Logic and Logical Philosophy
Volume 30 (2021), 711–739
DOI: 10.12775/LLP.2021.014

Luis Estrada-González, Alessandro Giordani, Tomasz Jarmużek, Mateusz Klonowski, Igor Sedlár and Andrew Tedder

Incorporating the Relation into the Language? A Survey of Approaches in Relating Logic

Abstract. In this paper we discuss whether the relation between formulas in a relating model can be directly introduced into the language of the relating logic it is interpreting, and present some stances on that problem. Other questions in the vicinity, such as what kind of functor would be the incorporated relation, or whether the direct incorporation of the relation into the language of relating logic is really needed, will also be addressed.

Keywords: incorporating relation; object language; relating logic; relating semantics

1. Introduction

Let us consider a language L_n consisting of a countable set of propositional variables PV (for convenience we will use letters: p, q, r, etc.) and with n propositional connectives, $c_1, \ldots, c_n$ (we will denote the arity of a connective c_i by $\mathrm{ar}(i)$). Suppose that in the semantics for L_n we have two non-empty domains of logical values and their sub-domains:

1. logical values for propositions DV_1 and the designated logical values D_1
2. connection values for ordered tuples of propositions DV_2 and the designated connection values D_2.[1]

[1] The logical values and the connection values might be considered to be a special kind of truth values; see Section 2.

Special Issue: Relating Logic and Relating Semantics. I. Guest Editors: T. Jarmużek & F. Paoli

Published online November 28, 2021

A semantics for language L_n is a *relating semantics* iff at least for one connective c_i the valuation of all complex propositions of the form $c_i(\varphi_1, \ldots, \varphi_j)$, where $j = \mathrm{ar}(i)$, in a world w requires not only valuations of pairs $(\varphi_1, w), \ldots, (\varphi_j, w)$ in DV_1, but also a valuation of j-tuples $((\varphi_1, \ldots, \varphi_j), w)$ in DV_2 [see Jarmużek, 2021; Jarmużek and Klonowski, 2021]. A valuation of j-tuples $((\varphi_1, \ldots, \varphi_j), w)$ in DV_2 can in a formal semantics represent various logical or non-logical relationships between $\varphi_1, \ldots, \varphi_j$ in a world w. Those relationships can be, for example:

- content relationships, for example, the relatedness relation;
- analytical relationships;
- causalities;
- temporal orderings;
- preference orderings;
- logical consequences of some logic

among many others.

We use a function with a co-domain DV_2 to evaluate either a relationship between $\varphi_1, \ldots, \varphi_j$ or a relationship of some objects to which we refer by means of $\varphi_1, \ldots, \varphi_j$ — for example, facts or states of affairs — in the relating semantics.

The name 'relating semantics' is justified since a valuation of j-tuples which receives a designated value induce j-ary relations among the formulas, which allow us to evaluate various relationships not necessarily expressible by means of extensional relationships. Clearly, if $DV_2 = \{1, 0\}$ and 1 is the designated value then an evaluation of relationship between $\varphi_1, \ldots, \varphi_j$ can be expressed by one j-ary relation over set of formulas. We usually call a relation over a given set of formulas a *relating relation* and use the symbol R to denote that relation. Finally, by 'relating logic' we mean any logic that is determined by some relating semantics [see Jarmużek, 2021; Jarmużek and Klonowski, 2021]. Here are some examples of relating logics:

1. classical mono-relating logic (see Section 4 and [Jarmużek and Klonowski, 2021; Klonowski, 2019; Jarmużek and Klonowski, submitted])
2. fragments of classical mono-relating logic:
   - relatedness logic [see Epstein, 1979, 1990],
   - dependence logic [see Epstein, 1987, 1990],
   - Boolean connexive logic [see Jarmużek and Malinowski, 2019a,b; Malinowski and Palczewski, 2021],

- Classical Propositional Logic (when R is assumed to be a universal relation).

A trend in logic during the last score and a half has been that of internalizing meta-theoretical notions and devices at the object-language level, in order to build ever more expressive logical systems, as in the case of display calculi [see Wansing, 1998, Chs. 2 and 3], labeled deductive systems [cf. Gabbay, 2014], hybrid logics [cf. Areces, Blackburn and Marx, 2001], the logics of provability, justification logics [cf. Artemov and Fitting, 2019], the **LFI**'s [see Carnielli, Coniglio and Marcos, 2007] or the more recent uses of atomic formulas representing their own being a premise, a conclusion, part of a conjunction, etc. [see Russell, 2017]. This issue is a special case of a more general problem, since for any interpretation (formal semantics) of some propositional language, we can ask about the possibility of introducing functor counterparts of the relations from the given interpretation. One can consider whether such a functor *can* or even *must* be introduced into the syntax for some reason. There is also the dual problem, although not as popular, of taking some properties out of the language or the logic to the metalanguage [see, e.g., Avron, 2014].

Another related topic is that of the inverse of the expressive power of a formal language or a logic. The expressive power of a language establishes what semantic structures can be expressed by it. The inverse to that would be what formal languages and logical systems can be determined by a semantics, what one could call the 'determination power' of a semantics, following Jarmużek [2021]. Relating semantics has a high determination power, as we have just seen.

In this paper we discuss another instance of the trend of internalizing meta-theoretical notions and devices at the object-language level. Namely, whether the relation between formulas in the relating model can be introduced into the language of of the relating logic it is interpreting. Accordingly, the main question addressed in this paper is the following one:

Q1. Can R be incorporated into the relating language?

Although such move can seem feasible, it remains to be seen what are the necessary and sufficient conditions to do so, and by means of what techniques and procedures. Thus, there are other related questions in the vicinity that will be considered:

Q2. What kind of functor is the incorporated R?

Q3. Can the functor counterpart of R be iterated or nested?

Q4. The direct incorporation of R into the language of relating logic is really needed?

The plan for the remainder of the paper is as follows. In Section 2, a positive answer to the question on the possibility of introducing the relations by means of special connectives is explored. The point of view expressed there is mainly endorsed by view of Alessandro Giordani. Section 3 is devoted to the question on the need of incorporating R into the language, and the views there correspond to Igor Sedlár and Andrew Tedder. Although no strong claim about the need is done there, it is argued that it may make the presentation of some logics simpler and more elegant. Moreover, it could facilitate a systematic comparison of relating logics with other logics. Finally, Section 4 presents another approach, mainly due to Tomasz Jarmużek and Mateusz Klonowski, to these questions. It is argued there that, in many cases, it is possible to incorporate the relation R into the language. Nonetheless, in all those cases there is no need to do so. Moreover, it is argued that there are other cases where such incorporation is not possible at all, and the question then is what consequences does this have for metalogical studies.

This paper collects and expands upon the views presented and discussed during the meeting "Do we really need relation R to be directly incorporated into the language of relating logic?".[2]

2. Incorporating relating relation directly into the language. The first perspective

The presentation of the first perspective is divided in three short parts: in the first one, a brief and formal introduction to basic relating semantics is provided (cf. Section 1); the second one is dedicated to sketch an argument against the possibility of internalization; finally, the third one is dedicated to present an interpretation of relating semantics that sheds light on the idea behind the representation of the relation into the language and to address the argument proposed in the second part.

[2] The meeting took place on February 26, 2021 and the recording is available through https://vc.umk.pl/playback/presentation/2.0/playback.html?meetingId=8389f9831f8d30a45207636ebb03baff0ecbb59f-1614329449410

2.1. Introducing relating semantics

In general, a logical framework is characterized by the way in which we select the aspects of a sentence that are crucial for identifying its truth value. In a relating setting the relevant aspects are:

1. the form of a sentence,
2. some relations between its components.

The set of relations between the components of a sentence constitutes a fundamental element in specifying a relating model and is intended to capture some semantic aspects the are crucial for an appropriate representation of the sense of the proposition expressed by that sentence.

