simple (PD) result of the form A xor B, A, B. The result is obtained by using conceptual principles about identity and the above circumstances. (The only logic required is the definition of an alternation.) The ship of Theseus is just one entity. The only alternatives for the identity of that one entity are ship A and ship B. Hence, the above alternation is the case. In virtue of being maintained, ship A is the ship of Theseus. In virtue of being constituted from the same materials into the same structure, ship B is the ship of Theseus.[3]

The Ship of Theseus could be defused to a hypodox by justifying (by *non-ad-hoc* reasoning) that neither continued maintenance nor being composed of the original components in the original way was sufficient for identifying the ship of Theseus. This would involve restricting those principles or arguing neither conclusively determines which is the ship

[3] Clark himself believes the Ship of Theseus *should* be represented by two arguments with incompatible conclusions, rather than one argument (Clark, 2012, pp. 160–161). He seems to think this is characteristic of an antinomy, as a subclass of paradoxes. Another case for two argument presentations of some other paradoxes such as Newcomb's problem and the Monty Hall problem is made by Olin (2003). Olin's point is that different authorities have staunchly defended each of two incompatible conclusions with respect to these problems. Olin does not actually use the label 'antinomy' but considers such two-argument paradoxes a philosophically significant type of paradox. (I note that Newcomb's problem, like the Ship of Theseus, proves a paradoxical dilemma; and I note that there is nowadays a consensus on the solution to the Monty Hall problem.) If Olin is right, one would expect that the same agent cannot consecutively defend both arguments as seeming sound. One finds similar definitions of 'antinomy' in some dictionaries, e.g., Cook (2009). Cook (2013, pp. 14–16) argues however that it is not significant whether an antinomy is presented as two arguments or one, because the two arguments can be combined into one argument. Moreover, paradoxes that Quine lists as paradigm 'antinomies' like the Liar, Russell and Grelling's can be presented as two arguments, although they are usually presented as one. They then trivially satisfy a two-argument definition of 'antinomy'. Furthermore, this issue seems irrelevant to definitions of paradoxes in terms of inconsistent sets whose elements seem individually acceptable (Lycan, 2010; Rescher, 2001; Schiffer, 2003). Finally, typically concluding with contraries is another superficial difference between the likes of the Ship of Theseus and the Liar (which typically concludes with the conjunction of contradictories). Again, this is a superficial difference if the contrary conclusions can be combined into one argument by adjunction. In any case, these matters do not seem to affect the fact that the likes of the Ship of Theseus, the Liar and Russell's paradoxes prove paradoxical dilemmas. Whether they do so in the same or separate proofs is irrelevant.

of Theseus. In either case, as a result {either Ship A xor Ship B is the ship of Theseus, it is underdetermined whether Ship A is the ship of Theseus, it is underdetermined whether Ship B is the ship of Theseus}. This result is a Dilemma over a Hypodox (DH). Remember 'U' stands for the operator 'It is underdetermined whether', then a dilemma over a hypodox, an (DH) result, has the general form {A xor B, U(A), U(B)}. With respect to a formal system implementing such a solution, these claims are made in the metatheory. These claims, particularly U(A) and U(B), may or may not be expressible in the objectlanguage.

In comparison, I outline a classification of other results with convenient labels in parentheses. These have some general forms that represent how these results differ from a paradoxical dilemma and each other.

Ideally, one justifies a determined (Det) result by finding fault with the proof of one or more of the members of the paradoxical dilemma so that the only results left are {A xor B, A} xor {A xor B, B}. Note that it follows from {A xor B, A} that B does not have a designated value. The Ship of Theseus would be determined if it could be justified that just one of its principles applies. I add the following minor point. A special case of a Determinate (Det) result worth labelling separately is a To Be Determined (TBD) result. That is, whether A xor B will be determined. Given such a situation, {A xor B} is either given or follows by still granted principles, and there is reason to accept that whether A xor B will somehow be determined.

'(Both)' labels a result when fault is found in the proof of A xor B, but not the proofs of A and B; so a (Both) result can prove {A, B} while denying A xor B. In our metatheory, we will express that denial as asserting ∼(A xor B), although this may not be expressible in the object-language. In some cases, a formal system implementing such a solution may not have an exclusive negation and may not have an 'xor' connective. A special case of this result, where A and B are contradictory, is a dialetheic result. A dialetheia is a true contradiction; nevertheless, this is not a paradoxical dilemma. That is, in a metatheory about the system one can assert each of {∼(A xor ∼A), A, ∼A}. This is clearly not the same as being committed to a paradoxical dilemma. Moreover, although some contradictions are asserted in some paraconsistent logics, such as LP, no paradoxical dilemma is asserted in LP. (PD) and (Both) results are incompatible. Note that a subclass of the (Both) result is the {A *or* B, A, B} result because that result is compatible with the more general {∼(A xor B), A, B} result. Moreover, a (Both) result is not necessarily dialetheic. The Ship of Theseus could be resolved to yield a (Both) result, if there is reason to reject the uniqueness of that ship. That would not necessarily be a dialetheic solution.

An (RSC) result is a *reductio* of a self-contradiction, S, that was purported to be possible in proving the paradoxical dilemma. The Barber paradox is resolved this way. Given the scenario about the circumstances (S) of the Barber, a paradoxical dilemma followed. But the circumstances are not possible, which is an (RSC) result for the Barber. I note that Quine (1976) classifies the Barber as *veridical* on the basis that this is its resolution.

A (DIS) result obtains when the paradox at issue is resolved by making a *distinction*. Or at least when the paradoxical dilemma is resolved by making a distinction such that whether A xor B depends on distinguishing whether C xor D. Given the situation {A xor B, C xor D, C $\longrightarrow$ A, D $\longrightarrow$ B} follows by still granted principles and a valid distinction. William James (1907) said a scholastic adage was 'whenever you meet a contradiction you must make a distinction'. I do not think this way is mandatory; it would be *ad hoc* to resolve the Ship of Theseus by distinguishing the rebuilt Ship of Theseus from the well-maintained Ship of Theseus. Nevertheless, making a distinction is one useful way of resolving a paradox among a number of ways. Consider for example Wrangham's *Goodness Paradox* (Wrangham 2019, Ch. 13). Some argue that humankind is naturally peaceful. Others argue that humankind is naturally violent. Apparently, there are authoritative reasons for both positions. Wrangham would distinguish between reactive aggression and proactive aggression. Then he argues: if one is talking about reactive aggression, humankind has low reactive aggression as a result of its evolution; but if one is talking about proactive aggression, humankind is prone to perpetrate proactive aggression.[4]

(MinI) for 'Minimally indeterminate' is a result where it is justified that whether A xor B is indeterminable. That is A xor B, but it is indeterminable which. Let 'D' stand for 'It is determinate whether ...'. Then, given the situation, {A xor B, $\sim$D(A), $\sim$D(B)}. The difference between (DH) and (MinI) is that a (DH) result can subsequently be

[4] I hear an objection that this is disambiguation, and therefore that the original argument was fallacious. But this would only be disambiguation if 'aggression' is ambiguous between 'reactive aggression' and 'proactive aggression', which it is not. Making a distinction to resolve a paradox, as in the case of the Goodness Paradox, is not the same as pointing to the fallacy of ambiguity in an argument.

resolved by justifying an additional principle that determines whether A xor B. However, if one succeeds in justifying a principle that resolves an indeterminacy then the matter was not indeterminate. For (MinI) says that there is no principle that determines the matter at all, so there won't be one subsequently justified and hence accepted that will determine the matter (not in the same system anyway).[5] Among other approaches, (MinI) might be used to represent an epistemic island, or the result of a supervaluation solution (cf. Fine, 1975). For example, a (MinI) result for the Ship of Theseus would be that it is an epistemic island such that Ship A xor Ship B is the Ship of Theseus but it is impossible to know which.

A (MaxI) result is maximally indeterminate. Whether A, B, and A xor B are all indeterminable. {∼D(A xor B), ∼D(A) & ∼D(B)}. Again, it has somehow been determined that no extra principle that determines the matter can be justified and hence granted.

A (None) result obtains if it is justified that some expressions in the paradoxical dilemma have no value, somehow do not actually express anything, or are not even in a language. There is nothing to determine because *none* of the members of the paradoxical dilemma have a designated or undesignated value, even if they are allowed in the language. Let 'V' stand for '... has a value' and angle brackets for some canonical convention for naming the expression they enclose. Then {∼V(<A>), ∼V(<B>)}. This could be the result of a value gap solution or a non-closure solution, that is the language or domain is not closed under some principle. It is difficult to see how this result could obtain in the case of the Ship of Theseus, but it has well-known applications to the Liar sentence as a result of some solutions to the Liar paradox inside formal systems that implement a non-closure solution or a gappy solution.

(Neither) is a different result where {∼A, ∼B} is somehow justified. This means that the proof of all the members of the paradoxical dilemma have been rejected *and* the negations of both A and B are proven.

(Revenge) is the last result in this list. Some of the best paradoxes have revenge paradoxes. A revenge paradox might result in a number of ways, for example {A xor B, ∼A, ∼B}. (Revenge) might result from proving A and B are not the case and finding faults in their original

[5] Personally, I think (MinI) is a hard resolution to justify, because obviously there is or was a principle that determined the matter if there was a (PD) result to resolve, even though that principle (or principles) incompatibly overdetermined the matter.

proofs, while nevertheless the reasoning for A xor B still follows. This result is still paradoxical. I note that, in classical logic, 'A xor B' is equivalent to '∼A xor ∼B'. So, if the granted principles include classical logic, then this result is still a paradoxical dilemma; that is, it would still be a (PD) result; but I am not stipulating whether the logic is classical, so I include a (Revenge) result as a separate result. (Nonetheless, I am open to eliminating this separate type of result if it can be proven that it is always reducible to another paradoxical dilemma (PD) result.)

This list of distinguishable results is not exhaustive, although it is sufficient for our purposes. Moreover, please note that some adjustments to other results must be made if the paradoxical dilemma is such that {A xor B xor C, A, B}. A corresponding dilemma over a hypodox would be {A xor (B xor C), U(A)), U(B xor C))}. Other adjustments are equally straightforward.

Besides, our taxonomy only compares and relates one general characteristic — different results of resolving a paradoxical dilemma. Other details of a particular solution will vary with particular characteristics of that solution. Accordingly, for example, the above taxonomy of results of resolving a paradoxical dilemma in different ways does not determine what results from treating a disjunction of a paradox and a hypodox according to a particular solution that does not defuse the paradox to a hypodox. In contrast, for a solution that defuses the paradox to a hypodox and uses truth-functional disjunction, the solution to this disjunction is itself a hypodox. Section 6.6 will exemplify this point with respect to a (Mini) solution in comparison to results of an (DH) solution for a Liar sentence disjoined with a Truth-teller.

