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A Monadic Second-Order Version of Tarski's Geometry of Solids

Abstract. In this paper, we are concerned with the development of a general set theory using the single axiom version of Leśniewski's mereology. The specification of mereology, and further of Tarski's geometry of solids will rely on the Calculus of Inductive Constructions (CIC). In the first, part we provide a specification of Leśniewski's mereology as a model for an atomless Boolean algebra using Clay's ideas. In the second part, we interpret Leśniewski's mereology in monadic second-order logic using names and develop a full version of mereology referred to as CIC-based Monadic Mereology (λ-MM) allowing an expressive theory while involving only two axioms. In the third part, we propose a modeling of Tarski's geometry of solids relying on λ-MM. It is intended to serve as a basis for spatial reasoning. All parts have been proved using a translation in type theory.

Keywords: mereology; monadic second-order logic; geometry of solids; calculus of inductive constructions; theorem prover; type theory

1. Introduction

In this paper, we revisit the formalization (Dapoigny and Barlatier, 2015) in a theorem prover of the logical system of Stanisław Leśniewski (1916) and extend it to Tarski's geometry of solids (1956a). A challenging problem is to minimize the set of axioms while enhancing expressiveness, by using a specification in a theorem prover (i.e., Coq) which will formally establish the consistency of the formalism. While set theory is the predominant axiomatic system that serves as a foundation for mathematics, its models reject sets containing the so-called *urelements*, i.e., elements of sets that are not themselves sets. Furthermore, it cannot account for

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proper classes; that is entities that are not members of another entity. Therefore, some more elaborate set theories have been put forward as more expressive extensions such as von Neumann-Bernays-Gödel set theory, Morse-Kelley set theory or Tarski-Grothendieck set theory. While the first two are conservative extensions of Zermelo-Fraenkel set theory the latter is not. In a quite different way, Leśniewski's mereology can also be considered as a "set" theory since we can express distributive as well as collective classes. We will show in a first part of the paper that a model of mereology[1] using set theory and formulated within type theory, is able to provide an expressive theory in which proper classes can be expressed as well as sets.

Among approaches that have been proposed in the last century, the most prominent foundations for a formal theory of part-whole relations have been Leśniewski's mereology (1916) and the calculus of individuals (Leonard and Goodman, 1940). Mereology is a theory which encapsulates three levels, a higher-order classical logic, a calculus of names and a part-of theory) while the formal framework of (Leonard and Goodman, 1940) known as the calculus of individuals is a nominalistic theory that has been shown to be equivalent to a monadic second-order language using the discreteness relation as its only predicate. While these theories are quite similar, Leonard and Goodman claim that Leśniewski's system is "rather inaccessible, lacks many useful definitions, and is set forth in the language of an unfamiliar logical doctrine and in words rather than symbols". More recent works have pointed out that these arguments fall short and many axiomatic basis have been investigated (see, e.g., Lejewski, 1967; Clay, 1974; Sobociński, 1984; Cocchiarella, 2001, to cite a few) showing that Leśniewski's approach bears more richness than the calculus of individuals to provide a sound and expressive axiomatic system (see Simons, 1987, p. 71). Furthermore, students of Leśniewski have extended the set of definitions in mereology, e.g., with overlap and discreteness. Clay (1974) has proposed a different approach from that of Goodman whose purpose was to adapt Leśniewski's mereology to set theory while at the same preserving its second level, Leśniewski's ontology (LO) which supports mereology. With respect to other theories, we get the uniformity of classes due to names (it means that there is no conceptual distinction in nature between class and set).

[1] In the following part of the paper, the term "mereology" will refer to Leśniewski's mereology.

The usual approach has been to reduce mereology to some set-based axiomatic version (e.g., Classical Extensional Mereology) that is centered only on one binary predicate which stands for the relation "being a part of" (see, e.g., Varzi, 1996; Casati and Varzi, 1999; Tsai, 2015). However, this choice has generated multiple issues from the theoretical side such as the dispute about Weak Supplementation (see Cotnoir, 2021), the development of versions that are weaker than mereology (see Pietruszczak, 2005, 2020), or that are not elementarily axiomatizable (see Pietruszczak, 2015). In the above approaches, authors have blurred the formal specification of ε in LO on which mereology relies and have lost a great part of its expressive power. Furthermore, they have lost the concept of a collection and the link between collection and (collective) class. We depart from this assumption and argue following Tarski (1956b) and Clay (1974) that mereology (including LO) is a model for a Boolean algebra without zero consisting of nominal variables and the composition of the $\mathtt{el}$ function with the ε relation (the so-called part-of relation). Every mereological structure can be transformed into a complete Boolean lattice by adding the zero element (its non-existence is a consequence of axioms for mereology). Alternatively, every complete Boolean lattice can be turned into a mereological structure by deleting the zero element. We will show that a mereological theory involving appropriate functions, each one composed with the ε relation, has many benefits. For that purpose, we propose a set-based model of mereology using the Calculus of Inductive Constructions. More precisely, we extend several works such as Clay's one (1974) using characteristic functions and monadic second-order logic from (Clay, 1974; Smirnov, 1987) to build a set-based translation of mereology in a restricted subsystem of CIC denoted λ-MM. Since LO corresponds to a version of higher-order logic with arbitrary finite types, the lower part of λ-MM will map a fragment of LO limited to useful definitions using definitional comprehension (see next subsection for details).

Whereas the decidability of mereological theories has already been investigated, it put strong limitations on expressiveness (see Tsai, 2013). Having a decidable model is crucial if one expects to extend it with e.g., topology or geometry in order to take in account the spatial fragment of formal ontologies. To summarize, the present approach consists in a specification of most lemmas and theorems of Leśniewski's ontology and mereology, while the first layer is substituted by the logic of type theory. The theoretical basis will be the support for an interactive or automatic proof search. The complete model expressed in Coq (partly solved with automated theorem provers using *CoqHammer*), will result in a library which can be reused in the future for several purposes.

Tarski (1956a) then developed among others, a geometry of solids based on Leśniewski's mereology. We will exploit λ-MM to axiomatize a geometrical theory without points. Our first version (Dapoigny and Barlatier, 2015) relied on the different approach in which Leśniewski's systems were directly expressed in type theory through a shallow embedding using the five axioms of mereology. The present approach focuses on a strongest theoretical model while at the same time relying on few axioms. In addition, the conceptualization is rather more elaborate through applying some ideas of Jaśkowski. Finally, a small application example is set forth.

In section 2, we briefly summarize Leśniewski's systems (LO and mereology). Starting from basic properties of sets in mathematics we go on to translate a fragment of LO and the basic axiom of mereology in λ-MM in section 3, while in section 4 we detail the full system of mereology λ-MM. In section 5 we apply the above system to investigate a version of Tarski's geometry of solids and discuss a basic extension. Finally, the conclusion summarizes and motivates important results together with their possible extension.

