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Leibnizian Logic of Possible Laws.
A Formal Framework Motivated by Hintikka That
Blocks Lovejoy's Principle of Plenitude

Abstract. The so-called Principle of Plenitude was ascribed to Leibniz by
A. O. Lovejoy in *The Great Chain of Being: A Study of the History of an
Idea* [9]. Its temporal version states that what holds always, holds necessarily (or that no genuine possibility can remain unfulfilled). This temporal
formulation is the subject of the current paper. Lovejoy's idea was criticised
by Hintikka. The latter supported his criticisms by referring to specific
Leibnizian notions of absolute and hypothetical necessities interpreted in a
possible-worlds semantics. In the paper, Hintikka's interpretative suggestions are developed and enriched with a temporal component that is present
in the characteristics of the real world given by Leibniz. We use in our approach the Leibnizian idea that change is primary to time and the idea that
there are possible laws that characterize worlds other than the real one.
We formulate a modal propositional logic with three primitive operators for
change, temporal constancy, and possible lawlikeness. We give its axiomatics and show that our logic is complete with respect to the given semantics
of possible worlds. Finally, we show that the counterparts of the considered
versions of the Principle of Plenitude are falsified in this semantics and the
same applies to the counterpart of Leibnizian necessarianism.

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Keywords: Leibnizian possible worlds; the principle of plenitude; logic of
change; temporal logic; philosophical logic

Introduction

In the years 1932–1933, Arthur O. Lovejoy gave a series of lectures at
Harvard University, first published in 1936 under the title *The Great*

Chain of Being: A Study of the History of an Idea [9].[1] The leitmotif of these historical considerations was the search in different philosophical systems for the principle that Lovejoy called the *Principle of Plenitude* expressed by the slogan that all the metaphysical universe is 'a *plenum formarum* in which all categories of Being are necessarily exemplified' [9, 55]. In two lectures (especially in V, but also in IX) he devoted special attention to Leibnizian metaphysics, believing that his Principle should be treated even as 'the essential characteristic' of the totality of all *monads*, as a consequence of the *Principle of Sufficient Reason*, resulting in the *Principle of Continuity* [9, 144]. In one of the many formulations of the Principle of Plenitude sought in Leibniz's texts, Lovejoy states that 'no truths concerning compossibility are contingent' [9, 171]. This claim, in a certain temporal specification we adopt, is the subject of our criticism. We are convinced that Lovejoy's approach to Leibnizian metaphysics is misguided and we draw inspiration from a comprehensive analysis of this issue, developed by Hintikka in [3, 4].

The research presented here follows the main path of Hinitkka's attack in [3] but with two further interpretative steps concerning Leibniz's philosophy. First, we take into account its temporal component. In fact, Hintikka also considered temporal paraphrases of Lovejoy's Principle but he did not specify their relationship to Leibniz's idea of time. From the temporal perspective, the mentioned 'truths concerning compossibility' are propositions which *always hold* in some possible world. We adopt this way of speaking and we take into account some special properties of time described by Leibniz. In particular, we work with Leibniz's original idea that time is secondary to change. Our second addition to Hintikka's considerations relates to his discussion of different kinds of necessity: *absolute* and *hypothetical*. Lovejoy considers only the former one and thus he runs counter to the intentions of Leibniz. In terms of the semantics of possible worlds, we would say that a given law is absolutely necessary in some possible world when it always holds in that world and it is the case in all other possible worlds. In turn, those laws that characterize some possible world, but do not hold in at least one of the others, are considered as hypothetically necessary in respect to the world that they describe, in case that this world is considered as actual. The contingent laws of our world are constant in it, but they are only

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[1] The book is still in circulation. Until 2001, twenty-two reprints had been issued, with a second edition published by Routledge in 2009.

hypothetically necessary. From our point of view, it is interesting that both types of necessity as well as contingency applied to laws are not reducible to mere temporal constancy in a given world, insofar as their meanings are explained from the perspective of other possible worlds than the one in which they hold. This is one of the key ideas of Hintikka that we adopt and so we introduce the notion of *possible lawlikeness* into our considerations. Propositions that are possible laws characterize possible worlds other than ours and they are hypothetically necessary in these worlds.

As a result of our research, we propose a formalization in which changeability, constancy, and possible lawlikeness are interrelated in a way motivated by Leibnizian metaphysics. It is based on the propositional modal logic $\mathsf{L}\mathcal{C}\square$ which was already associated with Leibniz's philosophy of time and change in [12]. $\mathsf{L}\mathcal{C}\square$ deals with two primitive concepts: change, symbolized by the operator $\mathcal{C}$ (*it changes whether*), and constancy expressed by $\square$. In $\mathsf{L}\mathcal{C}\square$ we introduce a new primitive operator $\hookrightarrow$, to be read: *it is a possible law that.* We tie it with $\mathcal{C}$ and $\square$, and we axiomatize a new logic $\mathsf{L}\mathcal{C}\square\hookrightarrow$. The resulting system is interpreted in the structures of so-called *states of possible worlds.* We show that it is complete with respect to our semantics. Finally, we return to Lovejoy's Principle expressed in terms of our logic. It turns out that the formulated versions of it are false.

We start our presentation with a few explanations of the Principle of Plenitude and a preformal description of Leibniz's motivations. Although we are not focused particularly on the exegesis of Leibniz's texts, we indicate those of Leibniz's threads that have led us to the construction of the logic $\mathsf{L}\mathcal{C}\square\hookrightarrow$.

1. The Principle of Plenitude

Lovejoy interpreted Leibnizian metaphysics as taking into account a picture of reality being, in a sense, a system of *monads* — eternal spiritual individuals — described especially in two theodicean treatises: 'The Principles of Nature and of Grace, based on Reason' [7, G. VI 598–606, 636–642] and 'The Monadology' [7, G. V 607–623, 644–652]. In order to find the idea of a 'chain of being', which would allow him to state that Leibniz's metaphysics meets the Principle of Plenitude, he referred to a

fragment of a letter allegedly written by Leibniz, supposedly addressed
to Bayle. Here we read the following

> All the different classes of beings which taken together make up the
> universe are [. . .] so closely united that it would be impossible to place
> other between any two of them, since that would imply disorder and
> imperfection. [. . .] it is necessary that all the orders of natural beings
> form but a single chain, in which the various classes, like so many rings,
> are so closely linked one to another that it is impossible for the senses
> or the imagination to determine precisely the point at which one ends
> and the next begins [. . .]. [9, 144–145]

On this basis Lovejoy considered the Leibnizian reality as the universe
of sets of all monads, forming, except the most perfect one which is *God*,
so to speak: a linear order of species. Taking just this view, he stated
that '[t]o all appearance reality is full, not only in its minor details but
also in its more general features' [9, 147].

In Leibniz's universe, the existence of the extremely perfect monad is
necessary and this follows directly from the fact that it is possible [7, G.
VII 261–262, 167–168]. By virtue of the Principle of Sufficient Reason,
God is the ultimate reason for all other monads to exist [7, G. VII 302–
308, 486–487]. Thus, in Lovejoy's opinion, it is precisely the Principle
of Sufficient Reason that implies the fullness of reality described by the
Principle of Plenitude, and the latter is supposed to guarantee that in
such an ordered reality there are no gaps. These suggested relationships,
however, are not modeled here.

Lovejoy considers his idea of the 'chain of being' with reference to the
Leibnizian idea of the *real world* [9, 171] while aware of the explicit state-
ment of Theofilus that 'there must be species which never did and never
will exist, since they are not compatible with that succession of creatures
which God has chosen' [8, 307]. He then addresses the Principle of Pleni-
tude to the whole universe of infinitely many possible worlds to which the
real world belongs. The real world according to Leibniz is a collection of
compossible complete concepts of *individual substances*, i.e. monads [7, G.
III, 572–576, 662]. The complete concepts are complexes of all (*relational*
and *internal*) *qualities* attributed to monads [7, Ca. 1680–1684, 268]. In
our interpretation, we follow Mates and we extend this vision to every
possible world. Thus, we consider all possible worlds as formed out of
compossible concepts [10, 69–78]. In the case of nonreal worlds, they
are complexes of properties that could be attributed to some individuals

if the latter existed [10, 64]. The relational notion of compossiblity is different from the *possibility in an absolute sense*. The absolutely possible concepts are those that are not self-contradictory, but not all such concepts are mutually compossible [7, G. III 572–576, 661–662]. There are possibilities which never happen, otherwise all non-actual objects would be impossible [7, F. de C. 178–185, 263–266] and this statement is explicitly rejected by Leibniz. Interestingly, it is just the contraposition of Hintikka's temporal version of the Principle of Plenitude, which we will identify in a moment as (PM). All compossible complete concepts, as Russell would say in [11, 79], are under the 'reign' of characteristic *contingent laws* determining specific 'general features' of their reality [7, G. II 47–59, 333]. Contraries of contingent propositions imply no contradiction [7, G. IV, 427–463, 310]. One could say in terms of the semantics of possible worlds that contingent laws are constancies that describe the real world but their negations are realizable in some other possible worlds. Examples of contingent laws are *truths of facts*, including the laws of motion, as well as cases of inductive empirical generalities [7, PA. VI, 266–272, 88]. On the contrary, the absolutely (or metaphysically) necessary laws — called by Leibniz *eternal truths* — are those whose negations are contradictory [7, F de C., 178–185, 264]. They are constancies realized in every possible world. As for the term 'necessary' itself, Leibniz notes in a letter to Clark [7, IX, 696] that it is ambiguous because it can mean either absolute or hypothetical necessity. These modalities are essentially different and they play the important role in Leibniz's defense of himself against necessarianism — a view according to which everything that is real is also necessary. Taking precisely this distinction, Leibniz stated that '[t]he present world is necessary in a physical or hypothetical sense, not absolutely or metaphysically.' [7, G. VII 302–308, 487].[2] Following Leibniz, with regard to the laws, we would say that hypothetically necessary are those constancies that are

