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Grounding and Propositional Identity: A Solution to Wilhelm's Inconsistencies

Abstract. By following a recent result of [Wilhelm, 2021], it can easily be shown that standard conditions for immediate partial grounding and relevant identity conditions for propositions are inconsistent with one another. This is an unfortunate situation for all grounding enthusiasts; however, by adopting the approach presented by Poggiolesi [2016a,b], which displays a more-fined grained use of negations, it can also be shown that consistency can be restored back.

Keywords: grounding; propositional identity; inconsistency; consistency

Introduction

In the paper "Grounding and propositional identity", Wilhelm [2021] has shown that standard conditions for immediate partial grounding and relevant identity conditions for propositions are inconsistent with one another. This is of course an unfortunate situation for all grounding enthusiasts. In this paper we first diagnose the source of the inconsistencies and then, by adopting the more recent grounding approach proposed by Poggiolesi [2016b, 2018], we show how to eliminate them.

From inconsistency to consistency

We focus on a formal language L that contains parentheses (,), atoms p, q, ... and the classical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. L contains a binary sentence operator $\approx$ for regimenting claims about ground-propositional identity: intuitively, $A \approx B$ says that A and B are the same

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proposition from a grounding perspective. And L contains the sentence operator $\trianglerighteq$ for regimenting claims about grounding: intuitively, $A \trianglerighteq B$ stands for A immediately partially grounds B. For the purposes of this paper, we do not allow $\trianglerighteq$ and $\approx$ to be in the scope of themselves or each other. As is standard, we assume that $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ obey the usual principles of classical logic, and that $A \approx B$ obeys the usual principles of identity: in particular, it is reflexive and conforms to a version of Leibniz's law.

We first list the grounding conditions and ground-identity conditions for propositions that Wilhelm [2021] uses to prove inconsistency results. For any propositional formulas A, B, C,

(1) $A \trianglerighteq (B \wedge C)$ if, and only if, $(B \wedge C)$ and $A \approx B$ or $A \approx C$,
(2) $A \trianglerighteq \neg(B \wedge C)$ if, and only if, A and $A \approx \neg B$ or $A \approx \neg C$,
(3) $A \trianglerighteq (B \vee C)$ if, and only if, A and $A \approx B$ or $A \approx C$,
(4) $A \trianglerighteq \neg(B \vee C)$ if, and only if, $\neg(B \vee C)$ and $A \approx \neg B$ or $A \approx \neg C$,
(5) $A \trianglerighteq \neg\neg B$ if, and only if, A and $A \approx B$.

In addition, we assume that for some atom p

(6) $p \wedge \neg(p \trianglerighteq p)$.

And finally we have, for any propositional formulas A, B,

(7) $A \wedge B \approx \neg(\neg A \vee \neg B)$,
(8) $A \vee B \approx \neg(\neg A \wedge \neg B)$.

Let us illustrate informally how to generate one of the inconsistencies identified by Wilhelm. At the centre is a tension concerning what (4) and (5) imply about the relationship between p and $\neg\neg p$. It follows from (4) that $\neg\neg p$ is a ground of $\neg(\neg p \vee \neg p)$ and from (1) that p is a ground of $p \wedge p$. However, since, by (7), $p \wedge p \approx \neg(\neg p \vee \neg p)$, it follows from (4) or (1) that $p \approx \neg\neg p$. By contrast, (5) states that p is a ground of $\neg\neg p$. By Leibniz's law, these two conclusions enter into contradiction with the assumption that p does not ground itself, (6). So the inconsistency highlights an equivocation about which 'grounding level' $\neg\neg p$ is on, in comparison with p. And any route out of it will have to clearly take a position on this question: is $\neg\neg p$ on a level higher than p, as implied by (5), or on the same level, as implied by (1), (4) and (7)? The account of grounding developed by Poggiolesi [2016b, 2018] comes down unequivocally in favor of the former, motivated by the counterintuitive consequences that conditions such as (4) generate.