2. Overview on Leśniewski's mereology

In the early 20th century, Leśniewski proposed a generalized two-valued propositional calculus (called protothetic) based on a single primitive, the equivalence construct. This logical system was the support of a second level called Leśniewski's Ontology (LO), and of the third level (mereology) (see Leśniewski, 1916), whose purpose was the description of the world with collective classes[2].

2.1. Leśniewski's ontology

LO can be characterized as a general theory dealing with the logical relationships between names (ε). The logical support of LO corresponds to the distributive interpretation of classes (like sets). It consists of (i) a

[2] A collective class does not require the unintuitive distinction between an individual and a totality.

primitive category N (i.e., names), (ii) a proposition forming function, i.e., the copula ε, which connects two variables in category N without imposing a type distinction between them, (iii) a single axiom controlling the behavior of terms in N, (iv) ontological definitions and (v) a rule for ontological extensionality. It results in a single axiom:

$$\forall Aa\Big(A \,\varepsilon\, a \leftrightarrow (\exists B(B \,\varepsilon\, A) \wedge \\ \forall CD((C \,\varepsilon\, A \wedge D \,\varepsilon\, A) \to C \,\varepsilon\, D) \wedge \forall C(C \,\varepsilon\, A \to C \,\varepsilon\, a))\Big)$$

The first conjunction of the right side of the equivalence prevents A from being an empty term, the second conjunction states the uniqueness of A while the latter refers to a kind of convergence (anything which is A is also an a). Any well-formed expression can be assigned to exactly one category, one of two primitive categories (truth values or N) or to one of a potentially infinite number of categories that are a combination of the two primitive categories. Only two constant names are introduced through definitions, i.e., the empty name (Λ) defined as the contradictory name and the universal name (V). Many relations are then introduced in LO such as name inclusion with three forms, i.e., weak inclusion ($\subset$), strong inclusion ($\sqsubset$) and partial inclusion ($\triangle$), nominal negation ($\mathfrak{neg}$), nominal conjunction ($\cap$), nominal disjunction ($\cup$):

$$\forall ab(a \subset b \leftrightarrow \forall A(A \,\varepsilon\, a \to A \,\varepsilon\, b))$$
$$\forall ab(a \sqsubset b \leftrightarrow \exists A(A \,\varepsilon\, a) \wedge \forall A(A \,\varepsilon\, a \to A \,\varepsilon\, b))$$
$$\forall ab(a \triangle b \leftrightarrow \exists A(A \,\varepsilon\, a \wedge A \,\varepsilon\, b))$$
$$\forall Aa(A \,\varepsilon\, \mathfrak{neg}\, a \leftrightarrow (A \,\varepsilon\, A \wedge \neg A \,\varepsilon\, a))$$
$$\forall Aab(A \,\varepsilon\, a \cap b \leftrightarrow (A \,\varepsilon\, A \wedge A \,\varepsilon\, a \wedge A \,\varepsilon\, b))$$
$$\forall Aab(A \,\varepsilon\, a \cup b \leftrightarrow (A \,\varepsilon\, A \wedge A \,\varepsilon\, a \vee A \,\varepsilon\, b))$$

In Leśniewski' systems, definitions are not meta-theoretical abbreviations, but rather introduce new symbols into the object language (they are part of the language itself). More precisely each definition should follow meta-rules to be well-formed. Two kinds of definitions are involved, i.e., "protothetic" definitions which introduce functions that generate propositions and "ontological" ones which introduce functions that generate names:

$$\forall x_1, ..., x_n(\phi(x_1, ..., x_n) \leftrightarrow \psi(x_1, ..., x_n))$$
$$\forall A, x_1, ..., x_n(A \,\varepsilon\, \varphi(x_1, ..., x_n) \leftrightarrow A \,\varepsilon\, A \wedge \psi(A, x_1, ..., x_n))$$

where $A \,\varepsilon\, A \wedge \psi(...)$ stands for the name to be constrained. It provides two information, (i) a propositional expression which requires the copula whose argument is the new defined type and (ii) an equivalence between this type and a given propositional expression.

LO relies on a single category N for names. A singular name (i.e., singleton) always occurs as the first argument of ε while the second argument is a name. A formula like $A \,\varepsilon\, A$ is a predicate proving that A is a singular name. As a consequence, two equalities are introduced, one for singular names called *singular_eq* and another for name referred to as *weak_eq*.

2.2. Mereology

Mereology built on LO introduces a few axioms which are not deducible from logical principles. It introduces collective classes, the name-forming functions "class of", "part of" and "element of" relations together with their properties. The most usual formalization introduces a mereological entity called "part" as a primitive. Mereology is developed on new primitive functions of which the most important is that of class, denoted kl. Two name forming functions are required, i.e., pt (for "part of") and el (for "element of") which all belong to the category of functions taking an argument in N and returning a value in N. It must be underlined that the returned name with the kl function is always a singular name while for the el function, it is a name. While the original theory starts with the pt primitive (i.e., proper part), it is also possible to start from the el (part) which is less restrictive. Axioms (A1) and (A2) assert respectively that the relation pt is asymmetric and transitive:

$$\forall PQ(P \,\varepsilon\, \mathsf{pt}\, Q \;\rightarrow\; \neg\, Q \,\varepsilon\, \mathsf{pt}\, P) \tag{A1}$$

$$\forall PQR((P \,\varepsilon\, \mathsf{pt}\, Q \;\wedge\; Q \,\varepsilon\, \mathsf{pt}\, R) \;\rightarrow\; P \,\varepsilon\, \mathsf{pt}\, R) \tag{A2}$$

Then, two definitions are needed. They introduce respectively the functions el and kl:

$$\forall PQ(P \,\varepsilon\, \mathsf{el}\, Q \;\leftrightarrow\; (P \,\varepsilon\, P \;\wedge\; (P = Q \;\vee\; P \,\varepsilon\, \mathsf{pt}\, Q))) \tag{D1}$$

$$\begin{aligned} &\forall Pa(P \,\varepsilon\, \mathsf{kl}\, a \;\leftrightarrow\; P \,\varepsilon\, P \;\wedge \forall Q(Q \,\varepsilon\, a \;\rightarrow\; Q \,\varepsilon\, \mathsf{el}\, P) \;\wedge \\ &\quad \forall Q(Q \,\varepsilon\, \mathsf{el}\, P \;\rightarrow\; \exists CD(C \,\varepsilon\, a \;\wedge\; D \,\varepsilon\, \mathsf{el}\, C \;\wedge\; D \,\varepsilon\, \mathsf{el}\, Q))) \end{aligned} \tag{D2}$$

The term $P = Q$ stands for the equality between singular names. Definition (D2) says that P is a collective class of objects a if and only if (i)

P is an object, (ii) every a is an element of P and (iii) for any Q, if Q is an element of P, then some element of Q is an element of some a.