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[2] Adams reconstructs the two contingency theories on the basis of Leibniz's writings falsifying the statement that Leibniz's philosophy is founded on a general necessitarian claim [1, 9–52]. The distinction between absolute and hypothetical necessity is considered in the first one which is based on the Leibnizian idea of truth [1, 16–19]. In the second one, the central point is the issue of God's freedom to choose the real world. On the basis of both of these theories, Adams believes that Leibniz managed to defend himself against his early necessarian position. We share this opinion and use concepts that are involved in the first of the reconstructed theories (and this is in line with Hintikka's approach). Cf. also [10, 117–121].

valid in some world provided this world is realized (that this happens by the decision of God). All contingent laws of the real world are just of this kind. However, in general, the laws that characterize other possible worlds may also be candidates for such necessary propositions. Leibniz is convinced of the existence of such laws when he writes that

> [...] as there exists an infinite number of possible worlds, there exists also an infinite number of laws, some peculiar to one world, some to another, and each possible individual of any one world contains in the concept of him the laws of his world.   [6, G II. 40, 43]

In fact, this Leibnizian idea motivates us to introduce in our formalization the aforementioned modality of possible lawlikeness which applies to constancies that occur in other possible worlds than the real one. Both actual contingent and other possible laws are '*laws* of the *general order* of this possible universe with which they are in accord' [7, G. II, 47–59, 333].

Given this outline of the above distinctions, we can now extract from the extended historical context the main thread of one of Lovejoy's numerous arguments that lead him to the 'discovery' of the alleged Leibniz-like Principle of Plenitude. It begins with the following line from Russell:

> What is called the 'reign of law' is, in Leibniz philosophy, metaphysically necessary, although the actual laws are contingent.   [11, 79]

First, as Hintikka points out, Lovejoy asserts the statement that the compossibility of concepts is nothing more than their possibility (1). Regardless of Leibniz's distinction, Lovejoy states that '[it] seems plain that compossibility does not differ in principle from possibility, in the traditional philosophical sense of the latter term' [9, 171]. As a result, Lovejoy states that from the perspective of the real world it must be the case that, all compossible concepts which are simply possible, are realized in it (because they form in it the 'chain of being') (2). Now, Lovejoy makes the next step by assuming that all reality is absolutely necessary (3), and so, all possible concepts (and laws about them) are just absolutely necessary (4). However, in view of the distinction between absolute and hypothetical necessity and the quoted passage of Leibniz [7, G. VII 302–308, 487], assumption (3) is also flawed and this fact is again noticed by Hintikka.[3] Anyway, Lovejoy follows this path and

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[3] One of Lovejoy's arguments for (3) is based on the idea that God acts according to the Principle of Sufficient Reason and in view of the fact that the real world is the most perfect one from infinitely many possible worlds, God was forced to actualize

he keeps in his conclusion from (3) the sense of metaphysical necessity used by Russell, even if he should have used the hypothetical sense. One could also say that even if Lovejoy would omit (3) (or he would weaken it using the notion of hypothetical necessity), the reasoning regarding (4) would be biased by the so-called *fallacy of the slipped necessity*.[4] Russell's formulation is not in doubt when understood that it is absolutely necessary that if anything belongs to the reign of a given law, then it is subject to that law. However, it does not follow from this that, in the case where this law is contingent, it applies to any element of its reign with absolute necessity. But Lovejoy just claims that in the frame of Leibnizian metaphysics all possibilities that belong to the reign of actual contingent laws are metaphysically necessary, and moreover, that the same applies to every possible world, regardless of which one was chosen by God [9, 171–172].

As we have already said, our formal reconstruction of Lovejoy's idea involves elements of the Leibnizian concept of time. We justify this approach by the fact that the Leibnizian real world itself contains a temporal component. As Leibniz states, individual concepts constituting the real world contain past, present and future qualities (predicates) of monads (individual substances) [7, Ca. 1680–1684, 268], which are subject to changes. For this reason, we look at the real world as a series of transforming *states* [7, G. VII 303, 487] which we understand as collections of complexes of all qualities that the monads possess simultaneously. Changes that consist in acquiring or losing qualities by monads cause the state of the world to change and, as a result, time begins to flow being a measure of change [7, GV. 139]. We consider so-called *objective external time* which begins with the creation of the real

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it. In fact, this justification is not convincing from the perspective of the second of Leibniz's contingency theories mentioned by Adams [cf. especially 1, 34–42]. We do not consider this issue here.

[4] The fallacy lies in the confusion between 'Necessarily, if P then Q' and 'if P then necessarily Q'. As Mates claims, Leibniz realized that reasonings of this type are incorrect [10, footnote 33, pp. 97–98, p. 117]. Using L for necessity, the fallacy can be associated with the use of schema (s^L) $L(A \to B) \to (A \to LB)$. When ($\mathsf{s}^L$) is added to classical propositional logic extended only by the necessity rule $\vdash A \Longrightarrow \vdash LA$, then the formula $A \to LA$ is derivable. In the case of the Lovejoy's reasoning as sketched above, A would represent the statement that *from the perspective of the real world all compossible concepts are simply possible* and B — *all compossible concepts are realized in the real world*. Sentence (2) would be of the form $L(A \to B)$ and conclusion (4): $(A \to LB)$.

world[5], which is 'an order of successions' [7, Letter to Clark IIIth, 682] and infinite in the future [7, G. II, 362].

The laws of any possible world understood in this way are to be true in all its successive states and so to be true at all times. Now it is clear that Hintikka's formulations can be regarded as temporal counterparts of Lovejoy's Principle of Plenitude:

$$\text{What holds always, holds necessarily.} \tag{PL}$$

$$\text{No genuine possibility can remain unfulfilled} \atop \text{through an infinite stretch of time.} \quad [3, 260] \tag{PM}$$

In our formalization, we refer to these two versions.

Our aim is to find the counterparts of (PL) and (PM) expressed in a language with primitive notions of change, constancy, and possible lawlikeness, and then to falsify these counterparts in the temporal semantics of possible worlds given for that language. First, however, we model some preformal intuitions that underlie it.

2. From one to many possible worlds

We accept the idea that there are infinitely many, mutually independent monads forming the set $M = \{\mathsf{m}_i\}_{i\in\mathbb{N}}$ and infinitely many of their qualities that are elements of $Prop = \{\mathsf{p}_k\}_{k\in\mathbb{N}}$. Monads can acquire and lose qualities never losing their existence. The nonempty set $a_1 \subseteq M \times Prop$ is the initial state of the real world.

In general, we consider the following structure: $\langle M, Prop, \boldsymbol{a}\rangle$, where $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{N}}$ and every $a_n \subseteq M \times Prop$. Successive states of $\boldsymbol{a}$ represent the Leibnizian real world, about which we assume at least two minimal conditions that must be met at the same time. Firstly, no state of the real world is empty, i.e., for all a_n: $a_n \neq \emptyset$. In this way we model the assumption that in every state of the real world there exist some concepts that are compossible in a_n. Secondly, we accept the idea that successive states of the real world are conditioned by changes in attributions; that is, for all a_n there are $\mathsf{m}_i, \mathsf{p}_k$ such that: $((\langle \mathsf{m}_i, \mathsf{p}_k\rangle \in a_n$ and $\langle \mathsf{m}_i, \mathsf{p}_k\rangle \notin a_{n+1})$ or $(\langle \mathsf{m}_i, \mathsf{p}_k\rangle \notin a_n$ and $\langle \mathsf{m}_i, \mathsf{p}_k\rangle \in a_{n+1}))$. These changes induce the flow of external time.

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[5] As Russell notes, this is one of the three senses of time, we can find in Leibniz's philosophy, in addition to time in God's mind and time based on the mutual *perceptions* of monads [11, 152].

Time begins with God's choice of the initial state of the real world. We consider time as linear. The unity of time is identified with its connectivity.[6]

Let us take now the propositional language formed out of: atomic propositions $At = \{\alpha_k^i\}_{i,k \in \mathbb{N}}$; truth connectives $\neg$, $\rightarrow$; modal operators: C – *it changes whether*, $\Box$ – *it is necessary that*; brackets: (,). Atomic constant α_k^i is to mean that *quality* p_k *is attributed to monad* m_i.

We accept the standard definitions of other classical connectives. The interpretation function $int\colon At \rightarrow M \times Prop$ assigns meanings to atoms: $int(\alpha_k^i) = \langle \mathsf{m}_i, \mathsf{p}_k \rangle$.