Imagine the above listed results together with a paradoxical dilemma (PD) form an hendecagon of opposition. Imagine (PD) is on the top vertex and each contrary result is at another vertex. ((TBD) can be listed outside the polygon beside (Det), as it is simply a variation of a (Det) result.) Consider the relations along sides and diagonals between vertices of the hendecagon. The result at a vertex is contrary with the results at other vertices, excepting that (DIS) is not strictly contrary with (Det). Let these last two form the base of the hendecagon. I think that each result is incompatible with the original paradoxical dilemma, and hence contrary to it (in the sense that one cannot be committed to both results at the same time in the same way). I already argued in the previous section that the paradoxical dilemma (PD) and the corresponding hypodox with the same alternation are incompatible. That is, Whether A xor B cannot be both incompatibly overdetermined (PD) and underdetermined (DH) in the same way at the same time given the same principles. I also argue it cannot be incompatibly overdetermined (PD) and determined (Det) at the same time. And, I would argue the same holds for (PD) with respect to the results positioned at other vertices.

Transitional relations are also of interest. Apart from (Revenge), the results of such transitions are explicit in the list above. Nevertheless, how the transition from a paradoxical dilemma to a particular result is achieved or how a transition from one result to another is achieved is left open. I have made only occasional short comments on some approaches that may be used to justify that result. Most transitions, except to (RSC), require an alteration to the set of granted principles. One can evidently do this within natural language reasoning, but normally altering the principles of a logical system would involve using a different logical system. The transitional relation from (PD) to (DH) is *defusion*, as mentioned; and the converse relation is *enfusion*. Enfusion involves accepting an extra principle (or extending the application of one); any transition to (Dis) requires accepting some distinction. The relations from (DH) to (Det) and from (DH) to (Dis) are also of interest. These represent preferred ways of resolving a hypodox. Conceivably, one might defuse a paradox (PD) to a hypodox (DH) by restricting one or more principles, and subsequently resolve that hypodox (DH) by extending another principle or accepting a new principle or distinction that determines the matter, yielding a (Det) or (Dis) result in the end. This is the two step approach that was mentioned in the introduction. It will feature again in the last section on Bertrand's chord paradox.

I will demonstrate using this taxonomy of hypodoxes, paradoxes, their relations and these contrary results of resolving a paradoxical dilemma to compare and critique proposed solutions to some paradoxes.

4. A pedagogic example: Frederica's 21st birthday – paradox, hypodox, determined date, or non-event?

This section expounds on a pedagogic example first put forward as a hypodox in my (2012) in contrast to W. S. Gilbert's birthday paradox (Gilbert and Sullivan, 1879).

Frederica has a dilemma about when to celebrate her 21st birthday. Born on 29th February, she has reason to celebrate on the 28th February or 1st March. Her initial preference is for both. However, her orthodox parents maintain:

(1) A person has one, and only one, birthday per annum.

By (1) and the given scenario about the circumstances, Frederica faces the dilemma of her birthday being on just one of 28th February or 1st March. At this stage, she believes she lacks any further principle to determine the matter, and thus that matter is underdetermined and she faces a hypodox (DH).

However, her parents subsequently each persist in giving her further counsel. This year, her American father rationalizes:

(2) Frederica's birthday is on 28th February, because she was born on the last day of February.

However, her English mother argues that:

(3) Frederica's birthday is on 1st March, because she was born on the day after 28th February.

If Frederica accepts (1), (2) and (3), she faces a contradiction, corresponding to a paradoxical dilemma, (PD) in our list above. From (1) it follows that Frederica's 21st birthday is on 28th February xor 1st March xor some other day of that year, and from (2) it is 28th February, but from (3) it is 1st March. (1) poses an alternation over which she seems forced to choose, and her dilemma is paradoxical in this case because the conjunction of (2) and (3) contradicts (1). (1) is a convention, a weak kind of principle. (2) and (3) are also conventions.

Frederica however argues that she is mindful of the scholastic adage that when faced with a contradiction, make a distinction (James, 1907, lecture 2). Frederica would distinguish between the 'birthday that in each non-leap year is 365 days after the last birthday and on the actual birth day and month in a leap year' and the 'birthday that in each non-leap year is 365 days before the next birthday and on the actual birth day and month in a leap year'. Then (2) is about the former, (3) about the latter, and (1) can be replaced by a principle saying there is just one of each kind of birthday per annum. Federica argues the apparent paradox is veridical and teaches that a distinction must be made to accommodate the anomaly in the calendar and avoid contradiction. Moreover, to assuage her pragmatic parents, Frederica claims her solution will work in the long run. Now Frederica argues she can celebrate each kind of birthday respectively on 28th February and 1st March. This corresponds to a (Dis) result in our hendecagon.

However, Frederica's parents reject her distinction as unpragmatic. Her mother says that Frederica's distinction has no empirical basis. Her father says that her distinction is not coherent with his partially ordered set of beliefs. Her parents say the apparent contradiction is a falsidical paradox because Frederica reasons using an erroneous convention. Her parents differ of course over which convention is erroneous. Her father says (3) is unwarranted; her mother says (2) is unwarranted. Each parent maintains a determinate (Det) result either way.

Suppose that Frederica is persuaded by her parents to maintain the convention (1) and confused as to which of (2) or (3) is unwarranted but now suspects both. If she had a reason to reject just one of (2) or (3), she would know when to celebrate her birthday. However, it now seems to Frederica that exactly one of (2) or (3) is not the case and yet there seems no basis for determining which. She considers this to be so, because she considers the reasons given for each of (2) and (3) equally cogent, but also she now considers neither to be conclusive; and yet there does not seem to be any other accepted principle that would determine which is correct. In this case, she now faces a hypodox (DH) again. Notice that she no longer faces a paradoxical dilemma (PD).

If she can now justify an additional principle (or find a reason) that proves that just one of (2) or (3) determines her birthday in a non-leap year, she would determine the matter, as per a (Det) result in our hendecagon.

As time passes, perforce of having thought about it so long, she somehow accepts (1) and that it is indeterminable which one of (2) or (3) applies to her birthday in a non-leap year to the exclusion of the other. She believes she will never know when to celebrate her 21st birthday. She feels despondently that when to celebrate her birthday is an inaccessible epistemic island. This result corresponds to (MinI) in our hendecagon.

Progressing past this slough, a scarier thought confronts Frederica. (1) might also be indeterminate if (2) and (3) are. If so, this result corresponds to (MaxI) in our hendecagon.

Frederica remonstrates to her mother about never being able to know when to celebrate. However, her mother chides Frederica for her reasoning. Her mother says there is a cogent reason for celebrating on 1st March, viz. Frederica was born on the day after 28th February and the reason given by her father, viz. that she was born on the last day in February, does not contradict that she was born on the day after 28th February. So how can it be unknowable that she should celebrate on 1st March when there is an uncontradicted cogent reason for it?

The next day, there is a glimmer of hope based on new information. Frederica reasons she might accept (1), but she now considers that whether (2) or (3) is correct is determined by some acceptable principle that she has evidence exists. This is because she has heard from her twin brother (who has been indentured with pirates till his 21st birthday) that he is coming home, and therefore she (mistakenly) believes he has found a principle that resolves the matter. Thus, she believes she is justified in a (TBD) result.

However, a subsequent missive from her brother disavows that there is any principle for deciding in favour of (2) or (3). So Frederica, on further reflection, instead rejects her parental convention about (1). She thus has different options. She returns to her preference for both (2) and (3); which from her new way of reasoning is now a (Both) result in our hendecagon. All ends more or less well, except that her brother arrives home after the festivities with further news.

Her brother informs her that killjoys, pirates and Quine do indeed deny (1). Yet worse, Quine replaces (1) with yet another competing convention:

(4) 'a birthday has to match the date of birth; and February 29 comes less frequently than once a year'. (Quine, 1976, p. 1)

Quine reasons as it were that birthdays are not *closed* under a per annum principle like (1). With respect to those born on 29th February, their birthday occurs usually every four years, given principle (4). ('Usually' because a century is only a leap year if it is divisible by 400.) Frederica's twin, Frederic was indentured to a pirate king till his 21st birthday, according to Gilbert & Sullivan's operetta *Pirates of Penzance*. Frederic reasons on the one hand that he is turning 21, having almost lived for 21 years; and on the other hand, 'reckoning by [his] natal day,' he has only had five birthdays and is therefore five 'and a little bit older'. Frederic is in a bind, because the pirate king maintains he is indentured till his 21st birthday. With respect to Frederica's original dilemma {(1), (2), (3)}, Sir Gilbert and Quine resolve it as an unprincipled self-contradiction. They deny that (1) is a convention. Thus, they consider (1), (2), and (3) only as conjointly self-contradictory assumptions on Frederica's part.

They maintain that Frederica's 21st birthday is neither on 29th February nor 1st March nor any other day that year. This is an (RSC) result in our hendecagon.

By Quine's and piratical reckoning, the twins' 21st birthday will be many more than 21 years pending. Quine concludes that a person can have lived 21 years and yet only have had 5 birthdays. He says this is an example of a veridical paradox, where the conclusion is after all true. It is a consequence of an anomaly in the calendar and Quine's premise (4) that a birthday has to match the date of birth.

However, Frederica believes her brother has been hoodwinked by an argument from authority. Neither pirates, Sir Gilbert nor Professor Quine are really authorities on birthdays. She vindicates her own position against Quine by researching the definition of 'birthday' in the Oxford English Dictionary (OED 2020b). In that authority a birthday is an anniversary, which itself is defined as an annual commemoration, except in some extended usages such as 'two-week anniversary' (OED 2020a). Also, from this reasonable search, Frederica finds nothing in the OED stating or entailing that a birthday must match the day and month of birth for those born on 29th February. Frederica is arguably correct; in any case, (1) and not-(4) is implied by OED definitions.[6]

This case is pedagogic. In my opinion the question of the date of Frederica's 21st birthday is hypodoxical, but clearly this is debateable. Nonetheless, I am not a pirate. Nor would I object to Frederica enjoying two birthdays for the price of one. Next is a lightweight case where I do maintain a particular result.