$$\forall PQa((P \mathrel{\varepsilon} \mathsf{kl}\, a \wedge Q \mathrel{\varepsilon} \mathsf{kl}\, a) \rightarrow P = Q) \tag{A3}$$

$$\forall Pa(P \mathrel{\varepsilon} a \rightarrow \exists Q(Q \mathrel{\varepsilon} \mathsf{kl}\, a)) \tag{A4}$$

Whereas axiom (A3) states the uniqueness of classes, axiom (A4) asserts that if some object is an a, then some object is the class of objects a. It has been shown that using any function of mereology (e.g., el), mereology can be formalized with a single axiom (see Lejewski, 1963; Sobociński, 1984). We will consider the single axiom with el as the primitive term referred to as (M) and inferentially equivalent to the four axioms above:

$$\begin{aligned}\forall AB\Big(A \mathrel{\varepsilon} \mathsf{el}\, B \leftrightarrow (A \mathrel{\varepsilon} A \wedge B \mathrel{\varepsilon} B \wedge \big(B \mathrel{\varepsilon} \mathsf{el}\, B \rightarrow \forall ab(B \mathrel{\varepsilon} a \wedge \\ \forall C(C \mathrel{\varepsilon} b \leftrightarrow \forall D(D \mathrel{\varepsilon} a \rightarrow D \mathrel{\varepsilon} \mathsf{el}\, C) \wedge \\ \forall D(D \mathrel{\varepsilon} \mathsf{el}\, C \rightarrow \exists EF(E \mathrel{\varepsilon} a \wedge F \mathrel{\varepsilon} \mathsf{el}\, D \wedge F \mathrel{\varepsilon} \mathsf{el}\, E))) \\ \rightarrow A \mathrel{\varepsilon} \mathsf{el}\, b)))\Big)\end{aligned} \tag{M}$$

The semantics of (M) is equivalent to four assumptions: (i) every individual is element of itself, (ii) if two individuals are element of each other, then they are identical, (iii) the relation el built with ε is transitive and (iv) if an individual is element of a name, this name corresponds to an individual.

3. Basic foundations

3.1. The monadic second-order translation

Following the idea of Clay (1969, 1974), i.e., using two axioms, we will be able to prove that mereology which relies on the part relation coincides with the model of a boolean algebra with its zero-deleted. We modify his work with a complete specification of characteristic functions in CIC and extend it with many relations composed with ε (proper part-of, collective class, collection, sub-collection, extern to, relative complement, etc.). Then we complete the synthesis by extending the works of Smirnov (1987) and Cocchiarella (2001) (which are limited to LO), to the full version of mereology and provide a monadic second order translation in Coq. This choice relies on the fact that monadic second order logic theory of sets is decidable (this is a consequence of the theory of atomic boolean algebras which has a recursive set of decidable completions (see, e.g., Makowsky, 2004)).

Leśniewski's version of LO corresponds to a version of higher-order logic with arbitrary finite types, and the lower part of λ-MM will map a limited subsystem of LO restricted to first-order construct using definitional comprehension. More precisely, to construct a usable definition for the ε relation, characteristic functions are of interest. They are functions defined on a set Σ that indicates membership of an element in a subset S of Σ, having the value true for all elements of S and the value false for those that are not in S. They make it possible to translate statements about sets to statements about functions. They will be at the heart of definitions as will be shown hereafter, i.e., we will specifically use characteristic functions to convert definitions into usable lemmas.

The specification of λ-MM includes all boolean connectives, first-order quantification and quantification over unary predicates. We assume a standard way for building the second-order one-place predicate calculus. Object variables are described with capital letters, while variables that refer to sets will use lowercase letters. The formulation of the monadic second-order language is based on a relational two-sorted language in which X and a range over the respective disjoint sets of objects and of second-order variables. More specifically, variables X, Y, Z denote objects while variables a, b, c denote sets of objects and are constrained to be names through typing (as a special case, variables A, B, C, etc. denote individual names). Atomic formulas belong to the sort B of boolean propositions. Complex formulas are constructed in a standard way using propositional binary connectives (i.e., $\in$, $\vee$, $=$) and quantifiers binding predicate variables. Formulae for the monadic second-order language, denoted ϕ, are given according to the following syntax:

$$\texttt{Terms}\ \tau ::= X \mid \mathit{true} \mid \mathit{false}$$

$$\texttt{MSL formulae}\ \phi ::= \tau = \tau \mid \tau \in a \mid \phi \vee \phi \mid \neg \phi \mid \exists X\ \phi(X) \mid \exists a\ \phi(a)$$

Formulae are closed under disjunction, negation and quantification over first or second-order variables. Other connectives are defined on the basis of classical equivalences. For example, universal quantification is derived from the existential form with: $\forall A\ \phi \leftrightarrow \neg \exists A \neg \phi$, implication $\phi \rightarrow \psi$, from $\neg \phi \vee \psi$, etc. Notice that binary predicate symbols may occur in formulae, however, only the unary ones may be quantified over. All formulae are interpreted in the standard set-based model of names, where V denote the universe of objects that are restricted to names. Individual name variables A, B, ... range over objects X, Y, ... such that each individual name corresponds unambiguously to a given singleton, $A = \{X\}$ and predicate variables range over sets of objects $a, b, \ldots \subseteq V$.[3] All formulas in the following are restricted to quantification over objects or set of objects, i.e., names.

3.2. The calculus of inductive constructions

We will begin by demonstrating that type theory is sufficiently expressive to replace protothetic. A qualitative measure relative to the Calculus of Inductive Constructions (CIC) which is, with some variants, the one implemented in the Coq theorem prover can be drawn. First we have to prove that CIC is at least as expressive as Church Simple Type Theory (STT) (see Church, 1940). Whereas STT has variables ranging over functions, together with binders for them, System F supports a mechanism of universal quantification over types which results in variables ranging over types, and binders for them. It yields that generic data types such as list, trees, etc. can be encoded. It follows that system F is more expressive than STT. Furthermore, CIC adds universes to the system F, which leads to an improvement of its consistency strength. Adding dependent types in CIC enhances the computational power but does not affect its consistency strength. As a result, the expressive power of CIC is higher than STT. Then, we have to prove that STT is at least as expressive as protothetic. In protothetic, quantification is allowed on propositional variables and variables of propositional functions to any degree. Propositional type theory, first studied in (Henkin, 1963; Kaminski and Smolka, 2008), is the restriction of simple type theory to a single base type that is interpreted as the set of two truth values[4]. It follows that protothetic is equivalent to a propositional type theory. Now, if we consider LO which extends protothetic, all symbols of LO can be substituted with variables that can be quantified, then LO is equivalent in expressive power to STT. It yields that protothetic as a sub-theory of LO is at most as expressive as STT. Combining the two claims, we derive that CIC is at least as expressive as protothetic. It turns out that using Coq's classical logic is justified.

[3] For readability, we shall use capital letters for names that are known to be individual names and lower case letters otherwise.

[4] Notice that CIC uses the Curry-Howard isomorphism.