Let us once again consider the language L_n and the set For_n of formulas in L_n defined in the standard way. A *general relating model* (a *model*, for short) is a triple:

$$\langle v, \{f_{c_i}\}_{i \leqslant n}, \{V_{c_i}\}_{i \leqslant n} \rangle,$$

where:

- $v\colon Var \longrightarrow DV_1$,
- $f_{c_i}\colon (DV_1)^{\mathrm{ar}(i)} \longrightarrow DV_1$,[3]
- $V_{c_i}\colon (For_n)^{\mathrm{ar}(i)} \longrightarrow DV_2$.

The basic idea is that f_{c_i} captures the extensional value of a sentence whose main connective is c_i, while V_{c_i} captures the semantic value which is dependent on the relations between elements of For_n. Here, DV_1 is a set of truth values, associated with a corresponding subset $D_1 \subseteq DV_1$ of designated values representing the ways in which a sentence can be true, while DV_2 is a set of truth values, associated with a corresponding subset $D_2 \subseteq DV_2$ of designated values representing the ways in which a relation between the elements of a sentence can subsist (cf. Section 1; see [Jarmużek, 2021] for a general introduction). In what follows we will work exclusively with the set of classical truth values $\{1, 0\}$, with designated value 1, and a set of classical connection values $\{1, 0\}$, with designated value 1, indicating that a relation holds.

The truth definition is based on the notion of extension in a model $\mathfrak{M}$. The *extension* $[\varphi]$ *of a formula* φ *in a model* $\mathfrak{M}$ is recursively defined as follows:

[3] Sometimes in a relating model we omit the family of truth functions $\{f_{c_i}\}_{i \leq n}$.

1. $[p]^{\mathfrak{M}} = v(p)$,
2. $[c_i(\varphi_1, \ldots, \varphi_{\mathrm{ar}(i)})]^{\mathfrak{M}} = f_{c_i}([\varphi_1]^{\mathfrak{M}}, \ldots, [\varphi_{\mathrm{ar}(i)}]^{\mathfrak{M}})$.

For instance, if c_i, for some $i \leq n$, is the conjunction $\wedge$ and DV_1 is the set of classical truth values $\{1, 0\}$, then the extension of a expression like $\varphi_1 \wedge \varphi_2$ is $[\varphi_1 \wedge \varphi_2]^{\mathfrak{M}} = \min([\varphi_1]^{\mathfrak{M}}, [\varphi_2]^{\mathfrak{M}})$, where $\min = f_\wedge = f_{c_i}$ is the function that returns the minimum of the extensions $[\varphi_1]^{\mathfrak{M}}$ and $[\varphi_2]^{\mathfrak{M}}$, in accordance with the idea that a conjunction is true precisely when both of its conjuncts are true.

The *truth of a formula* φ *in a model* $\mathfrak{M}$ is recursively defined as follows:

1. $\mathfrak{M} \models p$ iff $v(p) \in D_1$,
2. $\mathfrak{M} \models c_i(\varphi_1, \ldots, \varphi_{\mathrm{ar}(i)})$ iff
   2.1. $[c_i(\varphi_1, \ldots, \varphi_{\mathrm{ar}(i)})]^{\mathfrak{M}} = f_{c_i}([\varphi_1]^{\mathfrak{M}}, \ldots, [\varphi_{\mathrm{ar}(i)}]^{\mathfrak{M}}) \in D_1$,
   2.2. $V_{c_i}(\varphi_1, \ldots, \varphi_{\mathrm{ar}(i)}) \in D_2$.

For instance, if c_i is the conjunction $\wedge$ and DV_1 is the set of classical truth values $\{1, 0\}$, then the truth of a formula like $\varphi_1 \wedge \varphi_2$ is defined so that:

$$\mathfrak{M} \models \varphi_1 \wedge \varphi_2 \text{ iff } \min([\varphi_1]^{\mathfrak{M}}, [\varphi_2]^{\mathfrak{M}}) \in \{1\} \text{ and } V_\wedge(\varphi_1, \varphi_2) \in \{1\},$$

that is:

$$\mathfrak{M} \models \varphi_1 \wedge \varphi_2 \text{ iff } [\varphi_1]^{\mathfrak{M}} = [\varphi_2]^{\mathfrak{M}} = 1 \text{ and } V_\wedge(\varphi_1, \varphi_2) = 1.$$

It is then plain that assuming that $V_\wedge(\varphi_1, \varphi_2)$ is always 1 allows us to capture the standard truth conditions for a conjunction. A more interesting instance of such definition concerns a notion of *diachronic conjunction* $\wedge_{\mathbf{D}}$, which is the conjunction we usually exploit when telling a story. In this case, we want to represent the fact that a formula like $\varphi_1 \wedge_{\mathbf{D}} \varphi_2$ expresses the idea that the fact described by the first conjunct obtained before the obtaining of the fact described by the second conjunct. In order to do that, we can simply introduce a relation *Prec* of precedence and define the truth conditions for $\wedge_{\mathbf{D}}$ so that:

$$\mathfrak{M} \models \varphi_1 \wedge_{\mathbf{D}} \varphi_2 \text{ iff } [\varphi_1]^{\mathfrak{M}} = [\varphi_2]^{\mathfrak{M}} = 1 \text{ and } \mathit{Prec}(\varphi_1, \varphi_2) = 1.$$

As a result, we obtain that $\varphi_1 \wedge_{\mathbf{D}} \varphi_2$ is true if and only if φ_1 and φ_2 are both true and the fact described by φ_1 precedes the fact described by φ_2.

In light of the definition of the extension of a formula in a model, the intuitive idea suggests itself of introducing relation symbols in the language representing the relations in $\{V_{c_i}\}_{i \leqslant n}$. In so doing, we would be in a position to interpret such symbols via the elements of $\{V_{c_i}\}_{i \leqslant n}$, and so to explicitly define the relating connectives in terms of them. To stick to the previous example, we could:

1. introduce a relation symbol *Prec* in the language,
2. define $\mathfrak{M} \models Prec(\varphi_1, \varphi_2)$ so that $\boldsymbol{Prec}(\varphi_1, \varphi_2) = 1$,
3. define $\wedge_{\mathbf{D}}$ so that $\varphi_1 \wedge_{\mathbf{D}} \varphi_2 := \varphi_1 \wedge \varphi_2 \wedge Prec(\varphi_1, \varphi_2)$.

The main problem we have to address now is whether such a procedure is legitimate.

2.2. An argument against internalization

A basic argument against the legitimacy of internalizing the relations in $\{V_{c_i}\}_{i \leq n}$ rests on the consideration of the truth definition of the relating connectives. The truth definition suggests that there is a crucial difference between the two elements involved in characterizing the truth conditions of a composite formula. To be sure:

- the f_{c_i}, for any $i \leq n$, operates on *semantic* entities, being functions of type $(DV_1)^{\mathrm{ar}(i)} \mapsto DV_1$,
- the V_{c_i}, for any $i \leq n$, operates on *syntactic* entities, being functions of type $(For_n)^{\mathrm{ar}(i)} \mapsto DV_2$.

As a consequence, while *we are able* to find semantic entities that are interpreted via the f_{c_i}, *we are not able* to find semantic entities that are interpreted via the V_{c_i}, for any $i \leq n$. Since the V_{c_i} is itself operating on syntactic entities, it should be meaningless to try and find a representation of the V_{c_i} in the language.

The previous conclusion seems to be further supported by considering some paradigmatic instances of the $\{V_{c_i}\}_{i \leq n}$. As an example, in semantics of relatedness logics and semantics of dependence logics [Epstein, 1990], we work with subject matters and two prominent applications of these semantics are based on the relations of:

- subject matter *intersection* (used to interpret a first-degree entailment; see Epstein, 1990; Paoli, 1993, 2007),
- subject matter *inclusion* (used to interpret an analytic entailment; see Epstein, 1990; Paoli, 2007).