5. A lightweight legal example: The parking voucher paradox

Here is a *paradoxical* story of mine (comprised of a scenario of circumstances plus some related arguments). It ranks say 3 on Sainsbury's Richter scale of paradoxes from 1 to 10, wherein the Barber paradox

[6] I note that Frederica has anachronistically accessed version 3 of the OED. Curiously, the OED v2 definition of 'anniversary', read on its own, might appear to support Quine; but if one also reads the OED v2 definition of 'birthday', one will find that these two OED v2 definitions combined also support Frederica's case against Quine's convention (4). Conceivably, Quine may have accessed some other dictionary; but he cites no source for his interpretation of 'birthday'. Yet the conventional extension of this term is critical to discussion of Gilbert's paradox and Frederica's dilemma.

ranks 1 and the Liar and Russell's are examples of 10 (see Sainsbury, 2009, pp. 1–2).

I have some casual lecturing. I drive from my other workplace to the university and park my car. The sign reads 'Pay parking 9am to 5pm'. Not paying for parking my car within these times risks being fined. Accordingly, I pay for parking by putting coins into the parking voucher vending machine. The amount I have inserted will pay for parking till 4:59pm, so the machine tells me. I print the voucher that says I have paid for parking till 4:59pm. I place it on the top of my car's dashboard so that it can be clearly read through my car's windscreen. Then I dash to my lecture. It is my first lecture on paradoxes in 2009. 'Can I be legitimately fined?' I ask of the class as a segue into our topic. The initial student response is concern that the lecture finishes at 4pm. I assure students it will, but that I will be staying at the university past 5pm.

On the one hand, I rationalize that I cannot be legitimately fined. I argue there is no time between 4:59 and 5:00pm at which I can be fined. Parking vouchers can only be purchased in whole minutes. Certainly, there is no other whole minute between the times of 4:59 and 5:00pm, and one can park in that place freely at 5pm. Therefore, there is no minute before 5pm for which I did not pay for parking and by 5pm parking is free. In other words, I have paid for parking for minutes up to and including 4:59; once 4:59 has ended, I have not paid for parking; but at 5:00pm parking is free. This is apparently acceptable reasoning as to why I cannot be legitimately fined for parking in this place at 30 seconds after 4:59pm.

On the other hand, I am required to pay till 5pm and could be fined. Indeed, some students do argue that I can be fined for parking some seconds after 4:59 (i.e. some seconds after the time reaches 4:59) and before 5:00pm. Some even argue I can be fined for parking exactly at 5pm. Some also argue that I can be fined retrospectively after 5:00pm, as my car was evidently parked there at 5pm but with a ticket paid till 4:59pm. There is then some apparently acceptable reason why I can be legitimately fined for being parked in this place and only having a valid ticket till 4:59.[7]

[7] By serendipity, this situation had also actually occurred for me once years before, when I was parking to go to a logic reading group. Having paid for a parking voucher till 4:59, I put the situation to the group. Robert Meyer agreed that my reasoning for not being legitimately fined made sense and was bemused by the case; other logicians in the group thought that nevertheless I could be fined, but did not say

This is not simply a disagreement. Either argument is apparently acceptable on its own. Yet accepting both conclusions is apparently accepting a contradiction, given the common-sense premise ‘There is just one time from which parking is legally free’. There are a number of ways of representing this as a paradoxical dilemma. I prefer this one: {There is no nameable minute for which I did not pay for parking before 5pm xor parking is legally free only from 5pm, There is no nameable minute for which I did not pay for parking before 5pm, Parking is legally free only from 5pm}.

A parking inspector may decide to fine me or not, but the issue remains whether or not it is legitimate to fine me. The matter might be settled by some by-law; for argument’s sake, I assume it is not.[8]

The principles I applied are based on implications I drew from the sign and the ticketing being in minutes. I presume there is just one time from which parking is legally free. Then I reason that (1) What the sign says about having to pay for parking between 9am and 5pm implies I can be fined. However, I also reason that (2) I can be fined only if there is a nameable minute for which I did not pay for parking before 5pm, and that implies I cannot be fined.

Actually, my solution to this paradox is to maintain that none of the supporting arguments for (1) and (2) make either of these conclusions certain in this case. The argument for (2) infers from ticketing in minutes that there is no minute before 5pm for which I did not pay for parking. This reasoning is defeasible; but it is underdetermined whether it is controverted in this situation. Moreover, the reasoning from what the sign actually says to (1) is also defeasible. It is brought into question in this situation because ticketing is in minutes; but it is not controverted by that fact on its own. I maintain the sign and the ticketing in minutes do support but underdetermine which of (1) or (2) is the case because they do not provide indefeasible support for either (1) or (2), which are contraries in the given situation. The alternation ((1) xor (2)) is nevertheless supported by the situation. This is an example of a legal hypodox. It is underdetermined which of (1) xor (2) applies in this case;

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what was wrong with my reasoning. I think their view was based more on pragmatic cautionary considerations than pure reason.

[8] Some legal fiat may settle the matter. Roy Cook, in an informal conversation, pointed out there is probably some such by-law somewhere. Nevertheless, there may not be such a by-law, and the case as presented may be all there is to determine the matter.

something that could be addressed by a by-law, but as it happens in this case, is not; so the above contradiction can be defused to a legal hypodox. This is an example of an (DH) result.

6. A formal example: Defusing the Liar paradox to a hypodox

In this section I defuse a Liar and some Liar-like paradoxes to hypodoxes in a formal system. This is a formal example of defusing a paradox to a hypodox, which was reasonably requested by a referee. (A reader uninterested in a formal example but interested in a philosophical example may wish to skip to the next section.)

The Liar paradox will be resolved to a Liar hypodox, not the Truth-teller. That is, the Liar and the Truth-teller will be distinct hypodoxes in our system. I provide a formal deductive system in which the Liar sentence is either the case xor not, yet it is underdetermined which it is. Moreover, in a semantic model for this system, the Liar sentence may consistently either be true or not true. Defusing the Liar to a hypodox is a *partial* solution to the Liar. That is, the hypodoxes of this system are consistent but determining their semantic values is a residual issue. Various arguments motivating and supporting this defusion of the Liar and Liar-like paradoxes can be found in (Eldridge-Smith, 2019, 2020). The system presented here is similar to those systems. Each is a First-order Predicate Logic with Identity, a Truth predicate, Canonical names and Hypodoxes (1PLITCH) system. Accordingly, I label the system here 1PLITCHv4 ('v4' for 'version 4') or just 1PLITCH. I call its unquantified fragment '1PLITCHv3'. Fragments of 1PLITCHv3 and v4 without a truth predicate are classical. In 1PLITCH, the applications of a number of rules are restricted for some formulas containing the truth predicate. These are identity elimination (=E), (∀E) and (∃I). Since the diagonal theorem depends on (=E) being unrestricted in its proof, the diagonal theorem is not universally valid in our system. Also, the introduction and elimination of the truth predicate is restricted. Moreover, in reasoning with the semantic rules of 1PLITCH, substitution of identicals is subject to the same restrictions as (=E) in the deduction system.

PLITCHv4 implements a general solution to Liar-like paradoxes that use a truth predicate if the following conjecture is correct. (Eldridge-Smith (2019, 2020) argue in support this conjecture.)

Conjecture 6.1 (Conjecture on a dependency of Liar paradoxes). *In a system that restricts rules for introducing and eliminating its truth predicate,* T*, to instances where it predicates a canonical name of a sentence, then each formulation of the Liar or a Liar-like paradox depends on swapping, within the scope of an occurrence of* T*, a canonical name of a* T*-sentence and a non-canonical term.*

A *canonical name* is one from which the expression named can be mechanically determined and which can be mechanically formed for any given expression. Following Tarski (1936), let a *formulation* of a paradox be an exhibited derivation in some object language, even a fragment of a natural language. A *T-sentence* is one in which the 'T' predicate occurs. *Swapping a canonical name of a T-sentence and a non-canonical term* involves substitution, instantiation or generalization. This may happen tacitly where derived rules (or schemas) are used, as is conjectured to be the case in derivations of the Liar using Gödel's self-refential lemma.

I begin by introducing a language and the deduction system. Then I list a selection of Liar-like paradoxes and Truth-teller-like hypodoxes. In formulations of these I mark the inferences that fail in our deduction system with an asterisk. This '*' is attached to the name of the inference that fails. Thereby a formulation shows both a derivation and which inference is invalid in our deduction system. I give a semantics for 1PLTICHv4, briefly discuss the solution and make some comments on how it avoids some revenge strategies. A referee also reasonably wonders how this system might compare to a classification system based on a fixed-point semantics, particularly that of (Cook, 2020, 2022). I comment on this.

6.1. A language for 1PLITCH

Consider a first order language, $\mathcal{L}$, with identity extended with canonical names for its expressions, a truth predicate, T, and a syntactic self-predication function, sp. A canonical name for an expression ψ is represented by $<\psi>$. Please think of them as quote names for this basic language. A quote name is formed by prefixing the expression it names by an opening quote and suffixing it by a closing quote. Herein, angle brackets '$<$' and '$>$' are used to represent quote marks in the language.[9]

[9] Using arithmetization rather than quote-names with the sp-function presents some technical complications (Quine, 1995). Nevertheless, such functions can be used with arithmetization (Smullyan, 1957).

The *terms* of $\mathcal{L}$ include lower case letters 'a' through 'o' as individual constants, and variables 'x', 'y' and 'z'. These may be subscripted. As well, canonical names of the form $<\phi>$ are terms, where ϕ is a well-formed formula of the language. There are also functional terms, called *sp-names*, of the form $\mathrm{sp}(<\psi(\mathrm{x})>)$, where $\psi(\mathrm{x})$ is a 1-place open formula of the language. The above are the only terms of the language. Individual constants, variables and functional terms of the form $\mathrm{sp}(<\psi(\mathrm{x})>)$ are *non-canonical terms*. Individual constants, canonical names and functional terms of the form $\mathrm{sp}(<\psi(\mathrm{x})>)$ are *closed terms*. A term is *T-free* iff it contains no occurrence of 'T'. A canonical name is T-free iff the expression it names contains no occurrence of 'T'. An sp-name, $\mathrm{sp}(<\psi(\mathrm{x})>)$, is T-free iff the formula ψ contains no occurrence of 'T'. Clearly, individual constants and variables are T-free.

The n-place predicate letters of $\mathcal{L}$ include: One 0-place predicate letter (i.e. a closed sentence): 'Q'; three 1-place predicate letters 'C', 'G', and 'T'; and two 2-place predicate letters 'R', and '='. Except 'T' and '=', predicate letters may be subscripted. A predicate 'F' is introduced by definition as a notational abbreviation of '∼T'.[10]

If ψ is an n-place predicate letter and $t_1, \ldots, t_n$ are terms, then $\psi(t_1, ..., t_n)$ is an *atomic formula* of $\mathcal{L}$. (The identity predicate is infixed between two terms.) If all its terms are closed terms, $\psi(t_1, ..., t_n)$ is an *atomic sentence*. This applies to the T predicate as well. A sentence of the form $\mathrm{T}(\delta)$ where δ is a closed term is an *atomic T-sentence*. So T(a) and $\mathrm{T}(<\sim\mathrm{Ta}>)$ are both atomic formulae.