In CIC, it is known that proof irrelevance, propositional completeness and the law of excluded middle are all endorsed by the boolean model. The type of propositions *Prop* will be restricted to boolean values using the classical library of Coq to support classical logic. The representation relies on inductive types that are algebraic types with constructors able to define data structures. A basic type *object* provides us with a decidable equality over objects through the classical library. Then, N is defined as an inductive type that is a subset of the type object and whose constructor Caract requires proved objects. The type N corresponds to the set of values generated by constructor functions (e.g., the smallest set that is stable by functions). Notice that N cannot be defined as a simple type since a type is not able to represent a set (all its elements should be of the same type). Using CIC notation, the defined inductive type has name N while its constructor has the (indexed) type between braces:

DEFINITION 3.1. $\texttt{Ind}(N : \texttt{Type})\ \{\ \mathsf{Caract}\text{: } (\mathit{object} \to \texttt{Prop}) \to N\ \}$

The main benefit of such a definition is that any construct having the type N needs only to provide the argument of N, i.e., $(\mathit{object} \to \texttt{Prop})$ according to its expected properties. In the type theory, all variables reside either in *object* or in N for data and in *Prop* for truth values. Let us mention that with the above assumptions, we do not follow Leśniewski's rules of definition since they are expressed with the rules of type theory.

3.3. Formalizing definitions and lemmas

The LO translation only requires names but many lemmas or definitions use objects to state an equivalence with a boolean algebra without zero. For any set a, we define a characteristic function as a predicate χ which, given element X such that $\chi(a, X)$ is true if and only if object X belongs to the set a:

DEFINITION 3.2. $\chi(a : N,\ X : \mathit{object})\ : \mathit{Prop} :=$

$$(\texttt{let}(f) := a \texttt{ in } f\ X)\ a\ X = \mathit{True}$$

It is nothing other than the type-theoretical translation of the assertion: $X \in a$ which will be denoted $\texttt{In}\ a\ X$ in the following. The representation of the set-based characteristic function relies on a construct that checks whether a given object X appears in a and assumes that the returned result is true. Given a and b having type N, set inclusion and set equality are defined as usual:

DEFINITION 3.3.

$set\_incl\ (a\ b : N) : Prop := \forall\ X{:}object,\ \texttt{In}\ a\ X \to \texttt{In}\ b\ X$

DEFINITION 3.4.

$set\_eq\ (a\ b : N) : Prop := \forall\ X{:}object,\ \texttt{In}\ a\ X \leftrightarrow \texttt{In}\ b\ X$

At the interface between set theory and mereology, singletons play a crucial role (see Hamkins and Kikuchi, 2016; Lewis, 1991). We introduce the *singleton* function which maps every object X to its associated set, $singleton(X)$. In all the remaining part of the paper we will use ι as a symbol to denote the *singleton* function *à la* Russell. To prove that an object a is a singleton we have to feed the Caract constructor with a function which for all object a' returns the right truth value:

```
match eqobject_dec a a' with
    | left h ⇒ True
    | right h ⇒ False
    end
```

In summary, names are sets of objects while a singular name is a singleton set. Assuming basic axioms of set theory, the following equivalence rule can be easily derived:

LEMMA 3.1. $\forall\, X\ Y : object.\ \texttt{In}\ Y\ (\iota\ X)\ \leftrightarrow\ X = Y$

The set-based interpretation of individuals is given by the following definition, with a a given name which denotes a second-order variable:

DEFINITION 3.5. $Individual(a : N)\ :=\ \exists\, X : object.\ set\_eq\ \iota(X)\ a$

From this definition we can easily infer that any individual is a singleton:

LEMMA 3.2. $\forall\, X : object.\ Individual(\iota\, X)$

The set equality between individuals can be easily shown to reduce to an equality between objects X and Y such that $\exists\, X.\ set\_eq\ a\ \iota(X)$ and $\exists\, Y.\ set\_eq\ b\ \iota(Y)$.

LEMMA 3.3. $\forall\, X\ Y : object.\ set\_eq\ \iota(X)\ \iota(Y) \leftrightarrow\ X = Y$

PROOF. In the first case, applying definition 3.4, substituting the respective indicator functions and by transitivity, $X = Y$. In the second case, rewriting equality and applying reflexivity solves the goal. $\dashv$

We can extend the definition of the indicator function to a functional structure $\phi(a)$ which construct a name with the general form, in which $S(\iota(X),\, a)$ denotes a sentence expressing an appropriate property for $\phi(a)$:

DEFINITION 3.6. $\phi(a) : N :=$ Caract (`fun` $X{:}object \Rightarrow S(\iota(X), a)$)

For a generic structure `In` $(\phi\, a)X$ with $\phi\, a$ any expression involving the set a as a free variable and an object variable X, as an indicator function that is true only if X belongs to $\phi\, a$, we can show that the following theorem holds:

THEOREM 3.1. $\forall\, a : N. \forall\, X : object.$ `In` $(\phi\, a)\, X \;\leftrightarrow\; \eta(\iota\, X)\, (\phi\, a)$

PROOF. From lemma 3.2 the right member becomes: $Individual(\iota\, X) \wedge$ `In` $(\phi\, a)\, X$. Since X denotes an object, we can substitute `In`$(\phi\, a)\, X$ in the right member with *incl* $(\iota\, X)(\phi\, a)$. Then, using definition 3.9, it is easily rewritten as $\eta\,(\iota\, X)\,(\phi\, a)$ ⊣

All definitions will be designed according this scheme. In other words, any definition describes the properties that an object X should fulfill to be a member of the set of objects[5] which are set up by the defined function ϕ.

3.4. Formalizing the LO translation

The signature for the first part of λ-MM has the following structure:

$$
\begin{aligned}
&P_{LO} = <S_{LO}, \Sigma_{LO}> \ \texttt{with}:\\
&S_{LO} = \{Prop, N\}\\
&\Sigma_{LO} =\\
&\qquad \eta : N \rightarrow N \rightarrow Prop\\
&\qquad \Lambda : N\\
&\qquad V : N\\
&\qquad singular\_eq : N \rightarrow N \rightarrow Prop\\
&\qquad weak\_eq : N \rightarrow N \rightarrow Prop\\
&\qquad weakInclusion : N \rightarrow N \rightarrow Prop\\
&\qquad neg : N \rightarrow N\\
&\qquad n\_disjunction : N \rightarrow N \rightarrow Prop\\
&\qquad n\_conjunction : N \rightarrow N \rightarrow Prop
\end{aligned}
$$

Several authors have explored Leśniewski's logic of names and have proved that it is reducible to a monadic second-order version (see, among

[5] That is, a member of the corresponding name.

others, Smirnov, 1987; Cocchiarella, 2001). They restrict their versions to non-creative definitions by adding existentials. However, CIC definitions are submitted to rules that constrain each definition to be instantiated in each of its occurrences in lemmas or theorems; therefore no existentials are required. First we need to prove that the mapping translates Leśniewski's single axiom for the fragment of LO into a theorem of λ-MM. Due to space limitations, we only provide proof sketches for significant theorems.