The language of relatedness and dependence logics is the special case of the language L_3 with the following connectives: $\neg$, $\wedge$, $\rightarrow$. Subject matter intersection, as a binary relation R determined over the set of formulas, is constrained so that the following conditions are satisfied:

$$
\begin{aligned}
&R(\varphi, \varphi),\\
&R(\varphi, \psi) \text{ iff } R(\psi, \varphi),\\
&R(\varphi, \psi \wedge \chi) \text{ iff } (R(\varphi, \psi) \text{ or } R(\varphi, \chi)),\\
&R(\neg\varphi, \psi) \text{ iff } R(\varphi, \psi),\\
&R(\varphi, \psi \to \chi) \text{ iff } R(\varphi, \psi \wedge \chi),
\end{aligned}
$$

and the canonical interpretation of subject matter intersection, $R_\cap$, is such that $R_\cap(\varphi, \psi)$ iff $var(\varphi) \cap var(\psi) \neq \varnothing$, where $var(\chi)$ is the set of propositional variables of χ, for any $\chi \in For_n$. It is evident that $R_\cap$ is a relation relating purely syntactic entities, since two formulas are related by being such that the set of propositional variables occurring in one of them intersects the set of propositional variables occurring in the other.

Similarly, subject matter inclusion, binary relation R determined over the set of formulas, is constrained so that the following conditions are satisfied:

$$
\begin{aligned}
&R(\varphi, \varphi),\\
&(R(\varphi, \psi) \text{ and } R(\psi, \chi)) \Rightarrow R(\varphi, \chi),\\
&R(\varphi, \psi \wedge \chi) \text{ iff } (R(\varphi, \psi) \text{ and } R(\varphi, \chi)),\\
&R(\neg\varphi, \varphi),\\
&R(\varphi, \neg\varphi),\\
&R(\varphi, \psi \to \chi) \text{ iff } R(\varphi, \psi \wedge \chi),
\end{aligned}
$$

and the canonical interpretation of subject matter inclusion, $R_\supseteq$, is such that $R_\supseteq(\varphi, \psi)$ iff $var(\varphi) \supseteq var(\psi)$. Again, there is no doubt that $R_\supseteq$ is a relation relating purely syntactic entities, since two formulas are related by being such that the set of propositional variables occurring in one of them includes the set of propositional variables occurring in the other. It goes without saying then that these relations are defined in such a way that no direct syntactic representation of them is allowed.

The problem is therefore whether we are forced to conclude, based on such cases, that the attempt to internalize relations in relating semantics is definitely flawed.

2.3. A defense of the legitimacy of internalization

Let us further develop the relating semantics to show that the aforementioned conclusion can be resisted. The basic idea is to assume a fine-grained notion of content of a sentence according to which both the connective and the way of composition of a sentence affect its content. We can proceed in two steps:

1. introduce a set of propositional contents which is in one to one correspondence with For_n,
2. introduce a set of relations between contents which is in one to one correspondence with $\{V_{c_i}\}_{i\leq n}$.

As a consequence, all the relations on the set of contents are faithfully represented as relations on For_n. In particular, all functions of $\{V_{c_i}\}_{i\leq n}$ can be interpreted as functions that codify such relations, so as to allow us to justify the introduction of relating connectives interpreted on them.

Though formally sound, this solution might be put into question as to its being a sensible solution. In fact, someone could argue that there is nothing in the content of a sentence that legitimates the introduction of a set of relation symbols in one to one correspondence with $\{V_{c_i}\}_{i\leq n}$. Still, Giordani thinks we can defend the solution by considering the cases proposed above. On the one hand, it seems to be fully legitimate to introduce a relation symbol like $Prec$ for representing the fact that the fact represented by φ_1 precedes the fact represented by φ_2, since the relation of temporal precedence relates facts, not syntactic entities, and therefore it seems to be more appropriate to consider Prec as a semantic relation and $Prec$ as its syntactic counterpart. On the other hand, turning back to semantics of relatedness logics and dependence logics, what we can note is that the canonical definitions of subject matter intersection and subject matter inclusion are just ways of representing such relations within the language. Indeed, subject matter intersection and subject matter inclusion are first of all relations between subject matters, and only derivatively relations between formulas, so that $var(\varphi)\cap var(\psi)\neq\varnothing$ and $var(\varphi)\supseteq var(\psi)$ are just ways to canonically construe them.

Based on that, Giordani thinks that we are allowed to conclude both that it is legitimate to consider the $\{V_{c_i}\}_{i\leq n}$ introduced in relating semantics as based on more fundamental relations between sentential contents and that it is legitimate to represent of the relations between sentential contents via relation symbols in the object language interpreted by the elements of $\{V_{c_i}\}_{i\leq n}$. Besides, in respect of the resulting semantics, we could be able to derive an elementary proof of completeness along the lines of the following:

Proposition 2.1. *Let c be a connective of arity j and whose semantic definition is known and c_R be a connective whose semantic definition* *is obtained by the semantic definition of c and the introduction of an appropriate function V_{c_R}. Then:*

1. *if a truth lemma is provable with respect to $c(\varphi_1, \ldots, \varphi_j)$ and $R(\varphi_1, \ldots, \varphi_j)$, a corresponding truth lemma is also provable with respect to $c_R(\varphi_1, \ldots, \varphi_j)$, where $c_R(\varphi_1, \ldots, \varphi_j)$ is defined as $c(\varphi_1, \ldots, \varphi_j) \wedge R(\varphi_1, \ldots, \varphi_j)$, with R interpreted via V_{c_R},*
2. *if the logic of c and R is complete, the logic of c_R is also complete.*

The idea on which the proof could be based is roughly this one. Since a truth lemma is provable with respect to $c(\varphi_1, \ldots, \varphi_j)$ and $R(\varphi_1, \ldots, \varphi_j)$, we have that $\mathfrak{M} \models c(\varphi_1, \ldots, \varphi_j)$ iff $[c(\varphi_1, \ldots, \varphi_j)]^{\mathfrak{M}} = 1$, where $\mathfrak{M}$ is a canonical model for the logic under discussion. Thus, we obtain:

$$\mathfrak{M} \models c_R(\varphi_1, \ldots, \varphi_j) \text{ iff } (\mathfrak{M} \models c(\varphi_1, \ldots, \varphi_j) \text{ and } \mathfrak{M} \models R(\varphi_1, \ldots, \varphi_j)),$$
$$\mathfrak{M} \models c_R(\varphi_1, \ldots, \varphi_j) \text{ iff } ([c(\varphi_1, \ldots, \varphi_j)]^{\mathfrak{M}} = 1 \text{ and } \mathfrak{M} \models R(\varphi_1, \ldots, \varphi_j)),$$
$$\mathfrak{M} \models c_R(\varphi_1, \ldots, \varphi_j) \text{ iff } ([c(\varphi_1, \ldots, \varphi_j)]^{\mathfrak{M}} = 1 \text{ and } V_{c_R}(\varphi_1, \ldots, \varphi_j) = 1),$$

where V_{c_R} is defined so that $V_{c_R}(\varphi_1, \ldots, \varphi_j) = 1$ just in case $\mathfrak{M} \models R(\varphi_1, \ldots, \varphi_j)$.

3. Incorporating relating relation directly into the language. The second perspective

The second perspective goes along with the first one. It emphasizes that adding a connective directly corresponding to the relating relation is simple, elegant and connects relatedness logic with other well-known logics, for instance Fagin and Halpern's logic of general awareness [Fagin and Halpern, 1988]. In this section we demonstrate this point by outlining a version of epistemic logic with relatedness. We also consider a hypothetical counterargument to having a connective directly corresponding to the relating relation in the language, pointing to a language/metalanguage confusion and problems with nesting.