For any formulas A, B of our language, we describe their *satisfaction* on n state of $\boldsymbol{a}$ ($\models$) in the following way:

- $M, Prop, \boldsymbol{a} \models^n \alpha_k^i$ iff $int(\alpha_k^i) \in a_n$,
- $M, Prop, \boldsymbol{a} \models^n \neg A$ iff $M, Prop, \boldsymbol{a} \not\models^n A$,
- $M, Prop, \boldsymbol{a} \models^n A \rightarrow B$ iff either $M, Prop, \boldsymbol{a} \not\models^n A$ or $M, Prop, \boldsymbol{a} \models^n B$,
- $M, Prop, \boldsymbol{a} \models^n CA$ iff   either both $M, Prop, \boldsymbol{a} \models^n A$ and $M, Prop, \boldsymbol{a} \not\models^{n+1} A$, or both $M, Prop, \boldsymbol{a} \not\models^n A$ and $M, Prop, \boldsymbol{a} \models^{n+1} A$,
- $M, Prop, \boldsymbol{a} \models^n \Box A$ iff $\forall_{m \geqslant n}(M, Prop, \boldsymbol{a} \models^m A)$.

For $\Diamond$ we accept the semantic condition:

- $M, Prop, \boldsymbol{a} \models^n \Diamond A$ iff $\exists_{m \geqslant n}(M, Prop, \boldsymbol{a} \models^m A)$.

Just from conditions for $\Box$ and $\Diamond$ we know that for every state n of the real world $\boldsymbol{a}$:

$$\forall_{m \geqslant n}(M, Prop, \boldsymbol{a} \models^m A) \Longrightarrow M, Prop, \boldsymbol{a} \models^n \Box A, \qquad (\mathsf{P}\Box)$$
$$M, Prop, \boldsymbol{a} \models^n \Diamond A \Longrightarrow \exists_{m \geqslant n}(M, Prop, \boldsymbol{a} \models^m A). \qquad (\mathsf{P}\Diamond)$$

If operators $\Box$ and $\Diamond$ were regarded as formalizing the notions of necessity and possibility that occur in (PL) and (PM), respectively, then implications (P$\Box$), (P$\Diamond$) could be regarded as their formalizations. In connection with our semantics of the states of the real world, one could say that our $\Box$, $\Diamond$ express 'real' necessity and 'real' possibility (or 'genuine' necessity/possibility). The crux of all this construction, however,

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[6]   Our ideas are based on research conducted by Futch in [2]. He considers unity, linearity, and posteriority to change, as the essential attributes of time in Leibniz's philosophy. We do not consider here the issue of the justification of temporal reductionism based on the Leibnizian theory of causality, which plays an important part in Futch's analysis. In our formalization, the concept of change gets only an extensional description.

is that it is precisely in this one-world semantics, that the meanings of necessity and possibility are flattened to the temporal sense, just as is expressed by the truth conditions for $\square$ and $\lozenge$.

As Hintikka rightly points out in his counter-argument to Lovejoy's position, as long as we cannot relate these modalities to a world other than the real one, they can express only time constancy or instantaneous occurrence in this real world. But when interpreted like this, our temporal versions of the Principle of Plenitude become trivial and do not express what is crucial to them. Actually, they are mentioned to establish the relationships between constancy, instantaneous occurrence, and metaphysical necessity and possibility respectively.

To model the Leibnizian meanings of the considered modalities, we follow Hintikka's idea and we consider a structure $\langle M, Prop, \boldsymbol{W} \rangle$, where M, $Prop$ are as described above and $\boldsymbol{W}$ is at least a two-element set of Leibnizian possible worlds, such that every $\boldsymbol{w} \in \boldsymbol{W}$: $\boldsymbol{w} = \{w_n\}_{n \in \mathbb{N}}$, where every $w_n \subseteq M \times Prop$, $\boldsymbol{a} \in \boldsymbol{W}$.

We modify the interpretation of the formulas. We refer now to $\langle M, Prop, \boldsymbol{W} \rangle$ and we proceed in an analogical way as described for $\boldsymbol{a}$ -satisfaction, but we consider satisfaction in any $\boldsymbol{w} \in \boldsymbol{W}$.

Our language is extended now by the primitive operator $\hookrightarrow$, to be read: *it is a possible law that* with the following meaning:

- $M, Prop, \boldsymbol{W}, \boldsymbol{w} \models^n \hookrightarrow A$ iff $\exists_{\boldsymbol{u} \in \boldsymbol{W}/\{\boldsymbol{a}\}} \forall_{k \in \mathbb{N}} (M, Prop, \boldsymbol{W}, \boldsymbol{u} \models^k A)$.

Now we can see that constancy $\square$ in $\boldsymbol{a}$ does not imply lawlikeness in $\boldsymbol{w}$. As to contingent laws, they are $\boldsymbol{a}$-constancies the negations of which hold in some non-real world. In turn, the fact of being absolutely necessary guarantees $\boldsymbol{a}$-constancy and impossibility to be non-constant in any nonreal world.

Now it is also easy to see that the implications:

$$M, Prop, \boldsymbol{W}, \boldsymbol{a} \models^n \square A \implies M, Prop, \boldsymbol{W}, \boldsymbol{a} \models^n \neg \hookrightarrow \neg \square A, \quad (*)$$
$$M, Prop, \boldsymbol{W}, \boldsymbol{a} \models^n \hookrightarrow \lozenge A \implies M, Prop, \boldsymbol{W}, \boldsymbol{a} \models^n \lozenge A \quad (**)$$

do not follow from the assumed satisfaction conditions.

We come precisely to Hintikka's standpoint that in many possible worlds semantics, both absolutely necessary and contingent laws are not reducible to constancies in this world in which they are considered.

Our next step is to formulate a logic for the analysis proposed here and to find counterparts to (PL) and (PM) that are falsified in a certain generalization of the semantics outlined in this section.

3. Change, constancy, and possible lawlikeness formalized. Logic $\mathsf{L}\mathcal{C}\square\hookrightarrow$

Let us consider so-called *L-structures* which are tuples $\langle W, a \rangle$, where W is a set of Leibnizian possible worlds with at least two elements, and $a \in W$ is the real world. Every Leibnizian possible world $w \in W$ is a set $w = \{\mathrm{w}_n\}_{n \in \mathbb{N}}$ of its successive states.

Attributions of qualities to monads are coded using atomic expressions in our language. However, we do not note in this language any dependencies between the attributions of qualities (and also the monads themselves) and so the issue of the internal construction of states is irrelevant. Thus, states of Leibnizian possible worlds are points this time. For any *L*-structure we consider the interpretation function $\mathcal{I} \colon \bigcup W \to 2^{At}$ that assigns to every state the set of atomic formulas that are true in it. For L and $\mathcal{I}$ we define the satisfaction of formulas. We modify the condition for atomic formulas given in Section 2:

$$L, \mathcal{I}, w \models^n \alpha_k^i \quad \text{iff} \quad \alpha_k^i \in \mathcal{I}(\mathrm{w}_n)$$

and we proceed as before:

$$
\begin{aligned}
L, \mathcal{I}, w \models^n \neg A \quad &\text{iff} \quad L, \mathcal{I}, w \not\models^n A, \\
L, \mathcal{I}, w \models^n A \to B \quad &\text{iff} \quad L, \mathcal{I}, w \not\models^n A \text{ or } L, \mathcal{I}, w \models^n B, \\
L, \mathcal{I}, w \models^n \mathcal{C}A \quad &\text{iff} \quad (L, \mathcal{I}, w \models^n A \text{ and } L, \mathcal{I}, w \not\models^{n+1} A) \text{ or} \\
&\qquad (L, \mathcal{I}, w \not\models^n A \text{ and } L, \mathcal{I}, w \models^{n+1} A), \\
L, \mathcal{I}, w \models^n \square A \quad &\text{iff} \quad \forall_{k \geqslant n}(L, \mathcal{I}, w \models^k A), \\
L, \mathcal{I}, w \models^n \hookrightarrow A \quad &\text{iff} \quad \exists_{u \in W/\{a\}} \forall_{k \in \mathbb{N}}(L, \mathcal{I}, u \models^k A).
\end{aligned}
$$

For $\vee$, $\wedge$ and $\leftrightarrow$ the satisfaction conditions are standard.

We introduce also two other modal operators $\Diamond$, and $(u\mathcal{C})^k$. The first has a meaning analogous to the meaning given in Section 2:

$$L, \mathcal{I}, w \models^n \Diamond A \quad \text{iff} \quad \exists_{m \geqslant n}(L, \mathcal{I}, w \models^m A).$$

The modality $(u\mathcal{C})^k$ is to be read: *through k states it doesn't change that* and is understood in the following way:

$$L, \mathcal{I}, w \models^n (u\mathcal{C})^k A \quad \text{iff} \quad \forall_m(n \leqslant m \leqslant n + k \implies L, \mathcal{I}, w \models^m A).$$

Having any fixed interpretation $\mathcal{I}$, we say that the formula A is *valid in* L iff $L, \mathcal{I}, w \models^n A$, for every $w \in W$ and $n \in \mathbb{N}$. A is *logically valid* iff A is valid in every *L*-structure.

As we have already announced, we base the system of our new logic $\mathsf{LC}\square\hookrightarrow$ on the $\mathsf{LC}\square$ formulated in [12]. In the logic $\mathsf{LC}\square$ two primitive operators $\mathcal{C}$ and $\square$ are used. Originally, $\mathsf{LC}\square$ was meant to formalize the Leibnizian idea to treat the notion of change as primitive to time and to define by change linear infinite time with the first element.