Definitions of the universe V of names and the empty name are respectively such that:

DEFINITION 3.7. $V := \mathsf{Caract}\ (\texttt{fun}\ s{:}object \Rightarrow True)$.

DEFINITION 3.8. $\Lambda := \mathsf{Caract}\ (\texttt{fun}\ s{:}object \Rightarrow False)$.

from which we can easily derive:

LEMMA 3.4. $\forall\, a : N.\ incl\ a\ V$

The *eta* (η) relation between set variables provides the counterpart of the ε construct between names:

DEFINITION 3.9. $\eta\ (A\ \ b\ {:}N) : Prop := Individual\ \ A \wedge incl\ \ A\ \mathrm{b}$.

Among the consequences of definition 3.9, we can mention equality between names, which is:

LEMMA 3.5. $\forall\, A\,, B : N.\ \eta\, A\ B\ \wedge\ \eta\, B\ A \leftrightarrow\ set\_eq\ A\ B$

PROOF. For the first implication, by unfolding definition 3.9 we get two inclusions, *incl a b* and *incl a b* that should be both satisfied and using twice definition 3.3 results in 3.4 as expected. The second implication is similar. ⊣

According to these assumptions, the single axiom of Leśniewski's ontology becomes a theorem in λ-MM. Its proof requires a set of lemmas from Clay's work (we only supply relevant lemmas) leading to the following results. Firstly, unfolding definition 3.9 and using the reflexivity of inclusion, we obtain:

LEMMA 3.6. $\forall\, A : N.\ \eta\, A\ A\ \leftrightarrow\ Individual\ A$

Secondly, by lemma 3.6 and definition 3.9, we get:

LEMMA 3.7. $\forall\, A\, b : N.\ \eta\, A\, b\ \rightarrow\ \eta\, A\ A$

Using the distributivity of *set_eq* on inclusion, symmetry of *set_eq* and definition 3.9 we have:

LEMMA 3.8. $\forall\, A\, b\, C : N.\ \eta\, A\, b\ \wedge\ \eta\, C\ A\ \rightarrow\ \eta\, C\, b$

By *set_eq*, lemmas 3.7, 3.6, distributivity of *set_eq* over inclusion, symmetry and transitivity of *set_eq* and with definition 3.9 we obtain:

LEMMA 3.9. $\forall\, A\, b\, C\, D : N.\ \eta\, A\ b\ \wedge\ \eta\, C\ A\ \wedge\ \eta\, D\ A \rightarrow\ \eta\, C\ D$

Using a lemma about singletons we have:

LEMMA 3.10. $\forall\, a : N.\ X : \textit{object}.\ \eta(\iota\, X)\ a \leftrightarrow\ \texttt{In}\, a\, X$

From definition 3.9, lemma 3.10 and singleton equality, we get:

LEMMA 3.11. $\forall\, a : N.\, X : \textit{object}.\, (\forall\, A\, B : N.\ \eta\ A\, a \wedge \eta\ B\, a \rightarrow \eta\, A\, B) \wedge$
$\texttt{In}\, a\, X\ \rightarrow\ \forall\, Y : \textit{object}.\, \texttt{In}\, a\, Y\ \rightarrow\ X\ =\ Y$

From lemmas 3.11 and 3.7, symmetry of *set_eq*, distributivity of *set_eq* on inclusion, some lemmas about singletons and unfolding definitions 3.3, 3.4 and 3.9, we get:

LEMMA 3.12. $\forall\, A\, b\, c : N.\ \eta\, A\ b\ \wedge\ (\forall\, D : N.\ \eta\, D\ b\ \rightarrow\ \eta\, D\ c) \wedge$
$(\forall\, E\, F : N.\ \eta\, E\ b\ \wedge \eta\, F\ b\ \rightarrow\ \eta\, E\ F)\ \rightarrow\ \eta\, A\ c$

Using lemmas 3.7, 3.8, 3.9 and 3.12 we obtain:

THEOREM 3.2. $\forall\, A\, b : N.\ \eta\, A\ b\ \leftrightarrow\ \exists\, C : N.\ \eta\, C\ A \wedge$
$(\forall\, D : N.\ \eta\, D\ A\ \rightarrow\ \eta\, D\, b) \wedge$
$(\forall\, C\, D : N.\ \eta\, C\ A\ \wedge\ \eta\, D\ A\ \rightarrow\ \eta\, C\ D)$

The single axiom of LO has been proved in λ-MM without any axiomatic assumptions other than the usual set-based axioms. Then many lemmas are proved and we only give the following which states the equivalence between set equality and equivalence of names:

LEMMA 3.13. $\forall\, a\, b : N.\textit{set\_eq}\ a\ b\ \leftrightarrow (\forall A : N, \eta\ A\ a \leftrightarrow \eta\ A\ b)$.

Basic definitions introduced below form the foundations of λ-MM. They are respectively: singular equality, weak equality, weak inclusion, nominal negation, nominal disjunction, nominal conjunction, minimal and maximal existence.

DEFINITION 3.10. $\textit{singular\_eq}\ (A\ B{:}N) := \eta\ A\ B \wedge \eta\ B\ A$

DEFINITION 3.11. $\textit{weak\_eq}\ (a\ b\ {:}N) := \forall\ A\ :\ N\ .\ \eta\ A\ a \leftrightarrow \eta\ A\ b$

DEFINITION 3.12. *weakInclusion* $(a\ b:N) := \forall\ A : N\ .\ \eta\ A\ a \rightarrow \eta\ A\ b$

DEFINITION 3.13. $\mathtt{neg}$ $(a:N) : N :=$
$\mathsf{Caract}(\mathtt{fun}\ X\text{: } object \Rightarrow (Individual\ (\iota\ X) \wedge \neg(\eta\ (\iota\ X)\ a)))$

DEFINITION 3.14. *n_disjunction* $(a\ b{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X\text{: } object \Rightarrow (\eta\ (\iota\ X)\ a \vee \eta\ (\iota\ X)\ b))$

DEFINITION 3.15. *n_conjunction* $(a\ b:N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X\text{: } object \Rightarrow (\eta\ (\iota\ X)\ a \wedge \eta\ (\iota\ X)\ b))$

DEFINITION 3.16. *exists_at_least* $(a : N) := \exists\ A, \eta\ A\ a$

DEFINITION 3.17. *exists_at_most* $(a:N) := \forall\ B\ C, \eta\ B\ a \wedge \eta\ C\ a \rightarrow \eta\ B\ C$

In the following, weak inclusion will be replaced with the symbol $\subseteq$ if no confusion results. Since λ-MM is derived from a subsystem of LO, including the above axiom, then it stands as a model for the boolean algebra. More than an hundred of lemmas are inferred from these definitions.