3.1. An epistemic logic with relatedness

Consider a special case of propositional language L_4 containing two binary connectives $\wedge, R$, and two unary connectives $\neg, K$. We will also use the following abbreviation: $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$. The language under considerations is thus an extension of the basic language of classical modal logic, with a binary connective R and using K instead of $\Box$ to highlight the epistemic interpretation discussed below.

A *model* for the considered language is a structure $\langle W, R, V\rangle$, where W is a non-empty set (of, say, "possible worlds"), R is a function assigning to each $w \in W$ a binary "connection" relation $R(w)$ on formulas, and V is a valuation function from PV to subsets of W.[4] A *pointed model* is a pair $\langle \mathfrak{M}, w\rangle$ where $w \in W$ of $\mathfrak{M}$. The satisfaction relation between pointed models and formulas is defined as follows:

$$\begin{aligned}
&\langle \mathfrak{M}, w\rangle \models \varphi \text{ iff } w \in V(\varphi), \text{ if } \varphi \in PV,\\
&\langle \mathfrak{M}, w\rangle \models \neg\varphi \text{ iff } \langle \mathfrak{M}, w\rangle \not\models \varphi,\\
&\langle \mathfrak{M}, w\rangle \models \varphi \wedge \psi \text{ iff } (\langle \mathfrak{M}, w\rangle \models \varphi \text{ and } \langle \mathfrak{M}, w\rangle \models \psi),\\
&\langle \mathfrak{M}, w\rangle \models K\varphi \text{ iff } \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W,\\
&\langle \mathfrak{M}, w\rangle \models \varphi\, R\, \psi \text{ iff } \langle \varphi, \psi\rangle \in R(w).
\end{aligned}$$

A formula φ is *valid in a model* $\mathfrak{M}$ iff $\langle \mathfrak{M}, w\rangle \models \varphi$ for all w in $\mathfrak{M}$.

The models considered here are "universal" **S5** models, where the universal accessibility relation is left implicit, and are extended with a function assigning a binary relation $R(w)$ to each w in the model. Informally, $K\varphi$ means that some fixed agent "knows" that φ in the standard sense of epistemic logic; and $\varphi\, R\, \psi$ means that ψ is related to φ. The meaning of "related" is deliberately vague here, in the spirit of relating logic.

We note that modifications of this framework that use several agents and explicit accessibility relations for each agent are trivial modifications of the present framework. We stick to this one because of its simplicity and the fact that it is enough to make our case for the introduction of the relation into the language.

This framework is closely related to the logic of general awareness put forward by Fagin and Halpern [1988]. Setting aside the fact that Fagin and Halpern work with a group of agents and explicit equivalence relations, the only difference is that their language contains a unary operator A instead of our binary R and, correspondingly, the models they consider are **S5** models extended with a function A that assigns a

[4] The function R might be identified with the indexed by possible words family of binary relations determined over formulas introduced in [Jarmużek and Malinowski, 2019b] and explored in [Jarmużek and Klonowski, 2020].

set of formulas to each $w \in W$, which is a unary relation on formulas, instead of our binary relation.

Fagin and Halpern read $A\varphi$ as "the agent is aware of φ", and A can be used to define the *explicit knowledge* operator by setting $X\varphi := K\varphi \wedge A\varphi$, representing the idea that in order to "really know" that φ, the agent must possess information that supports φ — this is represented by $K\varphi$ — and they must also be aware of (the concepts contained in the statement corresponding to) φ.[5] Given the fact that $A(w)$ can be any arbitrary set of formulas, explicit knowledge avoids all of the problematic closure properties associated with the "implicit knowledge" operator K. For instance, it is not the case that if ψ follows from φ — i.e. if $\varphi \rightarrow \psi$ is valid — then $X\psi$ follows from $X\varphi$.

In the framework with explicit R, "relating versions" of all binary propositional connectives can be defined naturally. Let $\varphi\ F\ \psi$ be a formula built with the binary connective F. Define the "relating version" of F as $\varphi\ F_R\ \psi := (\varphi\ F\ \psi) \wedge (\varphi\ R\ \psi)$.

Relating versions of unary propositional connectives, such as Kp, can be defined using a fixed formula χ (e.g. $K_R\varphi := K\varphi \wedge (\chi\ R\ \varphi)$). In fact, the framework of Fagin and Halpern can be emulated in the relating framework: pick any formula χ and define $A\varphi := \chi\ R\ \varphi$.

Many natural properties of R turn out to be canonical and hence the logics of the corresponding classes of models trivially axiomatizable. For instance:

- $R(w) = R(u)$ for all w, u corresponds to $(\varphi\ R\ \psi) \rightarrow K(\varphi\ R\ \psi)$,
- symmetry of each $R(w)$ corresponds to $(\varphi\ R\ \psi) \rightarrow (\psi\ R\ \varphi)$; and so on.

In fact, on the assumption that $R(w) = R(u)$ for all w, u, and so we may speak of *the* R in a model, each R corresponds to taking $\varphi\ R\ \psi$ as an axiom iff $\langle \varphi, \psi \rangle \in R$, with some natural limitations.

3.2. A hypothetical counter-argument

One may object against the framework above that incorporating R into the language by means of R amounts to confusing language with metalanguage. Many natural interpretations of R correspond to relations between certain properties of sentences or formulas (such as variable

[5] Fagin and Halpern actually use X as a primitive operator satisfying $X\varphi \leftrightarrow (K\varphi \wedge A\varphi)$, but let us modify their original presentation here.

sharing, for example), such that it does not make sense to have a *connective* expressing this kind of relations *in the language*. A related worry is that, on such interpretations, formulas with nested occurrences of R do not seem to have a natural meaning.

Tedder and Sedlár's answer to this objection is indirect and twofold. They point out that (i) a similar objection has (unsuccessfully, we think) been raised against *relevance logics*, supposedly confusing the object-language connective of implication with the metalanguage entailment relation, and that (ii) a similar worry has not been raised, to the best of our knowledge, against the awareness logic of Fagin and Halpern.

As to (i), in the *Grammatical Propaedeutic* appendix of [Anderson and Belnap, 1975], Nuel Belnap responds to the language/metalanguage charge by noting that the choice between representing a natural language notion as a formula predicate or as a connective is not forced by the grammatical rules of natural languages. Rather, we have choices to make in representing features of natural language in our logical grammar, and those choices may be guided by considerations other than natural language grammar, e.g. theoretical virtues of the representation such as *simplicity*, *fruitfulness* and so on. The upshot is that, for purposes of representing features of natural language in logic, it pays to be "confused" about the object and metalanguage distinction, as there will be representations at both levels which are useful for various purposes. This kind of objection against relevance logic is considered irrelevant.

Tedder and Sedlár's point (ii) suggests that a similar issue seems to be considered irrelevant as an objection against using an object-language representation of propositional awareness in epistemic logic.

Be that as it may, we suggest considering the theoretical virtues and possible applications of logics "with an explicit R in the language", such as the epistemic logic outlined in the previous subsection, and weighing them against any possible objections with an open mind.

4. Not incorporating directly the relating relation into the language

In this section we argue that relating relation (1) need not be, and (2) in some cases it cannot be, directly incorporated into the language. In order to justify (1), we consider the classical mono-relating logic case and the definability of therelating relation in it. To motivate (2), we refer to

Boolean connexive logic, which is a fragment of classical mono-relating logic, and the fact that the relating relation is undefinable there. Finally, we present a semiotic analysis of the functor counterpart of the relating relation to explain why we think that such a functor cannot be a logical connective, contrary to what it was claimed in the previous sections.