Logic $\mathsf{LC}\square\hookrightarrow$ is defined as the smallest set which contains:

- all tautologies of classical logic;
- equivalences defining $(u\mathcal{C})^n$:

$$
\begin{aligned}
&(u\mathcal{C})^0 A \leftrightarrow A, \\
&(u\mathcal{C})^{n+1} A \leftrightarrow (u\mathcal{C})^n A \wedge \neg\mathcal{C}((u\mathcal{C})^n A);
\end{aligned}
\qquad\qquad ((u\mathcal{C})^n)
$$

- $\mathsf{LC}\square$ axioms of the following forms:

$$
\begin{aligned}
\mathcal{C}A \to \mathcal{C}\neg A, &\qquad\qquad (\mathcal{C}\neg)\\
\mathcal{C}(A \wedge B) \to \mathcal{C}A \vee \mathcal{C}B, &\qquad\qquad (\mathcal{C}\vee)\\
A \wedge \neg\mathcal{C}A \wedge \mathcal{C}B \to \mathcal{C}(A \to B), &\qquad\qquad (\mathcal{C}\to)\\
A \wedge B \wedge \mathcal{C}A \wedge \mathcal{C}B \to \mathcal{C}(\neg A \wedge \neg B), &\qquad\qquad (\mathcal{C}\wedge)\\
\square A \to (u\mathcal{C})^n A, \text{ for all } n \geqslant 0, &\qquad\qquad (\square u\mathcal{C})\\
\square A \to \neg\mathcal{C}\square A, &\qquad\qquad (\neg\mathcal{C}\square)
\end{aligned}
$$

- specific axioms for $\hookrightarrow$ with the following forms:

$$
\begin{aligned}
\neg\hookrightarrow A \leftrightarrow \hookrightarrow\neg\hookrightarrow A, &\qquad\qquad (\hookrightarrow\neg)\\
\hookrightarrow A \wedge \hookrightarrow B \leftrightarrow \hookrightarrow(\hookrightarrow A \wedge \hookrightarrow B), &\qquad\qquad (\hookrightarrow\wedge)\\
\hookrightarrow A \wedge \hookrightarrow(\hookrightarrow A \to B) \to \hookrightarrow B, &\qquad\qquad (\hookrightarrow\to)\\
\neg\mathcal{C}\hookrightarrow A, &\qquad\qquad (\hookrightarrow\mathcal{C})\\
\hookrightarrow A \leftrightarrow \hookrightarrow\square A, &\qquad\qquad (\hookrightarrow\square)\\
\hookrightarrow(\square A \vee \square B) \to (\hookrightarrow\square A \vee \hookrightarrow\square B); &\qquad\qquad (\hookrightarrow\square\vee)
\end{aligned}
$$

- equivalences for $\lozenge$:

$$
\lozenge A \leftrightarrow \neg\square\neg A. \qquad\qquad (\square\lozenge)
$$

The primitive rules are: modus ponens, extensionality rule

$$
A[B] \in \mathsf{LC}\square\hookrightarrow \ \text{ and } \ B \leftrightarrow D \in \mathsf{LC}\square\hookrightarrow \ \Longrightarrow \ A[D] \in \mathsf{LC}\square\hookrightarrow, \qquad (\texttt{rep})
$$

two specific rules for adding modalities $\neg\mathcal{C}$, $\hookrightarrow$:

$$
\begin{aligned}
A \in \mathsf{LC}\square\hookrightarrow \ \Longrightarrow \ \neg\mathcal{C}A \in \mathsf{LC}\square\hookrightarrow, &\qquad\qquad (\texttt{gen}\neg\mathcal{C})\\
A \to B \in \mathsf{LC}\square\hookrightarrow \ \Longrightarrow \ \hookrightarrow A \to \hookrightarrow B \in \mathsf{LC}\square\hookrightarrow, &\qquad\qquad (\texttt{mon}\hookrightarrow)
\end{aligned}
$$

and the rule of induction from $(u\mathcal{C})^n$ to $\Box$:

for any $n \geqslant 0$: $B \to (u\mathcal{C})^n A \in \mathsf{LC}\Box\hookrightarrow \implies B \to \Box A \in \mathsf{LC}\Box\hookrightarrow.$ $\quad(\omega)$

The Leibnizian concept of the priority of change over time is expressed in $\mathsf{LC}\Box$ in a way that for the temporal *next* operator $\bigcirc$ understood as

$$L, \mathcal{I}, w \models^n \bigcirc A \quad \text{iff} \quad L, \mathcal{I}, w \models^{n+1} A$$

the following equivalence is logically valid

$$\bigcirc A \leftrightarrow (A \leftrightarrow \neg \mathcal{C} A). \quad (\bigcirc)$$

Concerning our new axioms, we note that $(\hookrightarrow\neg)$ and $(\hookrightarrow\wedge)$ express the redundancy of $\hookrightarrow$ with respect to the negation and conjunction of $\hookrightarrow$-formulas. Schema $(\hookrightarrow\Box)$ describes the redundancy of $\Box$ in contexts with $\hookrightarrow$. $(\hookrightarrow\Box\vee)$ states that $\hookrightarrow$ may be distributed over the disjunction of $\Box$-formulas. The distribution of $\hookrightarrow$ over $\hookrightarrow$-implication is permitted when the antecedent is a $\hookrightarrow$-formula. This is expressed by $(\hookrightarrow\to)$ and we call it *the principle of weak slip of $\hookrightarrow$.* According to $(\hookrightarrow\mathcal{C})$, possible laws do not change.

As it is noted in [12], if we introduce to $\mathsf{LC}\Box$ the operator $\bigcirc$ using definition $(\bigcirc)$ we obtain the system equivalent to the $\Box$-fragment of temporal linear logic LTL extended by the following definition of change operator:

$$\mathcal{C} A \leftrightarrow (A \leftrightarrow \neg \bigcirc A).$$

Now we can state that $\mathsf{LC}\Box\hookrightarrow$ is sound and complete in respect to our semantics.

THEOREM 1 (Soundness). *If A is an $\mathsf{LC}\Box\hookrightarrow$ thesis, then A is logically valid.*

PROOF. The logical validity of: $(\mathcal{C}\neg)$, $(\mathcal{C}\vee)$, $(\mathcal{C}\to)$, $(\mathcal{C}\wedge)$ and admissibility of $(\mathsf{gen}\neg\mathcal{C})$, may be shown as in [13, 5–6]. Proofs of the logical validity of $(\Box u\mathcal{C})$ and $(\neg\mathcal{C}\Box)$ follow directly from the satisfaction of $\Box$. A proof of the admissibility of (ω) may be formulated indirectly. We prove the logical validity of: $(\hookrightarrow\to)$ and $(\hookrightarrow\Box\vee)$.

For $(\hookrightarrow\to)$ we assume indirectly that there exist $L, \mathcal{I}, w, k$ such that $L, \mathcal{I}, w \models^k \hookrightarrow A$; $L, \mathcal{I}, w \models^k \hookrightarrow (\hookrightarrow A \to B)$; $L, \mathcal{I}, w \not\models^k \hookrightarrow B$. We obtain that: (a) for some $w_0 \in W/\{a\}$: $\forall_n (L, \mathcal{I}, w_0 \models^n A)$; (b) for some $w_1 \in W/\{a\}$: $\forall_n (L, \mathcal{I}, w_1 \models^n \hookrightarrow A \to B)$; (c) $\forall_{w \in W/\{a\}} \exists_m (L, \mathcal{I}, w \not\models^m B)$. From (c) we

have $L, \mathcal{I}, w_1 \not\models^{m_0} B$ for some m_0. From (b) we have $L, \mathcal{I}, w_1 \models^{m_0} \hookrightarrow A \to B$. From (a), however, we obtain $L, \mathcal{I}, w_1 \models^{m_0} \hookrightarrow A$, so $L, \mathcal{I}, w_1 \models^{m_0} B$ which results in a contradiction.

For $(\hookrightarrow \Box \vee)$ we assume indirectly that there exist $L, \mathcal{I}, w, k$ such that (a) $L, \mathcal{I}, w \models^k \hookrightarrow (\Box A \vee \Box B)$ and (b) $L, \mathcal{I}, w \not\models^k \hookrightarrow \Box A \vee \hookrightarrow \Box B$. From (a) we obtain that: $\forall_n (L, \mathcal{I}, w_0 \models^n \Box A \vee \Box B)$, for some $w_0 \in W/\{a\}$, which in turn gives us $L, \mathcal{I}, w_0 \models^1 \Box A \vee \Box B$. Hence, $\forall_n (L, \mathcal{I}, w_0 \models^n A)$ or $\forall_n (L, \mathcal{I}, w_0 \models^n B)$. From (b) we have $\forall_{w \in W/\{a\}} \exists_n (L, \mathcal{I}, w \not\models^n \Box A)$ and $\forall_{w \in W/\{a\}} \exists_m (L, \mathcal{I}, w \not\models^m \Box B)$, hence $L, \mathcal{I}, w_0 \not\models^{n_0} \Box A$ and $L, \mathcal{I}, w_0 \not\models^{m_0} \Box B$. Next, we obtain $\exists_{j \geqslant n_0} (L, \mathcal{I}, w_0 \not\models^j A)$ and $\exists_{k \geqslant m_0} (L, \mathcal{I}, w_0 \not\models^k B)$, which results in a contradiction.