3.5. The Boolean model of mereology

$$
\begin{aligned}
&P_M = <S_M, \Sigma_M, Ax_M> \ \mathtt{with:}\\
&S_M = S_{LO}\\
&\Sigma_M = \Sigma_{LO}\ \cup\ \{\, el : N \rightarrow N\,\}\\
&Ax_M =\\
&\quad 1)\ \forall XY.\, X \leqslant Y\ \leftrightarrow\ (\mathtt{In}\,V\ X\ \wedge\ \mathtt{In}\,V\ Y\ \wedge\ (Y \leqslant Y\ \rightarrow\\
&\quad (\forall\,\beta\,\alpha.\ incl\,\alpha\,V\ \wedge\ incl\,\beta\,V \wedge \mathtt{In}\,\alpha\,Y\ \wedge (\forall\,Z.\,\mathtt{In}\,\beta\,Z\ \leftrightarrow\\
&\quad ((\forall W.\,\mathtt{In}\,\alpha\,W\ \rightarrow\ W \leqslant Z) \wedge\\
&\quad (\forall\,W.\,W \leqslant Z\ \rightarrow\ \exists\,ST.\,\mathtt{In}\,\alpha\,S\ \wedge T \leqslant\ W\ \wedge T \leqslant S))) \rightarrow\\
&\quad \exists L.\ set\_eq\ \beta\ (\iota L)\ \wedge\ X \leqslant L)))\\
&\quad 2)\ \exists\,XY.\,\neg(X = Y)
\end{aligned}
$$

All proofs for stating the equivalence between a boolean algebra without zero and the unique formula (M) for mereology (see subsection 2.2) are given in appendix A. We start with two axioms about a boolean algebra. The first one stands for a boolean algebra with zero deleted established in Clay's work (1969) while the second one assumes that the boolean lattice includes at least two elements. We first show that we can prove that the relation $\leqslant$ which generates this algebra is a partial order. Then, as proposed in (Clay, 1974), we assume the following type theoretical definition which defines el as a function depending on the partial order relation:

DEFINITION 3.18. el $(a : N) : N :=$ Caract (fun X: *object* $\Rightarrow$ (*Individual* $a \wedge \exists\ Y\ Z$:*object*, In $(\iota\ A)\ Y \wedge$ In $a\ Z \wedge Y \leqslant Z$)).

However, el is a function and it is its composition with the η relation which gives rise to what is usually called the "part of" relation in theories such as *Classical Extensional Mereology* (see, e.g., Varzi, 1996; Casati and Varzi, 1999; Cotnoir and Bacon, 2012). More generally, we will show that other functions can be composed with η in the same way, e.g., part, class, etc. This aspect is appealing since instead of having properties of a single relation, the composition offers both the properties of the η relation and those of the function. The adopted notation in the remaining part of the paper will comply with the following syntax: $\eta <arg_1> (\phi <arg_2>)$ in which $<arg_i>$ denote name arguments and $\eta\,\phi$ stands for the compound relation. We are now able to demonstrate that the el function such defined satisfies axiom (M). Therefore, mereology which relies on a single axiom has a boolean model without zero. This result has several impacts. First, it establishes the soundness of the model. Second, provided that we work on finite sets of names (as it is the case in most applications), the monadic version is decidable and can generate usable algorithms.

4. Modeling λ-MM

4.1. Formalizing minimal mereology

$$\begin{aligned} P_{mM} &= <S_{mM}, \Sigma_{mM}, Ax_{mM}> \ \texttt{with}: \\ S_{mM} &= S_M \\ \Sigma_M &= \Sigma_M \ \cup\ \{\mathsf{pt} : N \to N, \mathsf{kl} : N \to N\} \\ Ax_{mM} &= Ax_M \end{aligned}$$

To get a basic version of λ-MM, we provide the proper part (pt) and the class (kl) functions specified as axioms in (Leśniewski, 1916). They are built with characteristic functions:

DEFINITION 4.1. pt $(A : N) : N :=$ Caract (fun X:*object* $\Rightarrow$ (*Individual* $(\iota\ X) \wedge \eta\ (\iota\ X)($el $A) \wedge \neg($*set_eq* $A\ (\iota\ X))))$

DEFINITION 4.2. $\mathtt{kl}\ (a : N) : N := \mathsf{Caract}\ (\mathtt{fun}\ X{:}\mathit{object} \Rightarrow$
$(\mathit{Individual}\ (\iota\ X) \wedge$
$(\forall\ B, \eta\ B\ a \to \eta\ B\ (\mathtt{el}\ (\iota\ X))) \wedge$
$(\forall\ B, \eta\ B\ (\mathtt{el}\ (\iota\ X)) \to \exists\ C\ D, \eta\ C\ a \wedge$
$\eta\ D\ (\mathtt{el}\ C) \wedge \eta\ D\ (\mathtt{el}\ B))))$

For all values of X which satisfy right-hand sides of the definitions, we build the respective sets $\mathtt{pt}\,A$ and $\mathtt{kl}\,a$. Let us consider a collective class. If a denotes "books in my library" and $\mathtt{kl}\,a$ the class of "books in my library", in $\eta\,A\ (\mathtt{kl}\,a)$ A is an alias for "my library". The following three assertions must be true. First, A is the name of a single object that stands for "my library". Second, everything that is a book in my library is an element in my library. Finally, let D be an element of my library, e.g., a library shelf (meaning the books that are on it), then we have to find E which is among books in my library and such that some element on the library shelf is element of E. If E stands for the *Encyclopedia Britannica* whose volumes are on the library shelf, then the third volume is both an element of D and E.

A collective class only exists through its elements therefore it cannot be empty. Furthermore, the concept of collective class is not entirely resolved by its definition and more specifically, its uniqueness must be proved. Another problem relates to *Russell*'s antinomy (a set corresponds to any propositional function). We can observe that if a given object A is (one of) an a, i.e., $\eta\,A\ \ a$, then the class of a's must exist: if Francis Drake is a hero, then the class of heroes is existent. These important theorems can be derived in λ-MM. Firstly, by lemmas 3.7 and 3.9, we obtain:

LEMMA 4.1. $\forall\,A\,B\,c.\ \eta\,A\ B \wedge \eta\,B\ c \to \eta\,A\ c$

By $\mathtt{kl}$ definition and substitution of a with $(\iota\ A)$, the name of a class is always a singular name, i.e., a singleton:

LEMMA 4.2. $\forall\,A\,b.\ \eta\,A\ \ \mathtt{kl}\,b \to\ \mathit{Individual}\ A$

Moreover, rewriting $\eta(\iota\,A)(\mathtt{kl}\,a)$ in the left member of the class definition ($\mathtt{kl}$), we have:

LEMMA 4.3. $\forall A\,a.\ \eta\,A\ (\mathtt{kl}\,a) \to \exists\,B.\ \eta\,B\ a$

Now, by lemma A.17, we obtain:

LEMMA 4.4. $\forall\,A\,B.\ \eta\,A\ \mathtt{el}\,B \to\ \mathit{Individual}\ B$

The following lemmas are consequences of the collective class definition. Firstly, unfolding the $\mathtt{kl}$ definition, we get:

LEMMA 4.5. $\forall\, A\, B\, a.\ \eta\, A\, (\mathtt{kl}\, a) \rightarrow\ \eta\, B\ a \rightarrow\ \eta\, B\, (\mathtt{el}\, a)$

Secondly, using $\mathtt{el}$ and $\mathtt{kl}$ indicator functions with lemma 3.1 we have:

LEMMA 4.6. $\forall\, A\, B\, c.\ \eta\, A(\mathtt{kl}\, c) \rightarrow \eta\, B(\mathtt{el}\, A) \rightarrow \exists DE.\ \eta\, D\ c\, \wedge$
$\eta\, E(\mathtt{el}\, D) \wedge \eta\, E(\mathtt{el}\, B)$

From lemmas 3.7, 3.6 and A.5 we have that if a name is something, then it is an element of itself. This result departs from what is usually true of distributive classes assuming that a class never belongs to itself:

LEMMA 4.7. $\forall\, A\, b.\ \eta\, A\, b \rightarrow \eta\, A\, (\mathtt{el}\, A)$

From lemma A.5 we have that if a name exists, then the relation $\mathtt{el}$ built with η is reflexive:

LEMMA 4.8. $\forall\, A.\ \eta\, A\, A \rightarrow \eta\, A\, (\mathtt{el}\, A)$

By lemma 4.8 we have that any collective class name corresponds to an individual name:

LEMMA 4.9. $\forall\, a.\ \eta(\mathtt{kl}\, a)\, (\mathtt{kl}\, a)\ \rightarrow\ \eta(\mathtt{kl}\, a)\, (\mathtt{el}(\mathtt{kl}\, a))$

LEMMA 4.10. $\forall A\, B\, C.\ \eta\, A(\mathtt{el}\, B) \wedge \mathit{set\_eq}\ B\ C \rightarrow \eta\, A(\mathtt{el}\, C)$

PROOF. Using lemma 4.4, substituting `In` $b\ A$ with *incl* $(\iota\, A)\ b$ and inclusion of singletons with objects equality, then unfolding η, *Individual* and *set_eq* definitions and last, by applying theorem A.2. ⊣

Applying theorem 3.1 in $\mathtt{el}$ definition, by lemmas 3.3, 3.1 and A.3, we have that the relation $\mathtt{el}$ built with η is asymmetric:

LEMMA 4.11. $\forall\, A\, B.\ \eta\, A\, (\mathtt{el}\, B) \wedge \eta\, B\, (\mathtt{el}\, A) \rightarrow \mathit{set\_eq}\ A\ B$

By lemmas 3.6 and 4.8 we get:

LEMMA 4.12. $\forall\, A.\ \mathit{Individual}\ A \rightarrow\ \eta\, A\, (\mathtt{el}\, A)$

By unfolding definition 4.2, then applying lemmas 4.7, 4.12 and finally unfolding η definition, we obtain that any individual name is the class of its elements:

LEMMA 4.13. $\forall\, A\, a.\ \eta\, A\, a\ \rightarrow\ \eta\, A\, (\mathtt{kl}(\mathtt{el}\, A))$

From theorem A.5 and lemma 4.13 we obtain that if A is an element of B, then B is the class of elements of itself:

LEMMA 4.14. $\forall A\, B.\ \eta\, A\ (\mathsf{el}\, B) \to \eta\, B\ (\mathsf{kl}(\mathsf{el}\, B))$

From definitions of η, *incl*, *Individual*, el and applying lemma 3.1, theorem A.2 together with lemmas on singletons we get that the part-of relation el, built with η, is transitive:

THEOREM 4.1. $\forall\ A\, B\, C.\ \eta\, A\ (\mathsf{el}\, B) \wedge \eta\, B(\mathsf{el}\, C) \to \eta\, A\ (\mathsf{el}\, C)$

Unfolding pt's definition, and using lemmas 4.10 and 4.12, we get:

LEMMA 4.15. $\forall A\, B.\ Individual(A)\ \to\ \eta\, A\ (\mathsf{el}\, B)\ \leftrightarrow$
$(\eta\, A\ (\mathsf{pt}\, B)\ \vee\ set\_eq\ B\ A)$

By lemmas 4.10, 4.12, definition 4.1 and the symmetry of set equality we get that for each individual name, there exists an element of it:

LEMMA 4.16. $\forall A\, a.\ \eta\, A\ a\ \to\ \exists\, B.\ \eta\, B\ (\mathsf{el}\, A)$

The following theorem is relevant for proving the asymmetry of pt:

THEOREM 4.2. $\forall\ A\, B.\ \eta\, A\ (\mathsf{pt}\, B) \to\ \eta\, B\ (\mathsf{neg}(\mathsf{pt}\, A))$

PROOF. Using theorem 3.1 with $\phi\ A = \mathsf{pt}\, A$ and $\phi\ A = \mathsf{neg}\, A$ in respective definitions 4.1 and 3.13, then applying lemmas 4.4, 4.11 and symmetry of *set_eq* $\dashv$

Unfolding *eta*'s definition, and by theorem 4.2 we obtain that any individual is constrained to be a part of another individual:

LEMMA 4.17. $\forall A\, B.\ \eta\, A\, \mathsf{pt}\, B)\ \to\ Individual\ B$

By applying theorem 3.1 with $\phi\ A = \mathsf{pt}\, A$ in definition 4.1, we get:

LEMMA 4.18. $\forall\ A\, B.\ \eta\, A\ (\mathsf{pt}\, B) \to \eta\, A\ (\mathsf{el}\, B)$

So, using the pt definition and lemmas 4.11, 4.18, 4.15 and 4.17, it follows that the part relation built on η is asymmetric:

LEMMA 4.19. $\forall A\, B.\ \eta\, A\ (\mathsf{pt}\, B)\ \to\ \neg\ \eta\, B\ (\mathsf{pt}\, A)$

By substitution of the left members of kl and el definitions, applying theorem 3.1 and then lemma 4.6, we derive the following implication:

LEMMA 4.20. $\forall E\, a.\ \eta\, E\, a \to (\forall A.\ \eta\, A\ (\mathsf{kl}\, a) \leftrightarrow ((\forall B.\ \eta\, B\, a \to$
$\eta\, B(\mathsf{el}\, A)) \wedge (\forall B.\ \eta\, B(\mathsf{el}\, A) \to \exists C\ D.\ \eta\, C\, a \wedge \eta\, D(\mathsf{el}\, B)$
$\wedge\ \eta\, D\ \ \mathsf{el}\, C))))$

From lemmas A.5, 3.6, 4.12 and 4.20 we get:

LEMMA 4.21. $\forall A\, B\, a.\ \eta\, A\, (\mathsf{el}\, B) \wedge \eta\, B\, a \rightarrow \eta\, A\, (\mathsf{el}(\mathsf{kl}\, a))$