4.1. Classical-monorelating logic and the definability of relating relation

Classical-monorelating logic (CMRL) is the special case of relating logic obtained by extending classical logic with intensional counterparts of the binary extensional connectives [see Jarmużek and Klonowski, 2021; Klonowski, 2019; Jarmużek and Klonowski, submitted]. The *CMRL-language* is the special case of L_9 language that contains the standard logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and the following relating connectives: $\wedge^{\mathrm{w}}$ (relating conjunction), $\vee^{\mathrm{w}}$ (relating disjunction), $\rightarrow^{\mathrm{w}}$ (relating implication), $\leftrightarrow^{\mathrm{w}}$ (relating equivalence). The set of CMRL-formulas is defined in the standard way. We propose the following general reading for the relating connectives:

- $\varphi \rightarrow^{\mathrm{w}} \psi$ we can read as: *if φ then, what's related to it, ψ*
- $\varphi \wedge^{\mathrm{w}} \psi$ we can read as: *φ and, what's related to it, ψ*
- $\varphi \vee^{\mathrm{w}} \psi$ we can read as: *φ or, what's related to it, ψ*
- $\varphi \leftrightarrow^{\mathrm{w}} \psi$ we can read as: *φ iff, and what's related to it, ψ*.[6]

To interpret the CMRL-language we use a special case of a relating model (cf. Section 2). A *CMRL-model* is an ordered pair $\langle v, R\rangle$ such that $v\colon PV \longrightarrow \{1,0\}$ ($DV_1 = \{1,0\}$, with 1 as designated) is a classical valuation and R is a binary relation over CMRL-formulas ($DV_2 = \{1,0\}$, where 1 is designated).

We define the notion of truth in a CMRL-model in the standard way. We assume the classical truth conditions for formulas built up by means of the standard connectives. And for formulas built up by relating connectives we have:

$$\mathfrak{M} \models \varphi \wedge^{\mathrm{w}} \psi \text{ iff } [\mathfrak{M} \models \varphi \text{ and } \mathfrak{M} \models \psi, \text{ and } R(\varphi,\psi)],$$
$$\mathfrak{M} \models \varphi \vee^{\mathrm{w}} \psi \text{ iff } [\mathfrak{M} \models \varphi \text{ or } \mathfrak{M} \models \psi, \text{ and } R(\varphi,\psi)],$$

[6] We use the notational convention which is an effect of adding in the upper index of the standard binary connectives the letter w. The letter comes from the Polish words *wiązać*, *wiążący* which mean *to relate*, *relating*. Nevertheless, if in a language we use only one kind of implication, equivalence etc. we use usual symbols, without superscripts. In CMRL-language we have two kinds of connectives.

$$\mathfrak{M} \models \varphi \rightarrow^{\mathrm{w}} \psi \text{ iff } [\mathfrak{M} \not\models \varphi \text{ or } \mathfrak{M} \models \psi, \text{ and } R(\varphi, \psi)],$$
$$\mathfrak{M} \models \varphi \leftrightarrow^{\mathrm{w}} \psi \text{ iff } [\mathfrak{M} \models \varphi \text{ iff } \mathfrak{M} \models \psi, \text{ and } R(\varphi, \psi)].$$

Different conditions imposed on the relation can determine different CMRLs. The class of all models determines the logic **W**. Here is a sound and complete axiomatic system for **W** [Jarmużek and Klonowski, submitted; Klonowski, 2019]:

$$\text{Any truth-functional tautology w.r.t. } \neg, \wedge, \vee \tag{PL}$$
$$(\varphi \wedge^{\mathrm{w}} \psi) \rightarrow (\varphi \wedge (\varphi \rightarrow^{\mathrm{w}} \psi)) \tag{E$\wedge^{\mathrm{w}}$}$$
$$(\varphi \wedge (\varphi \rightarrow^{\mathrm{w}} \psi)) \rightarrow (\varphi \wedge^{\mathrm{w}} \psi) \tag{I$\wedge^{\mathrm{w}}$}$$
$$(\varphi \vee^{\mathrm{w}} \psi) \rightarrow (\varphi \vee \psi) \tag{E$\vee^{\mathrm{w}}$}$$
$$((\varphi \rightarrow^{\mathrm{w}} \psi) \wedge (\varphi \vee \psi)) \rightarrow (\varphi \vee^{\mathrm{w}} \psi) \tag{I$\vee^{\mathrm{w}}$}$$
$$(\varphi \rightarrow^{\mathrm{w}} \psi) \rightarrow (\varphi \rightarrow \psi) \tag{E$\rightarrow^{\mathrm{w}}$}$$
$$((\varphi \vee^{\mathrm{w}} \psi) \wedge (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow^{\mathrm{w}} \psi) \tag{I$\rightarrow^{\mathrm{w}}$}$$
$$(\varphi \leftrightarrow^{\mathrm{w}} \psi) \rightarrow ((\varphi \rightarrow^{\mathrm{w}} \psi) \wedge (\psi \rightarrow \varphi)) \tag{E$\leftrightarrow^{\mathrm{w}}$}$$
$$((\varphi \rightarrow^{\mathrm{w}} \psi) \wedge (\psi \rightarrow \varphi)) \rightarrow (\varphi \leftrightarrow^{\mathrm{w}} \psi) \tag{I$\leftrightarrow^{\mathrm{w}}$}$$
$$\varphi, \varphi \rightarrow \psi // \psi \tag{MP}$$

Let us briefly comment on our axiomatization. For every relating connective we presented the axiom schema that enables us to eliminate and the one that enables us to introduce a given relating connective. Notice that (E$\wedge^{\mathrm{w}}$) with (I$\wedge^{\mathrm{w}}$) and (E$\leftrightarrow^{\mathrm{w}}$) with (I$\leftrightarrow^{\mathrm{w}}$) leads to some definitions expressed in CMRL-language of $\wedge^{\mathrm{w}}$ and $\leftrightarrow^{\mathrm{w}}$ respectively. We can also define $\rightarrow^{\mathrm{w}}$ by means of other connectives, we have the following theses:

$$((\varphi \rightarrow \psi) \wedge ((\varphi \vee^{\mathrm{w}} \psi) \vee (\varphi \leftrightarrow^{\mathrm{w}} \psi))) \rightarrow (\varphi \rightarrow^{\mathrm{w}} \psi)$$
$$(\varphi \rightarrow^{\mathrm{w}} \psi) \rightarrow ((\varphi \rightarrow \psi) \wedge ((\varphi \vee^{\mathrm{w}} \psi) \vee (\varphi \leftrightarrow^{\mathrm{w}} \psi)))$$

The situation is different, however, in the case of $\vee^{\mathrm{w}}$. As we can see in the next subsection, $\vee^{\mathrm{w}}$ is not always definable by means of other connectives, for instance in logic **W** relating disjunction $\vee^{\mathrm{w}}$ is indefinable.