Proofs for remaining specific axioms for $\hookrightarrow$ may be formulated in an analogical way, using truth conditions from the definition of satisfaction.
$$\dashv$$

As to the completeness of $\mathsf{LC}\Box\hookrightarrow$, the proof is a modification of the completeness proof for $\mathsf{LC}\Box$ from [12]. $\mathsf{LC}\Box$ is complete in linear temporal structures with the first element. Our modification is based on the idea used in the completeness proof for the propositional dynamic logic PDL [5]. Having a $\mathsf{LC}\Box\hookrightarrow$ consistent formula A, we construct a model for A starting from finite consistent subsets built from the set of subformulas of formula A. As is always the case with a Henkin-style proof, the goal is to find for formula A, a model in which A is false.

Due to the idea from [5], we use the concept of maximal consistency relative to some set of formulas. We say that a set X is consistent iff there is no $\{B_1, \dots, B_n\} \subseteq X$ such that $\neg(B_1 \wedge \cdots \wedge B_n) \in \mathsf{LC}\Box\hookrightarrow$. X is Y-maximally consistent iff X is consistent, $X \subseteq Y$ and $\forall_{A \in Y}(A \notin X \implies X \cup \{A\}$ is inconsistent). Thus understood, a Y-maximally consistent set is finite, if Y is finite.

Let us call $Sub(A)$ the set of all subformulas of the formula A. We extend $Sub(A)$ by all negations of its elements and formulas $\hookrightarrow \alpha_k^i, \neg \hookrightarrow \alpha_k^i, \alpha_k^i, \neg \alpha_k^i$, where $\alpha_k^i \notin Sub(A)$:

$$Sub_\hookrightarrow^+(A) = Sub(A) \cup \{\neg A \; : \; A \in Sub(A)\} \cup \{\hookrightarrow \alpha_k^i, \neg \hookrightarrow \alpha_k^i, \alpha_k^i, \neg \alpha_k^i\}.$$

We also add to the set $Sub_\hookrightarrow^+(A)$ all $\Box$ and $\neg\Box$ closures of these formulas from $Sub_\hookrightarrow^+(A)$, which are preceded by $\hookrightarrow$:

$$Sub_{\hookrightarrow,\Box}^+(A) = Sub_\hookrightarrow^+(A) \cup \{\Box A \; : \hookrightarrow A \in Sub_\hookrightarrow^+\} \cup \{\neg\Box A \; : \hookrightarrow A \in Sub_\hookrightarrow^+\}.$$

We denote the set built of $Sub^+_{\hookrightarrow,\Box}(A)$ and: $\neg, \to, \mathcal{C}, \Box$ as $Fm(Sub^+_{\hookrightarrow,\Box}(A))$. It is worth noticing here that $Fm(Sub^+_{\hookrightarrow,\Box}(A))$ is formed without new $\hookrightarrow$-formulas, although there are $\hookrightarrow$-formulas in the set $Sub^+_{\hookrightarrow,\Box}(A)$.

We write $\vdash A$ instead of $A \in \mathsf{LC}\Box\hookrightarrow$. In the proofs we use two $\mathsf{LC}\Box\hookrightarrow$ theses:

$$\vdash \Box(\Box A \vee \Box B) \leftrightarrow (\Box A \vee \Box B), \qquad (\Box\vee)$$

$$\vdash \hookrightarrow A \leftrightarrow \hookrightarrow\hookrightarrow A. \qquad (\hookrightarrow\hookrightarrow)$$

Equivalence ($\Box\vee$) follows from the fact that $\Box$ has $\mathsf{S4}$ (in fact $\mathsf{S4.3}$ [12]) properties. We obtain ($\hookrightarrow\hookrightarrow$) from ($\hookrightarrow\wedge$) by B/A and (**rep**).

We assume that A^* is not a theorem. Thus, the set $\{\neg A^*\}$ is consistent. We extend $\{\neg A^*\}$ to a $Sub^+_{\hookrightarrow,\Box}(A^*)$-maximally consistent set and denote it as a_0. Next, we define $Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\Box}(A^*))$:

$$X \in Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\Box}(A^*)) \text{ iff } \{\hookrightarrow A : \hookrightarrow A \in \mathsf{a}_0\} \cup \{\neg \hookrightarrow A : \neg \hookrightarrow A \in \mathsf{a}_0\} \cup$$
$$\{\neg\Box A : \neg \hookrightarrow A \in \mathsf{a}_0\} \subseteq X \text{ and}$$
$$X \text{ is a } Sub^+_{\hookrightarrow,\Box}(A^*)\text{-maximally consistent} set. \quad (\text{R})$$

$Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\Box}(A^*))$ is nonempty, because of the Lindenbaum Lemma and the following lemmas on $\mathsf{LC}\Box\hookrightarrow$ consistent sets containing formulas with modalities $\hookrightarrow$ and $\Box$:

LEMMA 1. *If X is consistent and $\hookrightarrow F \in X$, then the set $\{\Box F\} \cup \{\neg\Box A : \neg \hookrightarrow A \in X\} \cup \{\hookrightarrow A : \hookrightarrow A \in X\} \cup \{\neg \hookrightarrow A : \neg \hookrightarrow A \in X\}$ is consistent.*

PROOF. We assume that X is consistent.

Case 1: $\{\neg\hookrightarrow B : \neg\hookrightarrow B \in X\} \neq \emptyset$. We assume indirectly, that there are formulas: $\hookrightarrow A_1, \ldots, \hookrightarrow A_n, \neg \hookrightarrow B_1, \ldots, \neg \hookrightarrow B_m, \hookrightarrow F$ belonging to X such that: $\vdash \hookrightarrow A_1 \wedge \cdots \wedge \hookrightarrow A_n \wedge \neg \hookrightarrow B_1 \wedge \cdots \wedge \neg \hookrightarrow B_m \wedge \neg\Box B_1 \wedge \cdots \wedge \neg\Box B_m \to \neg\Box F$. Applying ($\hookrightarrow\neg$) we obtain: $\vdash \hookrightarrow A_1 \wedge \cdots \wedge \hookrightarrow A_n \wedge \hookrightarrow \neg \hookrightarrow B_1 \wedge \cdots \wedge \hookrightarrow\neg \hookrightarrow B_m \wedge \neg\Box B_1 \wedge \cdots \wedge \neg\Box B_m \to \neg\Box F$. Let $A =: \hookrightarrow A_1 \wedge \cdots \wedge \hookrightarrow A_n$ and $B =: \hookrightarrow\neg \hookrightarrow B_1 \wedge \cdots \wedge \hookrightarrow\neg \hookrightarrow B_m$. Applying ($\hookrightarrow\wedge$) we obtain: $\vdash \hookrightarrow (A \wedge B) \wedge \neg\Box B_1 \wedge \cdots \wedge \neg\Box B_m \to \neg\Box F$. We derive $\vdash A \to B \Longrightarrow \vdash \neg \hookrightarrow \neg A \to \neg \hookrightarrow \neg B$ from (**mon** $\hookrightarrow$) and we have: $\vdash \neg \hookrightarrow \neg(\hookrightarrow (A \wedge B) \wedge \neg\Box B_1 \wedge \cdots \wedge \neg\Box B_m) \to \neg \hookrightarrow \neg\neg\Box F$. From ($\hookrightarrow\Box$) and (**rep**) we obtain: $\vdash \neg \hookrightarrow \neg(\hookrightarrow (A \wedge B) \wedge \neg\Box B_1 \wedge \cdots \wedge \neg\Box B_m) \to \neg \hookrightarrow F$, which is equivalent to: $\vdash \hookrightarrow F \to \hookrightarrow (\hookrightarrow (A \wedge B) \to \Box B_1 \vee \cdots \vee \Box B_m)$. Next, we use ($\hookrightarrow\to$) and classical logic to obtain: $\vdash \hookrightarrow F \wedge \hookrightarrow (A \wedge B) \to \hookrightarrow (\Box B_1 \vee \cdots \vee \Box B_m)$. A and B are conjunctions of formulas preceded by $\hookrightarrow$, so in virtue of ($\hookrightarrow\wedge$) we know, that $\vdash A \leftrightarrow \hookrightarrow A, \vdash B \leftrightarrow \hookrightarrow B$. The latter and by ($\hookrightarrow\wedge$) give us: $\vdash \hookrightarrow F \wedge \hookrightarrow\hookrightarrow A \wedge \hookrightarrow\hookrightarrow B \to \hookrightarrow (\Box B_1 \vee \cdots \vee \Box B_m)$.