For instance, conditions (4) and (7) imply that $\neg\neg p$ is a ground of $p \wedge p$, which is implausible.

Hence, Poggiolesi proposes to replace (4) by the following condition. For any propositional formulas A, B, C,

(4′) $A \trianglerighteq \neg(B \vee C)$ if, and only if, $\neg(B \vee C)$ and $A \approx B^*$ or $A \approx C^*$,

where the symbol $*$ is defined in the following way:

DEFINITION. For any propositional formulas A, A^*, is defined as follows:

$$A^* = \begin{cases} \neg^{n-1}E, & \text{if } A = \neg^n E \text{ and } n \text{ is odd} \\ \neg^{n+1}E, & \text{if } A = \neg^n E \text{ and } n \text{ is even} \end{cases}$$

where the main connective in E is not a negation, $n \geqslant 0$ and 0 is taken to be an even number.

Condition (4′) coincides with (4) in cases such as $\neg(p \vee q)$, identifying the partial grounds as $\neg p$ or $\neg q$. However it diverges precisely on cases such as $\neg(\neg p \vee \neg q)$: whereas (4) identifies the partial grounds as $\neg\neg p$ or $\neg\neg q$, according to (4′) the partial grounds are p or q. The grounds identified by (4′) thus do not enter in conflict with condition (5).

Analogous considerations apply to condition (2), which Poggiolesi replaces by (2′). For any propositional formulas A, B, C

(2′) $A \trianglerighteq \neg(B \wedge C)$ if, and only if, A and $A \approx B^*$ or $A \approx C^*$,

Conditions (2′) and (4′) have shown to be part of a solid logical grounding system [see Poggiolesi, 2018] and to be philosophically well-founded [see Poggiolesi, 2016a,b]. As we shall show, they are necessary conditions for the consistency result.

Note that (7) and (8) suffer from similar issues to (2) and (4). In a grounding framework, what counts for two propositions to be identical is not just that they have the same truth values, but also that they have the same grounds. However, $A \wedge B$ and $\neg(\neg A \vee \neg B)$ do not necessarily have the same grounds, and similarly for $A \vee B$ and $\neg(\neg A \wedge \neg B)$. For instance, taking $A = \neg p$ and $B = \neg q$ (for atoms p and q), $A \wedge B = \neg p \wedge \neg q$ has grounds $\neg p$ or $\neg q$ (by (1)), but $\neg(\neg A \vee \neg B) = \neg(\neg\neg p \vee \neg\neg q)$ has grounds $\neg\neg\neg p$ or $\neg\neg\neg q$ (according to both (4) and (4)′). The following conditions provide an adequate alternative. For any propositional formulas A, B,

(7′) $A \wedge B \approx (A^* \vee B^*)^*$,
(8′) $A \vee B \approx (A^* \wedge B^*)^*$.

Finally, under Poggiolesi's approach, the number of consecutive negations occurring before a formula plays an important role, as is clear from the use of * in the place of negation in conditions ($2'$), ($4'$), ($7'$), ($8'$). For this reason, Leibniz's law should be applied to formulas written in terms of * rather than straightforwardly in terms of negations. For an occurrence of a formula A in the formula C, let $C[(\neg)^m A]$ denote that it occurs in the immediate scope of precisely m negations (with $m \geq 0$).[1] Note that, by the definition (see p. 35), $(\neg)^m A$ is either identical to $(\neg)^{2n} A$ for some n, or it is identical to $(\neg)^{2n} A^*$ for some n. However, for any B such that $A \approx B$, substituting B for A in $C[(\neg)^m A]$ does not always yield the same proposition (up to $\approx$) as substituting B for A in the corresponding $C[(\neg)^{2n} A]$ or $C[(\neg)^{2n} A^*]$. For instance, with $A = p \wedge q$ for propositional atoms p and q, and $C = \neg A = \neg(p \wedge q)$, substituting $\neg(\neg p \vee \neg q) = B \approx A$ for A in $\neg A$ yields $\neg\neg(\neg p \vee \neg q)$. By contrast, $C = \neg A = \neg(p \wedge q) = A^*$, so substituting B for A here yields $B^* = \neg p \vee \neg q$. Since $\neg p$ and $\neg q$ are immediate partial grounds for both $C = \neg(p \wedge q)$ and $\neg p \vee \neg q$ (by ($2'$) and (3)), but not of $\neg\neg(\neg p \vee \neg q)$ (by (5)), it is clear that the latter substitution, but not the former, is the relevant one for Leibniz's law in a grounding context. Here we thus understand Leibniz's law to apply to the reformulation of formulas in terms of pairs of negations $(\neg)^{2n}$ and *.