In order to prove completeness theorem for the axiomatic system of **W** we use the fact that the relating relation is definable in the CMRL-language. Let us define the following abbreviation:

$$\varphi \looparrowright \psi := (\varphi \vee^{\mathrm{w}} \psi) \vee (\varphi \rightarrow^{\mathrm{w}} \psi). \tag{$\looparrowright$}$$

Formulas of the form of $\varphi \looparrowright \psi$ enable us to express in the CMRL-language that φ is related with ψ. The following fact holds [Klonowski, 2019; Jarmużek and Klonowski, submitted]:

Fact 4.1. *Let $\langle v, R\rangle$ be a CMRL-model and φ, ψ be CMRL-formulas. Then, $R(\varphi, \psi)$ iff $\langle v, R\rangle \models \varphi \looparrowright \psi$.*[7]

By fact 4.1 we get that any relational condition might be expressed by means of ($\looparrowright$) and so any logic that is CMRL and is defined by means of some relational properties can be axiomatized [Klonowski, 2019; Jarmużek and Klonowski, submitted]. Notice that in our axiomatic system we can prove the following kind of definitions of relating connectives which are useful in the proof of completeness [Klonowski, 2019; Jarmużek and Klonowski, submitted]:

$$(\varphi \wedge^{\mathrm{w}} \psi) \leftrightarrow ((\varphi \wedge \psi) \wedge (\varphi \looparrowright \psi)) \tag{Df $\wedge^{\mathrm{w}}$}$$

$$(\varphi \vee^{\mathrm{w}} \psi) \leftrightarrow ((\varphi \vee \psi) \wedge (\varphi \looparrowright \psi)) \tag{Df $\vee^{\mathrm{w}}$}$$

$$(\varphi \to^{\mathrm{w}} \psi) \leftrightarrow ((\varphi \to \psi) \wedge (\varphi \looparrowright \psi)) \tag{Df $\to^{\mathrm{w}}$}$$

$$(\varphi \leftrightarrow^{\mathrm{w}} \psi) \leftrightarrow ((\varphi \leftrightarrow \psi) \wedge (\varphi \looparrowright \psi)) \tag{Df $\leftrightarrow^{\mathrm{w}}$}$$

In the case of CMRL, neither axiomatization nor talking about the relating relation require the direct incorporation of the relating relation into the language. But this is not only true in CMRLs, but also in any other relating logic defined in the language with at least relating implication, relating disjunction and classical disjunction.

It seems that Fact 4.1 about the definability of the relating relation should end our discussion. One could say that even if we do not agree to represent the relation by a primitive functor, we can always add three connectives that allows us to define the abbreviation ($\looparrowright$). The problem is, however, that we cannot always add the required connectives to the language under consideration. Sometimes we work with a particular language without any new non-definable connectives in this language.

4.2. A fragment of classical-monorelating logic and the undefinability of relating relation

Suppose we are only interested in some fragment of CMRL defined in the language consisting of Boolean connectives $\neg, \wedge, \vee$ and relating im-

[7] Notice that we can also use $(\varphi \vee^{\mathrm{w}} \psi) \vee (\varphi \leftrightarrow^{\mathrm{w}} \psi)$ instead of $\varphi \looparrowright \psi$.

plication $\rightarrow^{w}$. Let us call such fragment the *Boolean logic with relating-implication* (BLRI). The syntax and semantics are defined as in the previous case except we consider only a fragment of the CMRL-language, viz. the BLRI-language, and relating relations are binary relations over BLRI-formulas.

Since we consider only the BLRI-language, we cannot arbitrarily add any functor counterpart of the relating relation or arbitrarily claim that the relating relation is definable. In fact, if we do not assume any relational properties, the relation need not be definable.

Still, not all properties secure definability. For instance, suppose that we would like to examine the BLRI-language to determine connexive logics. Consider a relation over BLRI-formulas satisfying the following conditions:

$$\sim R(\varphi, \neg\varphi) \tag{a1}$$

$$\sim R(\neg\varphi, \varphi) \tag{a2}$$

$$R(\varphi, \psi) \Rightarrow \sim R(\varphi, \neg\psi) \tag{b0}$$

$$R(\varphi \rightarrow^{w} \psi, \neg(\varphi \rightarrow^{w} \neg\psi)) \tag{b1}$$

$$R(\varphi \rightarrow^{w} \neg\psi, \neg(\varphi \rightarrow^{w} \psi)). \tag{b2}$$

and call it a *connexive relation*. Any logic defined by connexive relations is a *Boolean connexive logic*. We call **C** the smallest Boolean connexive logic. But then, the connexive relation cannot be defined in the language. To prove the undefinability of the connexive relation, we show that relating disjunction is not definable by means of other connectives in the logic **C** [cf. a simple modification of the proof of Fact 3.6 in Jarmużek and Klonowski, submitted]:

FACT 4.2. *Let us define:*

- *valuation v, $v(\varphi) = 1$ iff $\varphi = p$*
- *relating relation R, $R(\varphi_1, \varphi_2)$ iff $\varphi_1 = p$ and $\varphi_2 = q$*
- *relating relation Q, $Q(\varphi_1, \varphi_2)$ iff for some BLRI-formulas ψ_1, ψ_2, either $\varphi_1 = \psi_1 \rightarrow^{w} \psi_2$ and $\varphi_2 = \neg(\psi_1 \rightarrow^{w} \neg\psi_2)$ or $\varphi_1 = \psi_1 \rightarrow^{w} \neg\psi_2$ and $\varphi_2 = \neg(\psi_1 \rightarrow^{w} \psi_2)$.*

  Then:

1. *$R \cup Q$ and Q are connexive relations*
2. *$\langle v, R \cup Q\rangle \models p \vee^{w} q$ and $\langle v, Q\rangle \not\models p \vee^{w} q$*
3. *for every BLRI-formula φ such that the set of variables of φ is contained in $\{p, q\}$, either $\langle v, R \cup Q\rangle \not\models \varphi$ or $\langle v, Q\rangle \models \varphi$.*

By Fact 4.2, the relating disjunction is not definable in terms of other connectives in the BLRI-language. But then there is no BLRI-formulas schema that enables us to define every connexive relation. Otherwise, using some schema that defines every connexive relation the relating disjunction would be definable, i.e. it could be expressed by means of other connectives like in (Df $\vee^{\mathrm{w}}$).

It is worth noting, however, that we can still axiomatize **C** in spite of the undefinability result.[8] In fact, the axiom system consisting of the following axiom schemata and rule of inference is sound and complete with respect to **C** [see Klonowski, 2021]:

All the truth-functional tautologies written with $\neg, \wedge, \vee$.	$(\mathrm{PL}_{\neg,\wedge,\vee})$
$\neg(\varphi \to^{\mathrm{w}} \psi) \vee (\neg\varphi \vee \psi)$	(The modified elimination of $\to^{\mathrm{w}}$)
$\neg(\varphi \to^{\mathrm{w}} \neg\varphi)$	(Aristotle's Thesis)
$\neg(\neg\varphi \to^{\mathrm{w}} \varphi)$	(Variant of Aristotle's Thesis)
$(\varphi \to^{\mathrm{w}} \psi) \to^{\mathrm{w}} \neg(\varphi \to^{\mathrm{w}} \neg\psi)$	(Boethius' Thesis)
$(\varphi \to^{\mathrm{w}} \neg\psi) \to^{\mathrm{w}} \neg(\varphi \to^{\mathrm{w}} \psi)$	(Variant of Boethius' Thesis)
$\varphi, \neg\varphi \vee \psi // \psi$	(Material Modus Ponens)

In many cases, adding a new functor to a language will lead to the adoption of a substantially different language than the one that was supposed to be the subject of a given consideration. Thus, adding a functor counterpart of relating relation into the language will sometimes be simply forbidden. In many cases, if not all, we do not need to incorporate directly the relating relation into the language neither to talk about properties of relating relation, nor to consider the problem of axiomatization. Of course, while we do not need to incorporate the relation directly, we sometimes can do it. The question is, does it make sense? And what kind of functor is the counterpart of relating relation?

4.3. Incorporating the relation: a semiotic analysis

Before we try to answer by means of what kind of functor we can incorporate the relating relation, we first try to characterize semiotically the notion of propositional connective.

[8] In [Jarmużek and Klonowski, manuscript] it is shown that any relating logic defined by general relational conditions can be axiomatized.

φ	$c_0\varphi$	$c_1\varphi$	$c_2\varphi$
1	1	1	0
0	0	1	0

Table 1. Trivial connectives

In semiotic terms, a logical connective might be characterized either syntactically (more precisely, grammatically) or semantically. From a grammatical point of view, a propositional connective is a proposition-creating functor of propositional arguments. From a semantic point of view, a propositional connective is an expression whose interpretation in a given structure (valuation, relational structure, model) depends on the interpretation in the given structure of the component propositions connected by it. In other words, in order to interpret (in a given structure) a proposition built up by means of a propositional connective, we cannot ignore the interpretation (in the structure) of the component propositions.