Applying ($\looparrowright\looparrowright$) we obtain: $\vdash_{\looparrowright} F \wedge \looparrowright A \wedge \looparrowright B \to \looparrowright (\Box B_1 \vee \cdots \vee \Box B_m)$, and again applying ($\looparrowright\wedge$) and ($\looparrowright\looparrowright$) we obtain: $\vdash_{\looparrowright} F \wedge A \wedge B \to \looparrowright (\Box B_1 \vee \cdots \vee \Box B_m)$. From ($\Box\vee$) and (**rep**) we get: $\vdash_{\looparrowright} F \wedge A \wedge B \to \looparrowright (\Box B_1 \vee \Box(\Box B_2 \vee \cdots \vee \Box B_m))$. Applying ($\looparrowright\Box\vee$) and ($\Box\vee$) we obtain: $\vdash_{\looparrowright} F \wedge A \wedge B \to \looparrowright \Box B_1 \vee \looparrowright (\Box B_2 \vee \cdots \vee \Box B_m)$. If we apply ($\Box\vee$), (**rep**) and ($\looparrowright\Box\vee$), ($\Box\vee$) $m-1$ times, as we have done in last two steps, we obtain: $\vdash_{\looparrowright} F \wedge A \wedge B \to \looparrowright \Box B_1 \vee \looparrowright \Box B_2 \vee \cdots \vee \looparrowright \Box B_m$. Next, we apply ($\looparrowright\Box$): $\vdash_{\looparrowright} F \wedge A \wedge B \to \looparrowright B_1 \vee \cdots \vee \looparrowright B_m$, which is equivalent to: $\vdash_{\looparrowright} F \wedge \looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \looparrowright \neg \looparrowright B_1 \wedge \cdots \wedge \looparrowright \neg \looparrowright B_m \to \looparrowright B_1 \vee \cdots \vee \looparrowright B_m$. From ($\looparrowright\neg$) we infer: $\vdash_{\looparrowright} F \wedge \looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \neg \looparrowright B_1 \wedge \cdots \wedge \neg \looparrowright B_m \to \looparrowright B_1 \vee \cdots \vee \looparrowright B_m$. Thus, $\vdash_{\looparrowright} F \wedge \looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \to \looparrowright B_1 \vee \cdots \vee \looparrowright B_m$. We now have: $\{\looparrowright F, \looparrowright A_1, \ldots \looparrowright A_n, \neg \looparrowright B_1, \ldots, \neg \looparrowright B_m\} \subseteq X$. Thus, we finally obtain that X is inconsistent.

Case 2: for $\{\neg \looparrowright A \; : \; \neg \looparrowright A \in X\} = \emptyset$ the proof is analogical to case 1. $\dashv$

LEMMA 2. *For any* $\mathsf{w} \in \mathsf{W}$, *if* $\looparrowright A \in Sub^+_{\looparrowright, \Box}(A^*)$, *then:* $\looparrowright A \in \mathsf{w}_n$ *iff* $\exists_{\mathsf{u} \in W/\{\mathsf{a}\}} \forall_{k \in \mathbb{N}} A \in \mathsf{u}_k$.

PROOF. We assume that X is consistent.

Case 1: $\{\looparrowright A \; : \; \looparrowright A \in X\} \neq \emptyset$. We indirectly assume that there are formulas $\looparrowright A_1, \ldots, \looparrowright A_n, \neg \looparrowright B_1, \ldots, \neg \looparrowright B_m \in X$ such that: $\vdash_{\looparrowright} \looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \neg \looparrowright B_1 \wedge \cdots \wedge \neg \looparrowright B_m \to \Box B_1 \vee \cdots \vee \Box B_m$. Applying ($\looparrowright\neg$) and ($\looparrowright\wedge$) we obtain: $\vdash_{\looparrowright} (\looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \looparrowright \neg \looparrowright B_1 \wedge \cdots \wedge \looparrowright \neg \looparrowright B_m) \to \Box B_1 \vee \cdots \vee \Box B_m$. From (**mon** $\looparrowright$) we get: $\vdash_{\looparrowright\looparrowright} (\looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \looparrowright \neg \looparrowright B_1 \wedge \cdots \wedge \looparrowright \neg \looparrowright B_m) \to \looparrowright (\Box B_1 \vee \cdots \vee \Box B_m)$. We apply ($\looparrowright\looparrowright$) and obtain the following: $\vdash_{\looparrowright} (\looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \looparrowright \neg \looparrowright B_1 \wedge \cdots \wedge \looparrowright \neg \looparrowright B_m) \to \looparrowright (\Box B_1 \vee \cdots \vee \Box B_m)$. Next, we apply ($\looparrowright\wedge$), ($\looparrowright\looparrowright$) and ($\looparrowright\neg$) and infer: $\vdash_{\looparrowright} \looparrowright A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \neg \looparrowright B_1 \wedge \cdots \wedge \neg \looparrowright B_m \to \looparrowright (\Box B_1 \vee \cdots \vee \Box B_m)$. We now proceed in the same way as in Lemma 1. We apply ($\looparrowright\Box\vee$), ($\Box\vee$), ($\looparrowright\Box$) and obtain: $\vdash_{\looparrowright} A_1 \wedge \cdots \wedge \looparrowright A_n \wedge \neg \looparrowright B_1 \wedge \cdots \wedge \neg \looparrowright B_m \to \looparrowright B_1 \vee \cdots \vee \looparrowright B_m$. Because $\{\looparrowright A_1, \ldots, \looparrowright A_n, \neg \looparrowright B_1, \ldots, \neg \looparrowright B_m\} \subseteq X$, X is inconsistent.

Case 2: for $\{\looparrowright A \; : \; \looparrowright A \in X\} = \emptyset$ the proof is analogical to case 1. $\dashv$

Informally speaking, elements of $Con_{\mathsf{a}_0}(Sub^+_{\looparrowright, \Box}(A^*))$ may be treated as fragmentary descriptions of the initial states of different possible worlds containing the same $\looparrowright$-formulas, their negations, and also all formulas which are not necessary in the sense of $\looparrowright$, and are also not necessary in the sense of $\Box$.

The set a_0 has finitely many elements. Thus $Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\square}(A^*))$ is finite and every $X \in Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\square}(A^*))$ has finitely many elements. It is crucial for the next step of our proof.

We extend all elements of $Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\square}(A^*))$ and for any $\alpha^i_j \notin Sub^+_{\hookrightarrow,\square}(A^*)$ the set $\mathsf{a_0} = \mathsf{a}_0 \cup \{\alpha^i_j\}$ to $Fm(Sub^+_{\hookrightarrow,\square}(A^*))$-maximally consistent sets. (Adding α^i_j, to a_0 makes $\mathsf{a_0}$ different from every element of $Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\square}(A^*))$. This will be relevant in the proof of Lemma 7).

Next, we consider an enumeration of all formulas of $Fm(Sub^+_{\hookrightarrow,\square}(A^*))$: $A_0, A_1, A_2, \ldots$. For every $X \in Con_{\mathsf{a}_0}(Sub^+_{\hookrightarrow,\square}(A^*)) \cup \{\mathsf{a_0}\}$ we define the sequence (S^X_k) as follows:

$$S^X_0 = X,$$

$$S^X_{2k+1} = \begin{cases} S^X_{2k} \cup \{A_k\} & \text{if this is consistent,} \\ S^X_{2k} \cup \{\neg A_k\} & \text{otherwise,} \end{cases}$$

$$S^X_{2k+2} = \begin{cases} S^X_{2k+1} \cup \{\neg(u\mathcal{C})^m F\} & \begin{array}{l}\text{for some } m \geqslant 0 \text{ if } A_k = \neg\square F \\ \text{and } A_k \in S^X_{2k+1},\end{array} \\ S^X_{2k+1} & \text{otherwise.} \end{cases}$$

Every S^X_k is well defined in view of the following lemma:

LEMMA 3. *If X is finite, consistent and $\neg\square F \in X$, then there is an $m \in \mathbb{N}$ such that $X \cup \{\neg(u\mathcal{C})^m F\}$ is consistent.*

PROOF. We proceed indirectly. Since X is finite, let $\bigwedge X$ be the conjunction of all formulas from X. We have $\vdash \bigwedge X \to (u\mathcal{C})^m F$, for all $m \in \mathbb{N}$. Applying (ω) we obtain $\vdash \bigwedge X \to \square F$. However $\neg\square F \in X$, which entails that X is inconsistent. $\dashv$

For every $X \neq \mathsf{a_0}$: $\bigcup_n S^X_n$ is $Fm(Sub^+_{\hookrightarrow,\square}(A^*))$-maximally consistent and also $\bigcup_n S^{\mathsf{ao}}_n/\{\alpha^i_j\}$ is $Fm(Sub^+_{\hookrightarrow,\square}(A^*))$-maximally consistent. We know this from the Lindenbaum Lemma.

For every $\bigcup_n S^X_n$ we define sequence $(\mathsf{s}_n)_{n\in\mathbb{N}}$ in the following way:

$$\mathsf{s}_1 = \bigcup_n S^X_n, \quad \mathsf{s}_{n+1} = \{A \; : \; A \wedge \neg \mathcal{C}A \in \mathsf{s}_n\} \cup \{\neg A \; : \; A \wedge \mathcal{C}A \in \mathsf{s}_n\}.$$

The sequence that we denote as a is defined as above with $\mathsf{a}_1 = \bigcup_n S^{\mathsf{ao}}_n$. All sequences defined in this way form set $\mathsf{W} = \{\mathsf{a}, \mathsf{w}, \mathsf{w}', \mathsf{w}'', \ldots\}$.

Note that $\neg A^* \in \mathsf{a}_1$ because $\neg A^* \in \bigcup_n S^{\mathsf{ao}}_n$.