If Δ and Γ are formulas, so are $(\sim\Delta)$, $(\Delta \vee \Gamma)$, and $(\forall \mathrm{v}(\Delta))$, for each variable v. Strictly, nothing is a formula except by the above. Nevertheless, a conjunction ($\wedge$), implication ($\rightarrow$), biconditional ($\leftrightarrow$), and an xor connective ($\oplus$) are introduced by classical definitions, as well as the existential quantifier ($\exists$).

The *formulas* of $\mathcal{L}$ include, if β and γ are terms: $(\beta = \gamma)$, $(\beta = <\mathrm{T}(\beta)>)$, $\mathrm{T}(\gamma)$, $\mathrm{T}(<\mathrm{T}(\gamma)>)$, $\mathrm{S}(\gamma)$, $\mathrm{R}(\gamma, \beta)$, $\mathrm{sp}(\sim\mathrm{T}(\mathrm{x}))$, $(\beta = <\sim\mathrm{T}(\beta) \vee \mathrm{Q}>)$. In practice, parentheses are often elided.

A formula is *closed* iff it has no free variables. A *sentence* is a closed formula. A *T-sentence* is any sentence containing the T-predicate, even

[10] An identity premise for the Simple Liar is incompatible with defining falsity as truth of the negation. Let me illustrate this remark. As in section 6.3, 'l = Fl' is such an identity premise, where 'l' is an individual term. One cannot replace this occurrence of 'F' with the truth of the negation of 'l', because 'T∼l' is ill-formed.

inside a canonical name used in that sentence. Every sentence of the language is either a T-sentence or a *T-free sentence*.

6.2. The deduction system for 1PLITCHv4

1PLITCH has rules for introducing and eliminating its logical vocabulary: $\sim, \vee, \forall$, $=$, T. The self-predication function, sp, is also used to introduce some identities between non-canonical names of sentences and canonical names for those sentences. The rules for negation and conjunction introduction and elimination are classical rules. Assumptions may be introduced into a derivation. Undischarged assumptions are premises. Disjunction, material implication, and material equivalence are defined in terms of negation and conjunction in a classically standard way; as a result, classically standard rules for introducing and eliminating these binary connectives can also be used. The use of these sentential rules in derivations is often simply noted as 'SL' for 'Sentential Logic'. Existential quantification is defined in terms of negation and universal quantification in a classically standard way. The rules for $\forall$ introduction, identity introduction, and $\exists$ elimination are classically standard rules. Restrictions on Identity Elimination (=E) and universal quantifier elimination ($\forall$E) restrict their use for swapping a non-canonical name of a T-sentence with a canonical name of that sentence within the scope of an occurrence of the T-predicate. The restriction on ($\forall$E) also applies to ($\exists$I).

The rules TI and TE only introduce or eliminate the truth predicate where it is predicated of a canonical name of a closed sentence. In practice, its syntactic introduction or elimination will be governed by a derived schema below, used in effect as an axiom schema, which is a restricted form of Tarski's T-schema.[11]

(CT-Schema) $T{<}\phi{>} \leftrightarrow A$, where ϕ is a closed sentence for which ${<}\phi{>}$ is a canonical name.

Each and every instance of the above schema is derivable from no premises and is called a *T-biconditional*. The T-biconditionals are a subset of the T-sentences. The self-predication function is a syntactic function devised by Quine (1995).

- The self-predication of 'x is green' is ' "x is green" is green'.
- The self-predication of 'x is not true' is ' "x is not true" is not true'.

[11] The difference is that Tarski's T-schema can be used to predicate truth of any name of a sentence, canonical on non-canonical (Tarski, 1944).

The sp function for our language is defined as follows:

- $\mathrm{sp}(<\psi(\mathrm{x})>) = <\psi(<\psi(\mathrm{x})>)>$, where $\psi(\mathrm{x})$ is a formula with one free variable.[12]

Each and every instance of the above identity is derivable from no premises in 1PLITCH. Some formulations of the Liar paradox depend on a syntactic function, like the sp function, others depend on a premise. Our language and deductive system allow for the representation of both such formulations.

Consider the relevance of this function for our topic. In formalized English, the self-predication of 'the self-predication of (x) is not true' is 'The self-predication of "the self-predication of (x) is not true" is not true'. In 1PLITCH this can be formulated:

- $\mathrm{sp}(<\sim\mathrm{T}(\mathrm{sp}(\mathrm{x}))>) = <\sim\mathrm{T}(\mathrm{sp}(<\sim\mathrm{T}(\mathrm{sp}(\mathrm{x}))>))>$.

This is an identity that can be used in deriving a Liar paradox. It has the same form as: $\mathrm{a} = <\sim\mathrm{Ta}>$, which is an identity premise for a Liar. The constant, 'a' might be interpreted as denoting my favourite sentence; so that this premise represents the identity (made true in a model) that my favourite sentence is 'My favourite sentence is not true'. Nevertheless, the identity statement above that is an instance of the sp function is true by definition of the function. Moreover, consider:

- $\mathrm{sp}(<\mathrm{T}(\mathrm{sp}(\mathrm{x}))>) = <\mathrm{T}(\mathrm{sp}(<\mathrm{T}(\mathrm{sp}(\mathrm{x}))>))>$.

This functional identity has the same form as $\mathrm{e} = <\mathrm{Te}>$, which is an identity premise for a Truth-teller.

This sort of self-predication generally holds in 1PLITCH, as per the following lemma, which we obtain by the definition of the syntactic self-predication function:

LEMMA 6.1 (The Self-predication Lemma). *Given a syntactic function to the effect of self-predication, sp, and canonical naming of a language's expressions in the language, for any open sentence* $\psi(\mathrm{x})$ *in the language, the following identity is true*:

- $\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>) = <\psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>))>$.

[12] Remember our angle bracket expressions are quote names. There is more technical complexity if Gödel numbers are used (Quine, 1995). Eschewing this complexity, quote names are used in our language. They are sufficient to show that the Liar paradox can be defused to a hypodox. I note also that although our results do not depend on arithmetic, they are compatible with arithmetic of course.

An instance, related to Curry's paradox (using the truth predicate), is:

- sp(<T(sp(x)) → Q>) = <T(sp(<T(sp(x)) → Q>)) → Q>.

The above has the same form as:

- c = <Tc → Q>.

The first of these identities is assured in our system.

Some rules are restricted so that non-canonical names for a T-sentence are not swapped for canonical names within the scope of an occurrence of the T-predicate.

(=E) For any closed terms t_1 and t_2 and formula ϕ, provided both t_1 and t_2 are T-free, or t_1 does not occur in the scope of an occurrence of 'T' in ϕ, $\{\phi(t_1), t_1 = t_2\} \vdash \phi(t_2)$,

(∀E) For any formula ϕ, variable v, and closed term γ, $\forall v(\phi(v)) \vdash \phi(\gamma)$, where γ replaces all and any v in ϕ provided γ is not a canonical name for a T-sentence or no occurrence of the variable v occurs in the scope of 'T'.

In other words, the derivational rule for ∀ Elimination is to replace all occurrences of v with the same closed term, any closed term except a canonical name of a sentence containing 'T' if any of those occurrences of v is in the scope of T. The same restriction applies to the rule for ∃ Introduction.

(∃I) For any formula ϕ, variable v, and closed term γ, $\phi(\gamma) \vdash \exists v(\phi(v))$, where v replaces any γ in ϕ unless that occurrence of γ occurs in the scope of 'T' and γ is a canonical name for a T-sentence.

Notice that all classical uses of (=E) are deductively valid in the T-free fragment of 1PLITCH. Thus, the restriction on (=E) does not invalidate any classical theorems in that T-free fragment. A similar comment holds with respect to the quantificational rules of 1PLITCH. The restrictions on (=E) and (∀E) affect only instances involving sentences that were not part of the language of the classical calculus. Nevertheless, the restriction on (=E) is a little over restrictive and two rules are added to compensate.[13]

[13] I would add a third rule in an extension of this system to validate some more inferences that cause no paradoxes but are not validated by our restricted (=E) rule. However, it is a complex rule that we do not need for our present task of defusing Liar-like paradoxes to hypodoxes.

Reflexivity of identity is assured by the standard rule for =I. The deduction system also has a rule for symmetry of identity (Sym=), such that $t_1 = t_2 \vdash t_2 = t_1$, as well as a rule for transitivity of identity:

(Trans=) $\{t_1 = t_2, t_2 = t_3\} \vdash t_1 = t_3$.

Let us consider how the restrictions on our rules affect one method of diagonalization.

LEMMA 6.2 (The Restricted Diagonal Lemma). *Given canonical naming of a language's expressions and a diagonal function to the effect of self-predication,* sp, *then* ***for any T-free*** $\psi(\mathrm{x})$, *there is a closed formula* ϕ *such that:* $\phi \leftrightarrow \psi(<\phi>)$.

PROOF.

(1)	$\psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>)) \leftrightarrow \psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>))$	SL
(2)	$\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>) = <\psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>))>$	Self-predication lemma
(3)	$\psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>)) \leftrightarrow \psi(<\psi(\mathrm{sp}(<\psi(\mathrm{sp}(\mathrm{x}))>))>)$	(1) (2) = E **provided** ψ **is T-free**. ⊣

In particular, let $\psi(\mathrm{x})$ be ∼T(x), then, except that the use of = E in the proof above is invalid for this predicate, there would be some formula, λ, such that:

(5) $\lambda \leftrightarrow \sim\mathrm{T}(<\lambda>)$ – The Restricted Diagonal Lemma*.

This route to the Liar is invalid in 1PLITCH (cf. McGee, 1990, pp. 24–25, Theorems 1.2 and 1.3). Let us consider the Liar-like paradoxes we would defuse to hypodoxes in the next section.