With these elements in place, we now show that Poggiolesi's approach supports a consistent account of grounding and propositional identity.

Theorem. *Conditions* (1), ($2'$), (3), ($4'$), (5), (6), ($7'$), ($8'$) *are jointly consistent.*

Proof. In order to show that conditions (1), ($2'$), (3), ($4'$), (5), (6), ($7'$), ($8'$) are jointly consistent, we will construct a classical interpretation[2] mapping the formulas of L to the truth values T and F under which every condition is satisfied.

To this end, we first define several relations between propositional formulas (i.e. formulas not containing $\trianglerighteq$ or $\approx$). The relation $\cong$ is defined as follows. For any propositional formulas A and B, $A \cong B$ if, and only if, B is the formula A itself or it is A^*. The relation $\doteq$ is defined to be the closure of $\cong$ under conditions ($7'$) and ($8'$), according to the reformulation of formulas in terms of pairs of negations $(\neg)^{2n}$ and *

[1] So the next connective applied to $(\neg)^m A$ in C, if there is one, is not a negation.

[2] We follow [Poggiolesi, 2020] for constructing such interpretation.

discussed above. For any propositional formulas A and B, we say that A is an *immediate g-subformula* of B if, and only if, one of the following holds:

- $B \doteq \neg\neg C$ and $A \doteq C$, or
- $B \doteq (C \circ D)$ and $A \doteq C$ or $A \doteq D$.

We use $S(A)$ to denote the set of all immediate g-subformulas of the propositional formula A. Finally, we use $\models$ to denote the classical consequence relation (over propositional formulas).

Let us now define an interpretation $\mathcal{I}$. We begin with an arbitrary assignment of truth values to atoms such that there is some atom p with $\mathcal{I}(p) = T$. The propositional connectives are interpreted classically, with $\mathcal{I}(\neg A) = T$ iff $\mathcal{I}(A) = F$ and so on. The partial grounding and propositional identity connectives $\trianglerighteq$ and $\approx$ are treated by $\mathcal{I}$ as follows. For any propositional formulas A, B,

$\mathcal{I}(A \trianglerighteq B) = T$ if, and only if the following three conditions are satisfied:

(i) $A \in S(B)$;
(ii) $\mathcal{I}(B) = T$ if $B = C \wedge D$ or $B = \neg(C \vee D)$, $\mathcal{I}(A) = T$ otherwise;
(iii) it holds that:
- $\models A \to B$, or
- $\models \neg A \to \neg B$.

$\mathcal{I}(A \approx B) = T$, if, and only if, the following two conditions are satisfied:

(iv) $\models A \leftrightarrow B$, and
(v) $S(A) = S(B)$.

It is easy to check that conditions (1), (2′), (3), (4′), (5), (7′) and (8′) are satisfied by this interpretation. Moreover, (6) is satisfied for the atom p, and the relevant form of Leibniz's law holds. Since an inconsistency would contradict the classical interpretation of negation, it follows that (1), (2′), (3), (4′), (5), (6), (7′) and (8′) are jointly consistent. □

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