Lemma 4. *For any* $w \in W/\{a\}$ *and any* k: w_k *is* $Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$-*maximally consistent. Moreover, for any* $k \geqslant 2$, a_k *is* $Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$-*maximally consistent.*

Proof. We assume inductively that w_k is $Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$-maximally consistent and $\{A : A \wedge \neg \mathcal{C}A \in w_k\} \cup \{\neg A : A \wedge \mathcal{C}A \in w_k\}$ is not consistent. We have that there exists a finite set of formulas from $Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$ such that $\{A_1, \ldots, A_n, \neg \mathcal{C}A_1, \ldots, \neg \mathcal{C}A_n, \neg D_1, \ldots, \neg D_m, \mathcal{C}D_1, \ldots, \mathcal{C}D_m\}$ $\subseteq w_k$ and $\vdash \neg(A_1 \wedge \cdots \wedge A_n \wedge D_1 \wedge \cdots \wedge D_m)$. Let $A_1 \wedge \cdots \wedge A_n = A$ and $D_1 \wedge \cdots \wedge D_m = D$. We have $\vdash \neg(A \wedge D)$. Now because of (gen¬$\mathcal{C}$) we obtain $\vdash \neg \mathcal{C} \neg(A \wedge D)$ and next using ($\mathcal{C}$¬) $\vdash \neg \mathcal{C}(A \wedge D)$. Now from axiom ($\mathcal{C}\rightarrow$) and (rep) we get $\vdash \neg A \vee \mathcal{C}A \vee \neg \mathcal{C} \neg D$. From axiom ($\mathcal{C}\vee$) we obtain $\vdash \neg A \vee \mathcal{C}A_1 \vee \cdots \vee \mathcal{C}A_n \vee \neg \mathcal{C} \neg D$ and from ($\mathcal{C}$¬) we get $\vdash \neg A \vee \mathcal{C}A_1 \vee \cdots \vee \mathcal{C}A_n \vee \neg \mathcal{C}D$. Since $\{A, \neg \mathcal{C}A_1, \ldots \neg \mathcal{C}A_n\} \subseteq w_k$ we get $\neg \mathcal{C}D \in w_k$. From ($\mathcal{C}\wedge$) we obtain $\vdash \neg D_1 \wedge \cdots \wedge \neg D_m \wedge \mathcal{C}D_1 \wedge \cdots \wedge \mathcal{C}D_m \rightarrow \mathcal{C}(D_1 \wedge \cdots \wedge D_m)$. We have $\{\neg D_1, \ldots, \neg D_n, \mathcal{C}D_1, \mathcal{C}D_m\} \subset w_k$ so $\mathcal{C}(D_1 \wedge \cdots \wedge D_m) \in w_k$ and next $\mathcal{C}D \in w_k$. Since w_{k+1} is consistent, it is obvious that it is also $Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$-maximally consistent. The proof for a is analogous. ⊣

Using Lemma 4 we obtain:

Lemma 5. *For any* $w \in W$: $(u\mathcal{C})^k A \in w_n$ *iff* $A \in w_m$ *for any* m *such that* $n \leqslant m \leqslant n + k$.

Lemma 6. *For any* $w \in W$ *and any* $\Box A \in Fm(Sub^+_{\hookrightarrow,\Box}(A^*))$: $\Box A \in w_n$ *iff for any* $k \geqslant n$ *we have* $A \in w_k$.

Proof. We use the notation $\bigcirc^n A$ if there are n occurrences of $\bigcirc$ before the formula A. We employ the following thesis: $\Box \bigcirc^n A \rightarrow \bigcirc^n \Box A$. In the proof we need to use: $\Box(A \rightarrow \bigcirc A) \rightarrow (A \rightarrow \Box A)$, $\Box A \rightarrow \bigcirc \Box A$, $\Box A \rightarrow \bigcirc A$. (¬$\mathcal{C}\Box$) is to be used in the proof of all these theorems [sf. 12, 521-522]. We also need $\bigcirc(A \rightarrow B) \rightarrow (\bigcirc A \rightarrow \bigcirc B)$ and the generalization rule for $\bigcirc$: $\vdash A \Longrightarrow \vdash \bigcirc A$. We need for the latter only axioms for the operator $\mathcal{C}$, ($\bigcirc$) and (gen¬$\mathcal{C}$).

"$\Leftarrow$" We assume $\forall_{k \geqslant n} A \in w_k$ and $\neg \Box A \in w_n$. We use definition ($\bigcirc$) together with Lemma 4 to obtain: $\bigcirc^n A \in w_j$ iff $A \in w_{j+n}$. If $\bigcirc^{n-1} \Box A \in w_1$, then we have $\Box A \in w_n$ which is false by assumption. Thus $\neg \bigcirc^{n-1} \Box A \in w_1$. Next, we apply ($\Box$/) and obtain $\neg \Box \bigcirc^{n-1} A \in w_1$. From the definition of sequence (S_k) we obtain $\neg(u\mathcal{C})^m \bigcirc^{n-1} A \in w_1$ for some m, and $(u\mathcal{C})^m \bigcirc^{n-1} A \notin w_1$. From Lemma 5, we notice that for

some j such that $0 \leqslant j \leqslant m$: $\bigcirc^{n-1} A \notin \mathsf{w}_{j+1}$. Therefore, we have $k \geqslant n$ such that $A \notin \mathsf{w}_k$, which contradicts our assumption.

"$\Rightarrow$" We need $(\Box u\mathcal{C})$ and Lemma 5. $\dashv$

LEMMA 7. *For every* $\mathsf{w} \in \mathsf{W}$: *if* $\hookrightarrow A \in Sub_{\hookrightarrow,\Box}^+(A^*)$, *then* $(\hookrightarrow A \in \mathsf{w}_n$ *iff* $\exists_{\mathsf{u} \in W/\{\mathsf{a}\}} \forall_{k \in \mathbb{N}}(A \in \mathsf{u}_k))$.

PROOF. "*Rightarrow*" We assume that $\hookrightarrow A \in Sub_{\hookrightarrow,\Box}^+(A^*)$, $\hookrightarrow A \in \mathsf{w}_n$, and $\forall_{\mathsf{u} \in W/\{\mathsf{a}\}} \exists_{k \in \mathbb{N}}(A \notin \mathsf{u}_k)$. From $(\hookrightarrow\mathcal{C})$ we obtain $\hookrightarrow A \in \mathsf{w}_{n-1}$. Similarly we finally obtain $\hookrightarrow A \in \mathsf{w}_1$. If $\neg \hookrightarrow A \in \mathsf{a}_0$, then, because of (R): $\neg \hookrightarrow A \in \mathsf{w}_1$ (which is false by assumption). Thus $\hookrightarrow A \in \mathsf{a}_0$. Next, from Lemma 1 and (R) we know that there exists an u_1 with $\Box A \in \mathsf{u}_1$. Applying Lemma 6 we obtain $\forall_{n \geqslant 1} A \in \mathsf{u}_n$, which contradicts the assumption.

"$\Leftarrow$" We assume that $\hookrightarrow A \in Sub_{\hookrightarrow,\Box}^+(A^*)$, $\exists_{\mathsf{u} \in W/\{\mathsf{a}\}} \forall_{k \in \mathbb{N}}(A \in \mathsf{u}_k)$, and $\neg \hookrightarrow A \in \mathsf{w}_n$. From $(\hookrightarrow\mathcal{C})$ we obtain $\neg \hookrightarrow A \in \mathsf{w}_{n-1}$ and we finally obtain $\neg \hookrightarrow A \in \mathsf{w}_1$. If $\hookrightarrow A \in \mathsf{a}_0$, then because of (R): $\hookrightarrow A \in \mathsf{w}_1$, which is false by assumption. Thus $\neg \hookrightarrow A \in \mathsf{a}_0$. Now from Lemma 2 and (R) we obtain: $\forall_{\mathsf{u} \in W/\{\mathsf{a}\}}(\neg\Box A \in \mathsf{u}_1)$. Using Lemma 6 we obtain $\forall_{\mathsf{u} \in W/\{\mathsf{a}\}} \exists_{k \geqslant 1}(A \notin \mathsf{u}_k)$, which contradicts the assumption. $\dashv$

We consider the following structure $L^* = \langle \mathsf{W}, \mathsf{a} \rangle$. By Lemmas 4, 5, 6 and 7, we know that for every formula $A \in Fm(Sub_{\hookrightarrow,\Box}^+(A^*))$: structure L^* fulfills the following conditions:

(i) $\mathcal{C}A \in \mathsf{w}_n$ iff $(A \in \mathsf{w}_n$ and $A \notin \mathsf{w}_{n+1})$ or $(A \notin \mathsf{w}_n$ and $A \in \mathsf{w}_{n+1})$;
(ii) $(u\mathcal{C})^k A \in \mathsf{w}_n$ iff $\forall_m(n \leq m \leqslant n + k \Longrightarrow A \in \mathsf{w}_m)$;
(iii) $\Box A \in \mathsf{w}_n$ iff $\forall_{k \geqslant n}(A \in \mathsf{w}_k)$;
(iv) $\hookrightarrow A \in Sub_{\hookrightarrow,\Box}^+(A^*) \Longrightarrow (\hookrightarrow A \in \mathsf{w}_n$ iff $\exists_{\mathsf{u} \in W/\{\mathsf{a}\}} \forall_{k \geqslant 1}(A \in \mathsf{u}_k))$.

Conditions (i)–(iv) correspond to conditions for: $\mathcal{C}, \Box, \hookrightarrow$, and $(u\mathcal{C})^n$ from the definition of satisfaction.