6.3. The problem space and the deductive part of its defusion

Consider table 1 (Eldridge-Smith, 2008, p. 91 ff.). Let the listed identities hold, where Q is some other sentence. Eldridge-Smith (2008) gives formulations of each of these showing that the consequence in the third column follows given the "ordinary laws of logic", as Tarski (1936) puts it, and an instance of his well-known T-schema. Our "problem space" includes formulations using this naive reasoning to derive problematic consequences such as those listed in the table. Given this naive reasoning and the identities of sentences a through d in table 1:

Common name	Identity	Consequence	Given Q	Given ∼Q
Liar	a = <∼Ta>	Ta ↔ ∼Ta	over-determined	over-determined
Unquantified Epimenides	b = <∼Tb ∧ Q>	∼Tb ∧ ∼Q	Over-determined	b is false
Curry	c = <Tc → Q>	Tc ∧ Q	c is true	over-determined
ESP paradox / hypodox	d = <∼Td ↔ Q>	∼Q	over-determined	under-determined
Truth-teller	e = <Te>	T<Te> ↔ Te	under-determined	under-determined
Epimenidean Truth-teller	f = <Tf ∨ ∼Q>	Tf ∨ Q	under-determined	f is true
Curried Truth-teller	g = <Tg ∧ ∼Q>	∼Tg ∨ ∼Q	g is false	under-determined
ESP hypodox / paradox	h = <Th ↔ *Q* >	Q	under-determined	over-determined
Intrinsic truth	i =<Ti ∨ ∼Ti>	Ti ∨ ∼Ti	i is true	i is true
intrinsic falsity	j = <Tj ∧ ∼Tj>	∼Tj	j is false	j is false
Simple Liar	l = <Fl>	Tl ↔ Fl	over-determined	over-determined

Table 1. Some self-referential unquantified sentences in the problem space

> The Liar proves that sentence a is both true and false, whereas sentence b is provably false and therefore ∼Q, and sentence c is provably true, so Q follows; but d can be true or false and Q still follows!
> (Eldridge-Smith, 2008, p. 91)

For each row, in each circumstance in the table that is overdetermined, the sentence named by that letter is paradoxical; and in each circumstance that is underdetermined, the sentence named by that letter is hypodoxical.

For each of the above overdetermined cases, a derivation either uses the identity directly or uses an instance of the diagonal lemma.[14] However, as per the previous section, an instance of the diagonal lemma is

[14] Some other informal proofs in the literature appeal to 'meaning', stipulating that there is a sentence λ that means that 'λ' is not true. We instead restrict our attention to formal proofs appealing to substitution of identicals or diagnonalization. Still other proofs in the literature use stipulative definitions where we are using identities. Nevertheless, our formal language and system has been specified to use identities rather than definitions.

only valid if the corresponding derivation using (=E) is valid. That use of (=E) is invalid in our system. Here is a multi-conclusion formulation of the Liar — it may conclude with (3), (4), (5), (6), (8), (11), or (12). The derivation (1) to (3) is like that of Tarski's formulation (1936; 1944).

(1)	a = <∼Ta>	Premise
(2)	T<∼Ta>↔ ∼Ta	CT-schema
(3)	Ta ↔ ∼Ta	(1), (2) (=E)*
(4)	Ta ∧ ∼Ta	(3) SL (classical Sentential Logic)
(5)	(Ta ↔ ∼Ta) ∧ ∼(Ta ↔ ∼Ta)	(3) SL
(6)	Q	(4) or (5), SL
(7)	T<∼Ta> ⊕ Ta	(2) SL
(8)	Ta ⊕ Ta	(1), (7) (=E)*
(9)	∼Ta ↔ ∼Ta	SL
(10)	∼T<∼Ta> ↔ ∼Ta	(1), (9) (=E)*
(11)	T<∼Ta> ↔ ∼T<∼Ta>	(2), (4) SL
(12)	Ta ↔ ∼Ta	(1) T-schema*

For Tarski, (3) is a contradiction, since it is the negation of a classical theorem (given double negation). That is, an explicit contradiction, having the form $\phi \wedge \sim \phi$, can be derived as (5). The second conjunct of (5) is a logical theorem (given classical sentential logic); so, (3) is logically equivalent to (5) (given classical sentential logic). Thus, (3) is a contradiction or at least logically equivalent to one, given classical sentential logic. Tarski would restrict a formal language from formulating (1) and (2).

In 1PLITCH, (1) is well-formed. Besides, the Self-predication Lemma 6.1 validates identities with the same form as the identities of a through l in the table. As is well-known, the Liar need not rely on empirical premises. Also, (2) is a valid instance of the CT-schema. 1PLITCH incorporates classical sentential logic. 1PLITCH restricts (=E), so that (3) and its classical equivalent (8) do not follow from (1) and (2), nor from (1) and (7). For the same reason, (11) does not follow in 1PLITCH from (1) and (9). Remember also that Diagonalization is restricted in 1PLITCH (Lemma 6.2) in accord with our conjecture on a dependency of liar paradoxes (Conjecture 6.1).

Finally, the inference of (12) directly from (1) using the unrestricted T-schema is a semantic argument. But let's deal with it here. Seman-

(1)	k = <To>	Premise
(2)	n = <∼Tk>	Premise
(3)	o = <Tn>	Premise
(4)	T<To> ↔ To	CT-schema
(5)	T<To> ↔ T<Tn>	(3), (4) (=E)*
(6)	T<Tn> ↔ Tn	CT-schema
(7)	T<To> ↔ T<∼Tk>	(5), (6) SL, (2) (=E)*
(8)	T<∼Tk> ↔ ∼Tk	CT-schema
(9)	T<To> ↔ ∼T<To>	(7), (8) SL, (1) (=E)*

Table 2.

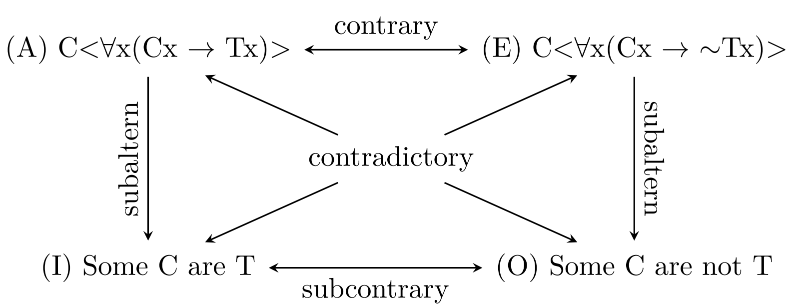
tically, (1) assures that ‘a’ is a name for ‘∼Ta’, albeit a non-canonical name. Given that ‘a’ names that sentence, (12) is an instance of the unrestricted T-schema, but it is not an instance of the CT-schema. This unrestricted use of the T-schema is invalid in 1PLITCH.

THEOREM 6.1. *Each of* Ta, Tb, Tc, Td, Te, Tf, Tg, Th, Ti, Tj *and* Fl *are underdetermined by the deductive system of 1PLITCHv4.*

PROOF. As per (3) above, the consequence listed in the table for the Liar is naively valid but invalid in 1PLITCH, viz. Ta ↔ ∼Ta. Similar formulations that would derive the consequences listed for identities b, c, d, f, g, h, j and l are invalid in 1PLITCH. For example, {c = <Tc → Q>, T<Tc → Q> ↔ (Tc → Q)} ⊬ Tc ∧ Q, where ‘⊢’ represents the consequence relation for 1PLITCH. The consequences tabled for e and i are theorems of 1PLITCH. Nevertheless, in 1PLITCH, Ti ∨ ∼Ti ⊬ Ti. This is clearly because of the restriction on (=E). Indeed, for this reason, 1PLITCH’s deductive system underdetermines each of Ta, Tb, Tc, Td, Te, Tf, Tg, Th, Ti, Tj and Fl regardless of whether Q or ∼Q is the case, given the tabled identities and corresponding instances of the CT-schema. For example, ∼(Tj ∧ ∼Tj) ⊬ ∼Tj. Moreover, Tj ⊬ T<Tj ∧ ∼Tj>. 1PLITCH does not determine any of Ta, Tb, Tc, Td, Te, Tf, Tg, Th, Ti, Tj and Fl by any of the means in the formulation of the Liar above, even if Q or ∼Q is given as a premise. The reader can work through the details modelled on the multi-conclusion formulation of the Liar. Given Conjecture 6.1, 1PLITCH does not determine any of them by any other means. ⊣

“Chain Liars” are avoided by the same restrictions, as per the example in table 2.

Thanks to the work of Prior, Geach and Church, we have a fair summation of the variations of the Cretan Liar paradox and how to generalize them (Prior, 1958, 1961). Eldridge-Smith (2008, 2022) gives hypodoxes corresponding to the original Epimenides and Geach's variation. Supposing all other Cretan statements to be false, then Epimenides the Cretan's statement (E) is paradoxical. However, if in those circumstances, Epimenides had instead said (I) 'Some Cretan statement is true', that statement would be hypodoxical. In contrary circumstances, supposing all other Cretan statements to be true, then if Epimenides had said (O) 'Some Cretan statement is false', this is paradoxical in those circumstances. This is Geach's variation. Eldridge-Smith also points out a hypodoxical dual for Geach's variation. If in those circumstances, Epimenides had instead said (A) 'Cretans always tell the truth', that statement would be hypodoxical. Had there been no other Cretan statement and Epimenides said just one of these things, then if it were (E) or (O) it would be paradoxical, or if it were (A) or (I) it would be hypodoxical. Here are these contingently paradoxical and hypodoxical statements arranged in a square of opposition.



While these scenarios and being able to pair all the above paradoxes and hypodoxes is interesting, that is a focus of Eldridge-Smith (2008, 2022). This article is focused on defusing the paradoxes to hypodoxes. Towards that end, I want to highlight some classical consequences of these paradoxes. Consequence 2 is commented on by Church (1946).

Common name	Premise	Consequence 1	Consequence 2
Epimenides	C<$\forall x(Cx \to \sim Tx)$>	$\sim \forall x(Cx \to \sim Tx)$	$\exists x(Cx \land Tx)$
Geach	C<∃x(Cx ∧ ∼Tx)>	T<∃x(Cx ∧ ∼Tx)>	∃x(Cx ∧ ∼Tx)

By Church's (1946) lights it seems paradoxical to conclude there is a true Cretan statement merely from the premise that a Cretan says that all Cretan statements are not true. (One can naively derive a contradiction if circumstances are such that all other Cretan statements are not true.) Here is a formulation of Church's variation of the Epimenides showing where an inference using (∀E) is invalid in 1PLITCH. This is typical of the way in which 1PLITCH defuses variations of the Epimenides.

(1)	$C<\forall x(Cx \to \sim Tx)>$	Premise
(2)	$T<\forall x(Cx \to \sim Tx)> \leftrightarrow \forall x(Cx \to \sim Tx)$	CT-schema
(3)	$\forall x(Cx \to \sim Tx)$	Assumption
(4)	$C<\forall x(Cx \to \sim Tx)> \to \sim T<\forall x(Cx \to \sim Tx)>$	(3) (∀E)*
(5)	$\sim T<\forall x(Cx \to \sim Tx)>$	(1), (4), →E
(6)	$\sim(\forall x(Cx \to \sim Tx))$	(5), (2), SL
(7)	$\sim(\forall x(Cx \to \sim Tx))$	(3)-(6) ∼I [3]
(8)	$\sim T<\forall x(Cx \to \sim Tx)>$	(2), (7) SL
(9)	$\exists x(Cx \wedge Tx)$	(7) PL

There are other examples, of course, not the least is Yablo's Paradox. (For hypodoxes for this and other examples using lists of sentences referring to each other, please see (Eldridge-Smith, 2008).)