Following these two semiotic aspects of propositional connectives we emphasize three properties:

- Grammatical category: propositional connectives connect (or precede) propositions, not terms or any other linguistic elements.
- Nesting: propositional connectives can be nested.
- Relative compositionality: the meaning of a propositional connective depends, at least partially, on an interpretation in a given structure of the parts connected (or the part preceded) by such connective.

This approach does not exclude from the universe of propositional connectives some trivial functors. For instance, consider the functors on Table 1.

These functors are propositional connectives, since their meanings, although trivial, cannot be specified without an interpretation of the part they precede.

Let us consider now a functor that is supposed to express that two states, which we refer to by propositions, are related:

$$\underbrace{\text{the fact that } \underbrace{\ldots\ldots}_{\text{proposition}} \text{ is related to the fact that } \underbrace{\ldots\ldots}_{\text{proposition}}}_{\text{proposition}}$$

Let us use the symbol R to represent such a functor in a formal language. Formulas of a propositional language with the standard connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ enriched by functor R in the usual way — i.e., if φ and ψ are formulas, so it is $\varphi\ R\ \psi$ — will be called *R-formulas*.

According to the intended interpretation of our functor, we use the following truth-condition for R-formulas built up by R:

$$\langle v, R\rangle \models \varphi\ R\ \psi \text{ iff } R(\varphi, \psi).$$

The functor R seems a propositional connective from a purely syntactic point of view. We can see that it can be nested. However, the meaning of R seems to be independent of the interpretation of any formulas. If so, we could not count it as a propositional connective.[9]

By contrast, as argued in Section 2.3, one could advance that what we can conclude is not that the meaning of $R(\phi, \psi)$ is independent of the content of ϕ and ψ, but only that it is independent of their extension/intension. Therefore, in a hyperintensional logic, where the content of a formula is richer than its extension/intension, the meaning of R would still be dependent on the way in which ϕ and ψ are interpreted. Indeed, what $R(\phi, \psi)$ says is, as stated above, that the fact described by ϕ is related via R to the fact described by ψ. So, if the fact described by a formula is related to its content, the interpretation of $R(\phi, \psi)$ will ultimately depend on the interpretation of the involved formulas.

Let us also notice that by Fact 4.1 any logic defined in the R-language can be translated into some CMRL. We can use the following translation:

$$\begin{aligned}
&\tau(\varphi) = \varphi, \text{ where } \varphi \in PV\\
&\tau(\neg\varphi) = \neg\tau(\varphi)\\
&\tau(\varphi * \psi) = \tau(\varphi) * \tau(\psi), \text{ where } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}\\
&\tau(\varphi\ R\ \psi) = (\tau(\varphi) \vee^{\mathrm{w}} \tau(\psi)) \vee (\tau(\varphi) \rightarrow^{\mathrm{w}} \tau(\psi))
\end{aligned}$$

[9] Let us notice that this tells Jarmużek and Klonowski's view apart from Sedlár and Tedder's regarding the case of entailment represented as a connective in relevance logic, as discussed in Section 3.2. Indeed, the relevance arrow that is a functor counterpart of entailment relation is a propositional connective — it can be nested and its meaning depends on an interpretation of the parts it connects — whereas the awareness operator of Fagin and Halpern cannot be a logical connective for the similar reasons as the functor R cannot be so.

In order to prove that relating models are faithful with respect to τ we also have to define a translation of the relating relation. Let R be a relating relation over R-formulas:

$$R^{\tau}(\varphi_1, \varphi_2) \text{ iff there are } R\text{-formulas } \psi_1, \psi_2 \text{ such that } R(\psi_1, \psi_2),$$
$$\tau(\psi_1) = \varphi_1 \text{ and } \tau(\psi_2) = \varphi_2.$$

By induction on R-formulas, we obtain:

FACT 4.3. *For any valuation* $v\colon PV \longrightarrow \{1, 0\}$, *any relating relation* R *over R-formulas, and any R-formula* φ, $\langle v, R\rangle \models \varphi$ *iff* $\langle v, R^{\tau}\rangle \models \tau(\varphi)$.

But maybe the functor counterpart of a relating relation should be rather a kind of predicate which is, in fact, suggested by the intended interpretation of it:

$$\underbrace{\underbrace{\ldots\ldots}_{\text{a name of proposition}} \quad \text{is related to} \quad \underbrace{\ldots\ldots}_{\text{a name of proposition}}}_{\text{proposition}}$$

Let us use the symbol R to represent such a functor in the formal language. By R-formulas we mean formulas of the propositional language with the standard connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ enriched by the functor R.

In this case, the functor counterpart of the relating relation connects names of propositions, not propositions themselves. We form the name of formulas by over-lining them. Then, for instance, we have the following R-formulas:

$$\neg(\overline{\neg p_1}\; R\; \overline{p_1})$$
$$\neg(\overline{p_1}\; R\; \overline{\neg p_1})$$
$$(\overline{p_1 \wedge p_2}\; R\; \overline{p_3}) \leftrightarrow ((\overline{p_1}\; R\; \overline{p_3}) \wedge (\overline{p_2}\; R\; \overline{p_3}))$$

but we would also have the following one:

$$\overline{\neg(\overline{\neg p_1}\; R\; \overline{p_1})}\; R\; \overline{\neg(\overline{p_1}\; R\; \overline{\neg p_1})}.$$

We have the following truth-condition for R-formulas built up by R:

$$\langle v, R\rangle \models \overline{\varphi}\; R\; \overline{\psi} \text{ iff } R(\varphi, \psi).$$

Let us notice that R can be nested in some sense, but nesting can occur in expressions with different grammatical categories, and thus it is not a kind of nesting that is distinctive feature of propositional connectives. But the second feature of propositional connectives, i.e. relative compositionality, seems to occur. The meaning of R in a sense depends on the interpretation of its parts, since it might be said that an expression of the form $\overline{\varphi}$ is interpreted as the name of φ. Nonetheless, the functor R is not a logical connective, because it does not connect or precedes propositions, but it is a proposition-creating functor of name-arguments.

5. Summing up and conclusions

In this paper we addressed first and foremost the following question:
Q1. Can R be incorporated into the relating language?
A positive answer was hinted at in 2.1, and then necessary and sufficient conditions for incorporating R were given in 4.1: (Boolean) disjunction, relating disjunction and relating implication must be expressible in the language. Thus yes, R can be incorporated under those conditions.

Nonetheless, the result is not a "definability ticket". It is not that we simply add the required connectives to have the relation incorporated, as they sometimes cannot be added without changing essentially the language, as opposed to merely expanding it. In this respect, it seems fair to conclude that R cannot be incorporated into the language without increasing its descriptive power, even if this, by itself, does not count against the legitimacy of incorporating R in order to get a richer language.

The next question addressed was
Q2. What kind of functor is the incorporated R?
In 2.1, it is suggested to internalize the relations in the model into the language by means of a special class of connectives, which can then be used to define, together with the usual connectives, the relating connectives. In Section 3 there is a proposal along the same lines, namely that the relations can be incorporated into the language through connectives.

The view on Section 4 does not take the R in $\varphi\ R\ \psi$ as a connective, though. The reason for the reluctance is that its evaluation conditions are not those of a connective: its value does not depend at all on the values of its components. To the claim that "[in relevance logics] it pays to be "confused" about the object and metalanguage distinction" it should be replied that scare quotes around 'confusion' are right, because the theoretical profits in relevance logic are not due to a confusion: the entailment arrow is a connective, at least under the criteria put forward in 4.3.