We can now come back to the proof of the main theorem:

THEOREM 2 (Completeness). *If* $A \notin \mathsf{LC}\Box\hookrightarrow$, *then* A *is not logically valid.*

Having a formula A^* which is not a theorem and the structure L^* we define $\mathcal{I}^*$ in such a way that for every $\mathsf{w} \in \mathsf{W}$: $\alpha_k^i \in \mathcal{I}^*(\mathsf{w}_n)$ iff $\alpha_k^i \in \mathsf{w}_n$. We prove by induction, with reference to conditions (i)–(iv), that for every formula $A \in Fm(Sub_{\hookrightarrow,\Box}^+(A^*))$ and every $\mathsf{w} \in \mathsf{W}$: $L^*, \mathcal{I}^*, \mathsf{w} \models^n A$ iff $A \in \mathsf{w}_n$. We have that $\neg A^* \in \mathsf{a}_1$. Thus $L^*, \mathcal{I}^*, \mathsf{a} \not\models^1 A^*$.

4. LC□↪ modalities and the 'plenitude' problem

Let us now indicate some properties of our three modal operators. We consider the following schemes with $* \in \{\mathcal{C}, \Box, \hookrightarrow\}$:

$$(\texttt{K}^*) \quad *(A \to B) \to (*A \to *B), \qquad\qquad (distribution)$$
$$(\texttt{ws}^*) \quad *(*A \to B) \to (*A \to *B), \qquad\qquad (weak\ slip)$$
$$(\texttt{s}^*) \quad *(A \to B) \to (A \to *B), \qquad\qquad (slip)$$
$$(\texttt{mon}^*) \quad \vdash A \to B \Longrightarrow\ \vdash *A \to *B, \qquad\qquad (monotonicity)$$
$$(\texttt{gen}_1^*) \quad \vdash A \Longrightarrow\ \vdash *A, \qquad\qquad (generalizations)$$
$$(\texttt{gen}_2^*) \quad \vdash A \Longrightarrow\ \vdash \neg * A, \qquad\qquad (generalizations)$$
$$(\texttt{v}_1^*) \quad *\top, \qquad\qquad (modal\ closures\ of\ logical\ truth)$$
$$(\texttt{v}_2^*) \quad \neg * \top. \qquad\qquad (modal\ closures\ of\ logical\ truth)$$

Only $\Box$ meets distribution $(\texttt{K}^*)$.

The schema $(\texttt{ws}^*)$ of weak slip is valid for $\Box$. The formula $(\texttt{ws}^\Box)$ is derivable from $(\texttt{K}^\Box)$ and $\Box A \to \Box\Box A$. For $\hookrightarrow$, the implication $(\texttt{ws}^\hookrightarrow)$ follows directly from our *principle of weak slip* $(\hookrightarrow\!\!\to)$. However, the formula $(\texttt{s}^\hookrightarrow)$ is not a thesis of LC□↪ and the same applies to $(\texttt{s}^\Box)$ and $(\texttt{s}^\mathcal{C})$. This means that our formalism is resistant to the aforementioned fallacy of slipped necessity with regard to our $\Box$ and $\hookrightarrow$.[7] The addition of $(\texttt{s}^\Box)$ to classical logic extended by $(\texttt{gen}_1^\Box)$, results in the derivability of $A \to \Box A$ (cf. footnote 4). The same happens if we add $(\texttt{s}^\hookrightarrow)$ to classical logic extended by $(\texttt{gen}_1^\hookrightarrow)$ which is derivable in LC□↪. Furthermore, we achieve the same result if we add $(\texttt{s}^*)$ for $* = \neg\mathcal{C}$ and generalization $(\texttt{gen}_2^\mathcal{C})$ to classical logic. In this case $A \to \neg\mathcal{C}A$ is derivable. It is noteworthy that both $\neg\mathcal{C}A \to A$ and $\hookrightarrow A \to A$ are not derivable in LC□↪.

The modalities $\Box$ and $\hookrightarrow$ are monotonic in the sense of $(\texttt{mon}^*)$. From this it follows that $(\texttt{gen}_1^\Box)$, $(\texttt{gen}_1^\hookrightarrow)$ are admissible, and $(\texttt{v}_1^\Box)$, $(\texttt{v}_1^\hookrightarrow)$ are logically valid. Modality $\mathcal{C}$ is not monotonic. Also, the rule $\vdash A \to B \Longrightarrow\ \vdash \neg\mathcal{C}A \to \neg\mathcal{C}B$ is not admissible. However, because $(\texttt{gen}_2^\mathcal{C})$ $(= (\texttt{gen}\neg\mathcal{C}))$ is admissible, so $(\texttt{v}_2^\mathcal{C})$ is valid.

Finally, let us look for adequate formulations of (PL) and (PM). We consider the following counterparts of them:

$$\Box A \to \neg \hookrightarrow \neg\Box A, \qquad\qquad (\text{P}\Box/\hookrightarrow)$$
$$\hookrightarrow \Diamond A \to \Diamond A. \qquad\qquad (\text{P}\hookrightarrow/\Diamond)$$

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[7] As Mates claims, this fallacy is evident in many reasonings of Aristotle and Leibniz seems to avoid it even though his distinction between absolute and hypothetical necessities comes just from Aristotle [10, 117–121].

In a free formulation ($\mathrm{P}\Box/\hookrightarrow$) states that if A holds always (is constant), then A is necessary (its negation cannot be a possible law). In turn, according to ($\mathrm{P}\hookrightarrow/\Diamond$), if A is possible (it may be a possible law), then A cannot remain unfulfilled through an infinite stretch of time (it will happen at some time). In fact, ($\mathrm{P}\Box/\hookrightarrow$) and ($\mathrm{P}\hookrightarrow/\Diamond$) express semantical conditions ($*$) and ($**$) respectively from Section 2.

Now, however, we can see that ($\mathrm{P}\Box/\hookrightarrow$) and ($\mathrm{P}\hookrightarrow/\Diamond$) are not theses of our logic. To falsify ($\mathrm{P}\Box/\hookrightarrow$) we consider a structure L' with domain $W = \{a, w\}$ and interpretation $\langle a, n \rangle \in \mathcal{I}''(\alpha_k^i)$ for all n, and $\langle w, n \rangle \notin \mathcal{I}''(\alpha_k^i)$ for all n. We obtain $L', \mathcal{I}''a \models^1 \Box\alpha_k^i$ because $\forall_n(L', \mathcal{I}''a \models^n \alpha_k^i)$, and $L', \mathcal{I}''a \not\models^1 \neg \hookrightarrow \neg\Box\alpha_k^i$ because $\forall_n(L', \mathcal{I}''w \not\models^n \alpha_k^i)$.

As to ($\mathrm{P}\hookrightarrow/\Diamond$) we note that it is inferentially equivalent to ($\mathrm{P}\Box/\hookrightarrow$). To close our considerations, let us observe that ($\mathrm{P}\Box/\hookrightarrow$) is derivable from the formula $A \rightarrow \neg \hookrightarrow \neg A$ which could be considered as Leibnizian necessarianism expressed in terms of our logic.

References

[1] Adams, R. M., *Leibniz: Determinist, Theist, Idealist*, Oxford University Press, 1994. DOI: 10.1093/0195126491.001.0001

[2] Futch, M., *Leibniz's Metaphysics of Time and Space*, Springer, 2008.

[3] Hintikka, J., "Leibniz on plenitude, relations, and the 'reign of law'" in S. Knuutila (ed.), *Reforging the Great Chain of Being. Studies of the History of Modal Theories*, Springer, 1981.

[4] Hintikka, J., "Was Leibniz's deity an akrates?", pages 85–108 in S. Knuutila (ed.), *Modern Modalities*, Kluwer, 1988.

[5] Kozen, D., and R. Parikh, "An elementary proof of the completeness of PDL", *Theoretical Computer Science*, 14 (1981): 113–118.

[6] Leibniz, G. W., *Leibniz-Arnauld Correspondence*, ed. and transl. by H. T. Mason and G. H. R. Parkinson, Manchester University Press, 1967. DOI: 10.1007/978-94-010-1426-7

[7] Leibniz, G. W., *Philosophical Papers and Letters*, ed. II, transl. and ed. by L. E. Loemker, The New Synthese Historical Library, vol. 2, Kluwer Academic Publishers, 1989.

[8] Leibniz, G. W., *New Essays on Human Understanding*, trans. and ed. (with notes) P. Remnant and J. Bennett. Cambridge, 1981.

[9] Lovejoy, A. O., *The Great Chain of Being. A Study of the History of an Idea*, Harvard U. P., 22th printing 2001. DOI: 10.4324/9781315132310

[10] Mates, B., *The Philosophy of Leibniz: Metaphysics and Language*, New York: Oxford University Press, 1986.

[11] Russell, B., *A Critical Exposition of the Philosophy of Leibniz*, Routledge, Taylor and Francis, 1992.

[12] Świętorzecka, K., and J. Czermak, "A logic of change with modalities", *Logique et Analyse*, 232 (2015): 511–527.

[13] Świętorzecka, K., and J. Czermak, "Some calculus for a logic of change", *Journal of Applied Non-Classical Logics*, 22, 1–2 (2012): 3–10. DOI: 10.1080/11663081.2012.682435

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