$(Y_1)\ \forall k > 1(\sim T(Y_k))$.
$(Y_2)\ \forall k > 2(\sim T(Y_k))$.
$\vdots$
$(Y_n)\ \forall k > n(\sim T(Y_k))$.
$\vdots$

Given such a list, part of the paradox may be derived as in table 3. Note that our language $\mathcal{L}$ cannot express this sort of quantification. (Also, to avoid confusion, this derivation uses quote marks for canonical naming.)

If our system was extended to represent this sort of quantification, lines (4) and (5) above show how the restrictions on our rules of inference would restrict the derivations of consequences that would otherwise lead to paradox.

In considering how 1PLITCH addresses the problem space of liar-like paradoxes, first, we have been inspired by Church to identifying paradoxical consequences that follow by naive reasoning involving the truth

(1)	$\forall k > n\ (\sim\mathrm{T}(\mathrm{Y}_k))$	Assumption
(2)	$\forall k > (n+1)\ (\sim\mathrm{T}(\mathrm{Y}_k))$	from (1) by arithmetic
(3)	$\mathrm{T}`\forall k > (n+1)\ (\sim\mathrm{T}(\mathrm{Y}_k))' \leftrightarrow$ $\forall k > (n+1)\ (\sim\mathrm{T}(\mathrm{Y}_k))$	CT-schema
(4)	$\mathrm{T}`\forall k > (n+1)\ (\sim\mathrm{T}(\mathrm{Y}_k))'$	(3), (2), SL
(5)	$\mathrm{T}(Y_{n+1})$	identity of list entry (Y_{n+1}), (4) (=E)*
(6)	$\sim T(Y_{n+1})$	(1) (∀E)*
(7)	$\sim (\forall k > n(\sim T(Y_k)))$	(1)-(6) ~I [1]
(8)	$\exists k > n(T(Y_k))$	(7) PL

Table 3.

predicate. Secondly, we have argued the deductive system of 1PLITCH avoids these consequences. Thirdly, we have demonstrated that a representative variety of unquantified variations of the Liar are underdetermined by this deduction system. Fourthly, in virtue of paradoxical consequences being avoided for a selection of quantified variations of the Liar that are expressible in 1PLITCH, I conjecture that quantified variations are also underdetermined.

6.4. Semantics

The semantics is designed to assure the inferences of 1PLITCH, including their restrictions. The semantics is standard for atomic formula without the truth predicate. It is also standard for disjunction and negation, except that the rules are expressed so that they cannot be instantiated using a non-canonical name for a sentence. As is standard, $\alpha = \beta$ is true iff those terms denote the same thing in the domain. The domain, D, includes the well-formed formulas of the language (wffs). So the domain is a set of objects and wffs, but the domain itself does not include names of either of these. An interpretation, I, assigns an interpretation to individual constants using a denotation function, den(). The denotation of canonical names is fixed; each canonical name denotes the well-formed formula for which it is a syntactically well-formed canonical name. In other words, for any formula, ψ, $\mathrm{den}(<\psi>) = \psi$. Also, the denotation of sp-terms is fixed in the semantics by rule 1b below, so that it accords with the sp syntactic function. A model of $\mathcal{L}$, M, determines which members of the domain satisfy T-free atomic formulas, that is, satisfy the n-place relations that are assigned to predicates by an interpretation, I. A set of sentences, E, is an extension of truth for $\mathcal{L}$, given M. The set E includes all atomic sentences made true by the model as well as sentences true in accord with the semantic rules below, sentences such as a Truth-teller may or may not be in E, as M does not determine this. Nevertheless, it is either in E or not. The semantic rules can be applied consistently either way. Accordingly, it can be asserted in the metatheory that a Truth-teller sentence is hypodoxical. Moreover, the Liar sentence may or may not be in E, and the semantic rules can still be applied consistently either way. Accordingly, it can be asserted in the metatheory that the Liar sentence is hypodoxical. Indeed, the semantic rules are such that each of Ta, Tb, Tc, Td, Te, Tf, Tg, Th, Ti, Tj and Fl could be consistently in E or not. Each is hypodoxical.

It must be borne in mind that E is a set of sentences, not names of sentences. Thus, if ‘a’ is a non-canonical name for ‘∼Ta’, ‘a’ cannot be a member of E because ‘a’ is a term, not a sentence. The sentence ‘Ta’ is either in E or it is not. If it is, that sentence is true; if it is not, ‘∼Ta’ is true and is in E. It will become apparent that the rule governing this, which corresponds to the CT-schema, also maintains our restriction on substitution of identicals in such cases. That is an interesting feature of the semantics for 1PLITCHv4. The difference among the hypodoxes in this semantics between Truth-teller hypodoxes and Liar hypodoxes is also interesting, and we will return to this when comparing the classification of hypodoxes in 1PLITCHv4 with another classification.

Here are our semantic rules with (labels), for a model, M, and set of sentences, E:

1. (Basis) For any atomic sentence, $\psi(t_1, t_2, ..., t_n)$, that is not an atomic T-sentence, $\psi(t_1, t_2, ..., t_n)$ is true in (M,E) iff $(\text{den}(t_1), \text{den}(t_2), \ldots, \text{den}(t_n)) \in I(\psi)$.

1a. In particular, for any atomic identity sentence and terms t and r, $t = r$ is true in (M, E) iff $\text{den}(t) = \text{den}(r)$.

1b. Also, any identity syntactically assured by the definition of the sp-function is true in the semantics. That is, for any one-place formula $\psi(\text{x})$, $\text{den}(\text{sp}(<\psi(\text{x})>) = \text{den}(<\psi(<\psi(\text{x})>)>)$.

2. (Sync) For any sentence (i.e. closed formula) ϕ for which $<\phi>$ is a canonical name, ϕ is true in (M,E) iff $\text{den}(<\phi>) \in \text{E}$.

3. (CT) For any sentence ϕ for which $<\phi>$ is a canonical name, $\text{T}<\phi>$ is true in (M,E) iff $\text{den}(<\phi>) \in \text{E}$.

4. (Negation) For any sentence ϕ for which $<\phi>$ is a canonical name, $\sim\phi$ is true in (M, E) iff ϕ is not true in (M,E).

5. (Disjunction) For any sentences ϕ and ψ for which $<\phi>$ and $<\psi>$ are respectively canonical names, $(\phi \vee \psi)$ is true in (M,E) iff ϕ is true in (M,E) or ψ is true in (M,E).

6. (Sentential definitions) Remember other sentential connectives are defined syntactically using $\sim$ and $\vee$.

7. ($\forall$). If in $\phi(v)$ no occurrence of v is in the scope of a T-predicate, $(\forall v)\phi(v)$ is true in (M,E) iff, for each closed term γ, $\phi(\gamma)$ is true in (M,E). If in $\phi(v)$ some occurrence of v is in the scope of a T-predicate, $(\forall v)\phi(v)$ is true in (M,E) iff, for each closed term γ that is not a canonical name of a sentence itself containing T, $\phi(\gamma)$ is true in (M,E).

8. ($\exists$ definition) Remember existential quantification is defined syntactically using $\sim$ and universal quantification.

The (Sync) rule assures members of E as truths, and that any sentence made true by the model, M, is a member of E. The (CT) rule assures that any sentence ϕ has the same truth-value as T$<\phi>$. The (CT) rule and the (Negation) rule avoid substitution of identicals by referring to a sentence using its canonical name on the right-hand-side of their biconditionals. The (CT) rule together with (Negation), (Disjunction) and (Sentential definitions) validate every instance of the CT-schema. Otherwise, use of these rules is reasonably clear.

In the semantics of PLITCHv4, each sentence, ϕ, and a sentence T$<\phi>$ have the same semantic value. However, if γ is a non-canonical name for the sentence ϕ, it will not necessarily follow that Tγ has the same value as ϕ. Here PLITCH differs from many other theories. If in the model, M, circumstances are true such that ϕ would have been naively liar-paradoxical, it is now hypodoxical in 1PLITCHv4: it is underdetermined by the deduction system (if our Conjecture 6.1 is granted), and either is a member of E or not, and it could be either. Moreover, if ϕ would have been naively liar-paradoxical, ϕ and T$<\phi>$ have the same semantic value but Tγ has the other semantic value. Furthermore, even though $(\gamma = <\phi>) \in$ E, the semantic rules do not license using substitution of identicals to obtain a contradiction from this and the deduction rules invalidate its use in such a case.

PLITCHv4 defuses the Liar to a hypodox. It validates all sentences and inferences that were valid in T-free classical logic (for $\mathcal{L}$). Nevertheless, it falls short as a theory of truth though. Without going into details, it needs further extension to validate some more inferences involving T-sentences that would not reintroduce paradox (Eldridge-Smith, 2020). It would also be ideal to further extend it to resolve the hypodoxes. Still, PLITCHv4 demonstrates that the Liar can be defused to a hypodox in a formal system.

6.5. Revenge Stratagems

Two typical stratagems for reformulating a Liar paradox against a particular solution are strengthening (which often involves using an exclusive negation) and forming a particular disjunction (cf. Rossi, 2019, pp. 213–214).

1PLITCH uses classical negation. I have used a strengthened form of the Liar as a standard example. ‘Is false’ is simply defined as ‘is not true’. Accordingly, $l = <Fl>$, is treated in much the same way as $a = <{\sim}Ta>$. Moreover, I have an objection to defining another falsity predicate, F^*, as truth of the negation, so that $F^*<A>$ iff $T<{\sim}A>$. $l_1 = F^*(l_1)$ is not syntactically well-defined. That is, $T({\sim}l_1)$ is syntactically ill-formed.[15]

The other typical stratagem for finding a Revenge Liar for a particular solution is to use its delimiting concept in a disjunctive predicate with ‘... or is not true’. If it is claimed that the Liar sentence is meaningless, then what about ‘This sentence is either meaningless or is not true’? If it is claimed that the Liar sentence lacks a truth-value (i.e. has a truth-value gap), what about ‘This sentence lacks a truth-value or is not true’? If it is claimed the Liar sentence is indeterminate, what about ‘This sentence is either indeterminate or is not true’? If that sentence is indeterminate then it is true; but if it is true, it is not indeterminate. So it is not indeterminate. But then, if it is true, it is not true; and if it is not true, it is true. If these sentences are not expressible in the object-language implementing the solution, they are nevertheless expressible in a metatheory about that solution using natural language as a metalanguage. Let us consider such an example for our present theory or some extension of it. My least favourite sentence is ‘My least favourite sentence is either underdetermined or not true’. Let U be a predicate ‘... is underdetermined’. Consider the argument from table 4.