That view has the consequence that the awareness operator would not be a connective either, and that it is qualitatively distinct from the standard epistemic operators. Compare the satisfaction conditions of each:

$$\langle \mathfrak{M}, w\rangle \models K\varphi \text{ iff } \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W,$$
$$\langle \mathfrak{M}, w\rangle \models A\varphi \text{ iff } \varphi \in A(w), \text{with } A(w) = W \times \mathcal{P}(L_4)$$

Unlike the case of K, there is no satisfiability relation for formulas on the right-hand side of satisfiability relation for the awareness functor A. Note that the claim is not that the relations in the metalanguage cannot be internalized, but rather that it can be so through a connective. Absence of problematization here is not absence of problem.

Further properties of R are worth investigating. For example,
Q3. Can R, the functor counterpart of R, be iterated?
Independently of the status of R — whether it is a predicate, a connective or something else — none of the proposals in this paper puts restrictions on the iteration of R. Nonetheless, a word of caution is mandatory here regarding the demand of nesting as a necessary condition for being a connective. Consider Anderson and Belnap's logic of tautological entailment, $\mathbf{E}_{\text{fde}}$ [see Anderson and Belnap, 1975, Ch. 15]. There, an expression like $p \to (p \to p)$ would not even be well-formed. There are at least three options in facing this situation:

(i) Nesting is required, so the arrow in $\mathbf{E}_{\text{fde}}$ is not a connective.
(ii) Nesting is not required, relative compositionality is enough, so the arrow in $\mathbf{E}_{\text{fde}}$ is a connective.
(iii) Nesting is not required in the same logic, but there must be a logic containing the original one where the connective can be nested; then the arrow in $\mathbf{E}_{\text{fde}}$ is a connective, because although it cannot be nested *in* $\mathbf{E}_{\text{fde}}$, it can be nested in some extensions of $\mathbf{E}_{\text{fde}}$, notably $\mathbf{E}$ and $\mathbf{R}$.[10]

This is not the place to even starting an attempt to solve this issue. The first option seems highly counter-intuitive, but avoiding it is not easy[11], as we will see in commenting the other two options. The second

[10] Other noteworthy extensions of $\mathbf{E}_{\text{fde}}$ are presented in [Paoli, 2007].

[11] This is in fact the line taken in [Humberstone, 2011, p. 1156], who declares the arrow in $\mathbf{E}_{\text{fde}}$ is not a propositional connective.

one requires a principled way to distinguish vocabulary that belongs to the stock of connectives from those that do not, and it remains to be seen whether relative compositionality, or a suitable surrogate of it, can be regarded as both a necessary and sufficient condition to identify connectives. Moreover, there is the problem that relative compositionality might be defined in such way that it induces nesting.[12] The third requires as well a principled characterization of those logics that can be regarded as proper logics to evaluate whether a given piece of vocabulary is nestable. As far as we know, this option has not been addressed in the literature.[13]

Finally, the following question was also addressed:

Q5. The direct incorporation of R into the language of relating logic is really needed?

Contrary to the appearances, there is more room for discussion here. As it has been said in 4.1, in a language containing disjunction, relating disjunction and relating implication, there cannot be a question about the need of incorporating R: it is already definable there. But if in language like that all what can be done in terms of the functor counterpart of R can be done without it, there is no need to consider it explicitly among the primitive notions. In that sense, such functor is not needed.

But simplicity about the number of primitive notions and signs — let us call it 'parsimony' here — is one among many other theoretical virtues. In fact, simplicity about the overall presentation of a theory — let us call it 'elegance' — might be as important as parsimony itself. As it has been argued by Baker [2003], while these two forms of simplicity are frequently conflated, it is important to treat them as distinct. One reason for doing so is that considerations of parsimony and of elegance typically pull in different, even opposite directions. Using extra signs may allow a theory to be formulated more simply, while reducing the stock of signs of a theory may only be possible by making it more complex, at least from

[12] Scott [1973] discussed at some length the problem of "unlimited closure under connectives"; Booth [1991] could serve as a basis for a conception of language that can make room for the arrow as a connective already in $\mathbf{E}_{\mathrm{fde}}$. A similar question arises, for example, in positional logic, where the realization operator is not nested in basic systems. However, it can be nested in the extensions of the basic systems. See [Jarmużek and Tkaczyk, 2019] or the special issue on positional logic [Jarmużek and Kupś, 2020].

[13] The question on the iterability of R invites the question on the iterability of R itself. We hope this is addressed in further investigations.

the point of view of tractability by human beings in a particular state of logical research.[14]

Something may not be absolutely necessary or necessary in itself, but necessary for something else, or for something to be obtained or to be done. In these latter cases one can speak rather of *usefulness* and *fruitfulness*, which might be as important as parsimony and elegance. There is a branch of logic that studies formal languages on its own, and clearly there is nothing wrong about that. But there are other logical enterprises, such as studying what arguments are valid, or comparing different logics. Part of the logician's method for studying validity (or comparing logics) is to represent valid arguments (or collections of them) with certain languages, but in such enterprises, these languages (and the functors that in part define them) are logic's tools, not its subject matter. The particular linguistic items with which logicians are concerned at a particular point in the development of logic are just a reflection of the logicians' progress (up to that point) in systematically doing something else, like classifying valid arguments or comparing logics.

Moreover, one cannot ignore the cognitive dimension of formal languages — see [Dutilh Novaes, 2012] on this — nor their ergonomic features — that is, their relative suitability for certain tasks for agents like us [see Blackwell, 2008; Barceló, 2016].[15] Whether by culture, training, acquaintance or whatnot, different presentations of logics might trigger different trains of thought even for the same person, and the discussion on Section 3 is an example of that. While if one is told that R can be internalized as $(\varphi \vee^{\mathrm{w}} \psi) \vee (\varphi \rightarrow^{\mathrm{w}} \psi)$ and then one just says 'OK', using a special notation for this, such as $\varphi\ R\ \psi$, or even $\varphi \looparrowright \psi$, could suggest links with other logics and topics — such as epistemic logics — and problems — for example, the nature of connectives and the classification of functors — that are not directly triggered by the more parsimonious notation.[16]

[14] For more on the goal-oriented nature of simplicity in logic, in particular in logical notation see [Bellucci et al., 2018] and [Woleński, 1989, Ch. V].

[15] For example, the history of logic and philosophy of mathematics in the twentieth century would have been different if Frege's theory in the *Grundgesetze* would have immediately been reduced to its minimum, i.e. the underlying language and a sole rule of derivation: from any set of premises whatsoever obtain any conclusion.

[16] Nonetheless, there are issues that are invited by both the more parsimonious expression and the one including extra signs. For example, both raise questions about the direction of the relation: Is *φ and ψ are related* the same as *φ is related to ψ*?

Acknowledgements. We want to thank the participants in the meeting "Do we really need relation R to be directly incorporated into the language of relating logic?" for their stimulating discussions and suggestions. The first author was supported by the PAPIIT project IN403719 and also by an IDUB mobility award.

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For the more parsimonious approach, this question arises because of the implication involved in the definition; in the other case, because the relation between $\varphi \, R \, \psi$ and $\psi \, R \, \varphi$ is almost a natural question to ask.

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LUIS ESTRADA-GONZÁLEZ
Institute for Philosophical Research
National Autonomous University of Mexico
Mexico
loisayaxsegrob@gmail.com

Alessandro Giordani
Faculty of Letters and Philosophy
Catholic University of Milan
Italy
alessandro.giordani@unicatt.it

Tomasz Jarmużek and Mateusz Klonowski
Department of Logic
Nicolaus Copernicus University in Toruń
Poland
{jarmuzek,mateusz.klonowski}@umk.pl

Igor Sedlár
Institute of Computer Science
Czech Academy of Sciences
The Czech Republic
sedlar@cs.cas.cz

Andrew Tedder
Institute of Philosophy I
Ruhr-Universität Bochum
Germany
ajtedder.at@gmail.com