For this derivation, the inferences of lines 6, 12 and 14 would be invalid in an extension of 1PLITCH. While this is not a full defence against any conceivable attempt to formulate a revenge Liar paradox, it does indicate how the defusion of the Liar to a hypodox could be maintained in response to similar attempts.

[15] This objection relates to another argument for my restriction on substitution of identicals given in (Eldridge-Smith, 2020).

(1)	$u = <Uu \vee \sim Tu>$	Premise (where U is a predicate)
(2)	$T<Uu \vee \sim Tu> \leftrightarrow (Uu \vee \sim Tu)$	CT-schema
(3)	Uu	Assumption
(4)	$Uu \vee \sim Tu$	3 SL
(5)	$T<Uu \vee \sim Tu>$	2, 4 SL
(6)	Tu	1, 5 (=E)*
(7)	$\sim Uu$	6, If a sentence is true, it is not underdetermined.
(8)	$\sim Uu$	3-7 ~I [3]
(9)	$T<Uu \vee \sim Tu>$	Assumption
(10)	$Uu \vee \sim Tu$	9, 2, SL
(11)	$\sim Tu$	10, 8 SL
(12)	$\sim T<Uu \vee \sim Tu>$	11, 1 (=E)*
(13)	$\sim T<Uu \vee \sim Tu>$	9-12 ~I [9]
(14)	$\sim Tu$	13, 1 (=E)*
(15)	$Uu \vee \sim Tu$	14 SL
(16)	$T<Uu \vee \sim Tu>$	2, 15 SL
(17)	Tu	1, 16 (=E)

Table 4.

6.6. Comparison with a Fixed-point classification

A Liar-like sentence in 1PLITCHv4 is a hypodox. If t_n is a non-canonical name for a sentence ϕ, then if ϕ was a liar-paradoxical sentence, then in 1PLITCHv4, Tt_n has a different truth-value than $T<\phi>$, however, both are underdetermined by M. In some cases, this is given certain circumstances. Namely, the circumstance in which it was naively paradoxical. In 1PLITCHv4, if ϕ is a Truth-teller hypodox, then Tt_n has the same truth-value as $T<\phi>$, however, both are underdetermined by M. Intrinsic truths are such that $T(t_n)$ is underdetermined by M, while ϕ is true; and intrinsic falsehoods are such that $T(t_n)$ is undetermined by M, and ϕ is false. 1PLITCHv4 thus exemplifies defusing Liar-like paradoxes to hypodoxes in a formal system. It meets the objective of this article. I note that I would recommend extending 1PLITCHv4 before discussing what such a system means for a theory of truth. Nevertheless, we can compare the above categories with a more sophisticated classification.

In a Kripkean fixed-point semantics, the fixed points are partially ordered. At the lowest fixed point, grounded sentences are true(T) xor false(F), and ungrounded sentences are in a gap or take a third value.

Let us say, following others, that those sentences that do not take a value of T or F at a fixed-point are *pathological* (ρ). The pathological sentences themselves are classified according to the semantic values they can take in certain higher fixed points. In connection with classifying pathological sentences, a number of researcher's have progressed adding intensional operators to a fixed-point semantics, including (Rosenblatt and Szmuc, 2014) and (Tourville and Cook, 2020). I will make a brief comparison with Cook's recent classification of pathological sentences.

Cook (2020, 2022)) gives a classification of intensional statuses of pathological sentences based on the semantic statuses a pathological sentence eventually obtains at higher fixed points. The characterisation of these intensional statuses is a sophisticated piece of work. I will give only a simplified characterisation in table 5, merely sufficient for our present comparison. Let σ be an assignment function that is used to designate an indefinite fixed point in a fixed-point semantics. Also, let σ^* be the valuation of sentences and names of sentences at σ (Cook, 2022, p. 86). Accordingly, '$\exists\sigma\sigma^*(\gamma) = $ T' says that there is a fixed-point σ such that the sentence that 'γ' denotes is true at the fixed point σ. Fixed-points are partially ordered, with a minimal or initial fixed point, for a model, M. Moreover, let σ_1 and σ_2 be fixed points such that $\sigma_2 \geq \sigma_1$. In the following table, what is listed in a row as the basis for an example sentence belonging to a classification can be applied generally to other sentences belonging to that classification.

All of Ta, Te, Ti, Tj, Tc$_c$, Tb$_c$, Tu$_c$ and their negations are hypodoxical in 1PLITCHv4. This reflects the simplicity of 1PLITCHv4, for better or worse (cf. Rosenblatt and Gallovich, 2022). Depending on one's purpose, a four-valued system might provide a better comparison (Da Ré et al., 2020). Nevertheless, 1PLITCHv4, having so restricted substitution of identicals, the truth values of sentences like Ta, Ti, Tj, Tc$_c$, Tb$_c$, and Tu$_c$ may differ from the truth values of sentences using canonical names. Both 1PLITCHv4 and Cook's semantics are intensional semantics, and yet both are also compositional.

Since all sentences like Ta through Tl and Tc$_c$, Tb$_c$ and Tu$_c$ are hypodoxical and thus are either true or not true in the semantics of 1PLITCHv4, any complex sentence of which they are truth-functional components will also be either true or not true in the semantics of 1PLITCHv4. For example, if Te$_1$ and Te$_2$ are both true in (M,E), then $\text{T}<(\sim\text{Tu}_c \vee \text{Te}_1) \wedge \text{Te}_2>$ will be true in (M,E).

There is nevertheless some accord between the two classifications over

Classification	Example	Identity	basis
Paradoxical	Liar	$a = <\sim Ta>$	$\forall\sigma(\sigma^*(a)=\rho)$, a is pathological at every fixed point
Semi-classical	Truth-teller	$e = <Te>$	$\exists\sigma(\sigma^*(e)=T)$ and $\exists\sigma(\sigma^*(e)=F)$ and $\forall\sigma_1(\exists\sigma_2(\sigma_2{}^*(e)=T$ xor $\sigma_2{}^*(e)=F)$
Semi-true/ Strictly unboundedly true	Tautology-teller	$i =<Ti \vee \sim Ti>$	$\forall\sigma_1(\exists\sigma_2(\sigma_2{}^*(i)=T))$
Semi-false/ Strictly unboundedly false	Contradiction-teller	$j = <Tj \wedge \sim Tj>$	$\forall\sigma_1(\exists\sigma_2(\sigma_2{}^*(j)=F))$
Non-true		$c_c = <\sim Tc_c \vee Te>$	$\exists\sigma(\sigma^*(c_c)=\rho)$ and $\exists\sigma(\sigma^*(c_c)=F)$ and $\forall\sigma_1(\exists\sigma_2(\sigma_2{}^*(c_c)=\rho$ xor $\sigma_2{}^*(c_c)=F))$
Non-false		$b_c = <\sim Tb_c \wedge Te>$	$\exists\sigma(\sigma^*(b_c)=\rho)$ and $\exists\sigma(\sigma^*(b_c)=T)$ and $\forall\sigma_1(\exists\sigma_2(\sigma_2{}^*(b_c)=\rho$ xor $\sigma_2{}^*(b_c)=T))$
Unstable	Instability-teller	$u_c = <(\sim Tu_c \vee Te_1) \wedge Te_2>$, where $e_1 = <Te_1>$ and $e_2 = <Te_2>$	$\exists\sigma_1(\forall\sigma_2\geq\sigma_1)(\sigma_2{}^*(u_c)=\rho)$ and $\exists\sigma_1(\forall\sigma_2\geq\sigma_1)(\sigma_2{}^*(u_c)=T)$ and $\exists\sigma_1(\forall\sigma_2\geq\sigma_1)(\sigma_2{}^*(u_c)=F)$

Table 5. Cook's classification of pathological sentences in an intensional fixed-point semantics

the semi-true, -false and -classical with respect to the canonical names of such expressions. Sentences that are semi-classical in an intensional fixed-point semantics are a subclass of the hypodoxes in 1PLITCHv4. Sentence i is semi-true in an intensional fixed-point semantics; by comparison in 1PLITCHv4, Ti is hypodoxical, $Ti \vee \sim Ti$ is a valid theorem, and $T<Ti \vee \sim Ti>$ is true. In the case of j, it is semi-false in an intensional fixed-point semantics; by comparison in 1PLITCHv4, Tj is hypodoxical, $Tj \wedge \sim Tj$ is false, and $T<Tj \wedge \sim Tj>$ is false.

7. A philosophical critique of various solutions to Bertrand's chord paradox referencing our classification of results and demonstrating an advantage of defusing it to a hypodox as an intermediate step towards Jaynes' solution

Let us consider Bertrand's chord paradox (Bertrand, 1889, p. 4).

> What is the chance that a randomly inscribed chord of a circle is longer than the side of an inscribed equilateral triangle?
>
> (Clark, 2012, p. 22)

Using three methods of randomly drawing such a chord, Bertrand demonstrates that the probability is a half, a third and a quarter. Thus, there are at least three different answers. The paradox fits a paradoxical dilemma. The definite description implies uniqueness: there is just one probability that a random chord of a circle is longer than an equilateral triangle inscribed with its points on the circle. However, there are mutually exclusive answers. Furthermore, there are reasons supporting each of these answers that the probability is a half, a third and a quarter. The set of these answers and the statement that there is just one such probability form a paradoxical set of the form of (PD), even though there are at least three alternatives in this case. (There may be arguments for other alternatives, but that will make no essential difference to our commentary.) Let the probability in question be P(Q). There are arguments for each member of {P(Q) is a half xor a third xor a quarter xor some other value, P(Q) = half, P(Q) = third, P(Q) = quarter}.

Bertrand himself considers the problem ill-posed. His thought seems to be that a principle of probability, the principle of indifference, fails to apply in this case even though it seems as though it does. There are a number of ways this could be so. Here are three of them. First, I take Bertrand to mean that the problem actually has no answer (None). Secondly, the application of the principle of indifference is refuted by the contradiction itself. This option is expounded by Shackel (2007). I will take issue with this option shortly. Thirdly, some argue that the principle of indifference can only apply given an additional stipulation about how the probability is to be calculated. 'The trouble with the original question is that it fails to specify *how* the chord is to be randomly selected' says Clark (2012, p. 23). Thus, Clark supports Marinoff's (1994) solution, which Shackel summarizes as follows. 'The claim is that the paradox poses a problem whose identity is indeterminate, and which can be resolved into a number of distinct determinate problems which are not themselves ill-posed in the primary sense' (Shackel, 2007, p. 163). I take this strategy to be first reducing the paradox to an indeterminate result, such as (MinI) or (MaxI) in our hendecagon; then making a distinction to obtain determinate results similar to (DIS) in our hendecagon. This approach has two inadequacies. First, if the issue is that the probability is indeterminate, then neither restricting nor adding a principle should resolve the issue. That is, strictly speaking 'indeterminate' implies there is no way of determining the probability. Yet making a distinction is in effect restricting principles or adding a principle. At least it could be construed that way. The matter would be quite different if the problem posed were first construed as a hypodox, that is, if the issue was that the probability is underdetermined for lack of a principle determining the matter. Then a distinction could be made that had the nett effect of adding a principle to resolve the hypodox. Secondly, simply saying 'When the method of random selection is adequately specified, a determinate answer is available' (Clark, 2012, p. 24), only partially addresses the issue. A complete solution should explain *why* the method of random selection needs to be specified. (Clark (2012, p. 255) himself makes essentially this same point about what is required of a solution to the Two Envelope Paradox.)

There are other options for resolving this paradox. If an alternate solution could explain why the presumption that there is one probability is unreasonable, one could argue for rejecting the alternation but retaining multiple values for P(Q) yielding a result like (Both) in our list.

Jaynes (1973) takes a somewhat different approach. He defends a determinate solution, a (Det) result in our hendecagon. He considers that the problem is after all well-posed. He appeals to additional principles (unknown to Bertrand and not accepted by Clark and Marinoff) concerning maximum entropy and transformation groups (Jaynes' own enhancement of the principle of indifference). Jaynes argues that there is a unique solution and that the other answers are incorrect. That is, Jaynes' solution entails the alternation, because he upholds one answer (a half) and finds fault with the reasoning for the other erstwhile purported answers. To correctly calculate the probability in question a chord drawn at random must have a uniform chance of being drawn. The fault he finds is that the other methods do not actually uniformly distribute the chance of drawing any individual chord, whereas he argues that the method supporting a probability of a half does uniformly distribute the chance of drawing any individual chord.

Jaynes' proposed solution seems to demonstrate how a paradox could be resolved by deploying or accepting additional principles; but how would this be a viable solution, given that a paradox is *already* incompatibly overdetermined? Jaynes' additional principles purportedly show that all but one answer is actually wrong. Given the relevant alternation, these additional principles, if accepted, would be ones that constrain the use of other principles or methods that when unconstrained result in incompatible conclusions. Although conceivable this would require non-monotonic reasoning. To see this, consider a case where A xor B is accepted. Principle P1 is invoked in an argument for A, while principle P2 is invoked in an argument for B. Both P1 and P2 are granted principles (in the case in question they are reasonable methods of random distribution). Then, it is questioned whether these arguments meet some proposed additional criterion, based on acceptance of an additional principle P3. In the case being considered, this is whether the probability of drawing a chord is uniformly distributed based on acceptance of the principles of maximum entropy and transformation groups. Finally, it is argued that P1 and not P2 conforms to this additional criterion.

One could try to argue this is simply a case of pointing to a fallacy in the reasoning for other answers. I think this oversimplifies the matter. Fallacious reasoning is proven by giving an analogous counterexample that exposes the fallacious reasoning or by justifying an exception (as in the case of division by 0). What happens, if one accepts Jaynes' solution, is that one accepts an additional criterion for random selection because one accepts that this follows from accepting the principles of maximum entropy and transformation groups. Apparently, these additional principles only make a difference in indenumerable sample spaces, or even more particularly Bertrand-like paradoxes. Jaynes' solution is not advanced by giving an independent counterexample, but by arguing for the relevance of these principles and that these principles (not some counterexample) imply that all but one answer to Bertrand's conundrum are wrong.

However, if our reasoning is monotonic, how can additional principles void a contradiction obtained with less principles? There seem to be two conceivable explanations. First, perhaps Jaynes was not accepting a new principle or extending the principle of indifference but was replacing that concept. Then, the other methods of calculating the probability were incorrectly validated by the principle of indifference. That is, Jaynes's revised principles invalidate the reasoning for all but one competing answer. This is perhaps what Jaynes has in mind. But if this is correct, Jaynes is committed to his revised principles being the principles that actually validate all cases where the principle of indifference was considered adequate. Secondly though, it is conceivable that the principle of indifference is not sufficient to determine this case involving indenumerable possibilities. That is, the problem is a hypodox potentially resolved by Jayne's additional principles. The resolution is then done in two steps. First, argue that the case goes beyond determination by the principle of indifference, that is, it is consistently underdetermined by that principle alone, and so is hypodoxical. This corresponds to an (DH) result, effectively *defusing* the paradox to a hypodox. Then secondly Jaynes's solution resolves this hypodox by accepting additional principles to uniquely determine the matter.

Shackel (2007) criticises both Marinoff's and Jaynes's approaches, and argues that the paradoxical argument *undermines* the principle of indifference. He says he supports Bertrand's original position that the problem is ill-posed. The paradoxical argument shows that the principle of indifference will not yield a unique probability. How on the one hand can Shackel say the problem is ill-posed and yet on the other hand admit the paradoxical argument? For Shakel, the argument is a *reductio* of the principle of indifference (given that Shackel rejects Marinoff-style disambiguation in general). This is confirmed by Shackel in his abstract. Given the principle, one has a contradiction, on which basis the principle is refuted, says Shackel. If Shackel is correct, the paradox is veridical, like the Barber, in terms of Quine's classification. His result would be an (RSC) result, relative to our list of results, if it were a reductio of an assumption or purported circumstance. That is, if there were some additional justification for thinking the principle in question had an exception or some clear counterexample. Without that, I think using a paradoxical argument to refute an accepted principle, the principle of indifference, is *ad hoc*. Besides, given Jayne's additional principles, one does not have a contradiction and therefore has no basis for a reductio argument.

Whether or not any of these solutions is correct, each is an example of a result of resolving a paradox or a hypodox. I noted that if it is a paradox, Jaynes' solution needs a justification for why accepting an additional principle will reduce the number of good answers to just one. I am arguing that this can be done in two steps though. First, arguing for the insufficiency of existing principles to conclusively determine the matter, so that as it stands it is underdetermined and hypodoxical. Secondly, arguing for additional or extended principles that determine just one answer, and in this way resolving the hypodox. Although Jaynes himself did not put it this way, Jaynes (1968) can be reconstrued as arguing for the insufficiency of the principle of indifference in cases like Bertrand's paradox, and then in his (1973) arguing that his extended principles address this.

Interestingly, Jaynes himself thought that such paradoxes are best considered as *overdetermined* rather than *underdetermined*, modelled on another paradox for which he did not have a solution,

> von Mises' water-and-wine problem [...] Here we are told that a mixture of water and wine contains at least half wine, and are asked: What is the probability that it contains at least three-quarters wine? On the usual viewpoint this problem is underdetermined; nothing tells us which quantity should be regarded as uniformly distributed. However, from the standpoint of the invariance group, it may be more useful to regard such problems as *overdetermined*; so many things are left unspecified that the invariance group is too large, and no solution can conform to it. (Jaynes, 1973, p. 490 with his italics)

However, finding a solution to this problem *as a hypodox* may point to another additional principle required. Mikkelson (2004) takes this approach of accepting an additional principle to resolve this problem. Although Mikkelson does not call the problem *underdetermined* and a *hypodox*, I take the liberty of suggesting that his solution to von Mises water into wine conundrum may be construed as a solution to this problem as a hypodox. *Pace* Jaynes, viewing this problem as *inconsistently overdetermined* does not allow this option for resolution, unless one's reasoning is non-monotonic. The choice between a paradox (PD) and its incompatible hypodox (DH) is exclusive. A distinct conundrum like Bertrand's Chord problem or von Mises Water into Wine problem cannot at once be both a paradox and a hypodox.

Let us review the above considerations in comparison to the Ship of Theseus paradox. The Ship of Theseus is not resolved by the method of making a distinction. On the one hand, it is not sufficient to resolve that paradox to say that *the ship of Theseus* is *underspecified*, and to say one needs to specify the principle (or method) by which the ship is identified, whether as composed of the original planks or being well-maintained. If it is given that two ships have a claim to the label for different reasons, 'the ship of Theseus' is a non-unique identifier (despite its incorporation of ‘the’). Therefore, it does not follow that one individual object both has original planks and has been completely renovated. Most database designers (unlike philosophers) would simply establish a unique key to distinguish the two items named ‘the ship of Theseus’ in their database. But such a “solution” misses the crux of the paradox – it is a paradox of identity over time, not terms and inferences that presume co-reference.

Arguably, as Shackel characterises it, Marinoff’s solution relies on ‘the probability of a randomly inscribed chord of a circle being longer than the side of an inscribed equilateral triangle’ not being a unique identifier and subsequently distinguishing terms for each incompatible answer. In general, a paradox is not fully resolved by disambiguating the name of something where what is expected is an explanation of why there is not a unique *thing*.

On the other hand, arguably, although identity through time is usually consistently determined by constant maintenance of a thing or being composed of original parts, in cases like the Ship of Theseus, these principles seem to inconsistently overdetermine which is the thing. In Bertrand’s Chord Paradox there is a principle, the principle of indifference, that usually consistently determines the answer, but in this case, seems to inconsistently overdetermine which is the answer to this probability conundrum.

As I was saying, if Jaynes had argued first that Bertrand’s chord problem is actually underdetermined by the principle of indifference, Jaynes could then argue that the problem can be determined by accepting additional principles to address the lack of a principle to determine a unique answer. As Jaynes work stands, taken literally, he construes the problem as apparently inconsistently overdetermined and is not using non-monotonic reasoning. Hence, he must argue that his principles replace the principle of indifference.

As Shackel (2007) points out, one expects a solution to such conundrums, not just an *answer*. As a candidate resolution, underspecification seems to be an answer but not a complete solution. As a second candidate resolution, a reductio without some additional justification also seems to be an answer that is an incomplete solution. As a third candidate, Jaynes’ answer would be a solution either if it were first reasonable to *defuse* the paradox to a hypodox (DH) or if he somehow independently justified his additional principles always replacing the principle of indifference (Det). I hope my critique of some proposed solutions to Bertrand’s paradox by comparing their results has illustrated the utility of this partial taxonomy, account of hypodoxes, and how it may sometimes be useful to *defuse* a paradox to a hypodox.

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