Unfolding definition 3.16 then, using theorem 4.3, lemma 4.3 together with lemmas about empty names we obtain that any collective class exists provided that its related name exists:

LEMMA 4.22. $\forall a\ ,\ \mathit{exists\_at\_least}\ \ a \rightarrow \mathit{exists\_at\_least}\ (\mathsf{kl}\, a)$

From lemmas 3.6, 4.7, 4.21 and 4.4 we get that each collective class denotes a singular name:

LEMMA 4.23. $\forall\, A\, a.\ \eta\, A\, a\ \rightarrow\ \eta(\mathsf{kl}\, a)(\mathsf{kl}\, a)$

By applying lemmas 4.7 and 4.21, we obtain that if a singular name is one of the plural name a, then it is also an element of the class of a's:

LEMMA 4.24. $\forall\, A\, a.\ \eta\, A\, a\ \rightarrow\ \eta\, A\, (\mathsf{el}(\mathsf{kl}\, a))$

Using lemma 4.23, we can now prove the fourth axiom of Leśniewski's mereology, i.e., that the class of a's exists provided that a exists:

THEOREM 4.3. $\forall\, A\, a.\ \eta\, A\, a\ \rightarrow\ \exists\, B.\ \eta\, B\, (\mathsf{kl}\, a)$

Unfolding definition 3.10, and by lemmas 4.3, 4.23 and 4.1, we get:

LEMMA 4.25. $\forall A\, a.\ \eta\, A\, (\mathsf{kl}\, a)\ \rightarrow\ \mathit{set\_eq}\ A\, (\mathsf{kl}\, a)$

By lemmas 3.8, 4.25 and 3.5, the third axiom of Leśniewski's mereology, which states the uniqueness of classes, is proved as follows:

THEOREM 4.4. $\forall\, A\, B\, a.\ \eta\, A\, (\mathsf{kl}\, a)\ \wedge\ \eta\, B\, (\mathsf{kl}\, a) \rightarrow\ \mathit{set\_eq}\ A\ B$

Many other theorems about collective classes can be proved (see the link at the end of this section) We will give below some important lemmas about the proper part (pt). The second axiom of Leśniewski's mereology expresses the transitivity for the part relation built on η is proved according to the following theorem:

THEOREM 4.5. $\forall\, A\, B\, C.\ \eta\, A\, (\mathsf{pt}\, B) \wedge \eta\, B\, (\mathsf{pt}\, C) \rightarrow \eta\, A\, (\mathsf{pt}\, C)$

PROOF. We use theorem 3.1, with $\phi\ A = \mathsf{pt}\, A$ in definition of pt, and lemmas 4.1, 4.11 and 4.10. ⊣

Finally, we obtain some useful lemmas. Firstly, unfolding definitions 3.18 and 3.4 and by lemmas 4.1, 3.6 and 4.8, we get:

LEMMA 4.26. $\forall A\,B.\ (\forall\,C.\ \eta\,C\ (\mathsf{el}\,A)\ \rightarrow\ \exists\,P.\ \eta\,P\ (\mathsf{el}\,C)\wedge \eta\,P\ (\mathsf{el}\,B)\rightarrow(\forall\,D.\ \eta\,D\ (\mathsf{el}\,A)\ \rightarrow\ \exists\,P\,Q.\ \eta\,P\ (\mathsf{el}\,D)\wedge \eta\,Q\ (\mathsf{el}\,B)\ \wedge\ \eta\,P\ (\mathsf{el}\,Q)\wedge\ \eta\,Q\ (\mathsf{el}\,A))$

Secondly, using lemma 4.14 and definition 4.2, we have:

LEMMA 4.27. $\forall\,A\,B.\ \eta\,A\ (\mathsf{el}\,B)\ \leftrightarrow\ \exists\,a.\ \eta\,B\ (\mathsf{kl}\,a)\ \wedge\ \eta\,A\ a$

4.2. Formalizing full mereology

$$
\begin{aligned}
P_{\lambda\text{-MM}} &= < S_{\lambda\text{-MM}}, \Sigma_{\lambda\text{-MM}}, Ax_{\lambda\text{-MM}} > \ \texttt{with}:\\
S_{\lambda\text{-MM}} &= S_{mM}\\
\Sigma_{\lambda\text{-MM}} &= \Sigma_{mM}\ \cup\ \{\ \mathit{ext} : N \rightarrow N, \mathit{coll} : N \rightarrow N,\\
&\qquad \mathit{SubColl} : N \rightarrow N, \mathit{relCompl} : N \rightarrow N \rightarrow N\}\\
Ax_{\lambda\text{-MM}} &= Ax_{mM}
\end{aligned}
$$

From all lemmas and theorems detailed above, many other lemmas are provable stating for instance that a class built on an empty name does not exist, the class of a and the class of class of a denote the same object unlike in set theory, any individual cannot be its proper part, etc. Additional functions such as exterior, collection, sub-collection and relative complement (see, e.g., Sinisi, 1983) can be introduced as characteristic functions (the symbol $\cap$ refers to nominal conjunction as defined in 3.15):

DEFINITION 4.3. $\mathit{ext}\ (A : N) : N := \mathsf{Caract}\ (\texttt{fun}\ X{:}\mathit{object} \Rightarrow (\mathit{Individual}\ (\iota\ X) \wedge \mathit{Individual}\ (A \wedge \neg(\exists\ C, \eta\ C\ (\mathsf{el}\ A) \wedge \eta\ C\ (\mathsf{el}\ (\iota\ X))))))$

DEFINITION 4.4. $\mathit{coll}\ (a : N) : N := \mathsf{Caract}\ (\texttt{fun}\ X{:}\mathit{object} \Rightarrow (\mathit{Individual}\ (\iota\ X) \wedge \forall\ B, \eta\ B\ (\mathsf{el}\ (\iota\ X)) \rightarrow \exists\ C\ D, \eta\ C\ a \wedge \eta\ D\ (\mathsf{el}\ C) \wedge \eta\ D\ (\mathsf{el}\ B) \wedge \eta\ C\ (\mathsf{el}\ (\iota\ X))))$

DEFINITION 4.5. $\mathit{SubColl}\ (A : N) : N := \mathsf{Caract}\ (\texttt{fun}\ X{:}\mathit{object} \Rightarrow (\mathit{Individual}\ (\iota\ X) \wedge \forall\ B, \eta\ B\ (\mathsf{el}\ (\iota\ X)) \rightarrow \eta\ B\ (\mathsf{el}\ A)))$

DEFINITION 4.6. $\mathit{relCompl}\ (B\ C{:}N) : N := \mathsf{Caract}\ (\texttt{fun}\ X{:}\mathit{object} \Rightarrow (\mathit{Individual}\ (\iota\ X) \wedge \eta\ B\ (\mathit{SubColl}\ C) \wedge \eta\ (\iota\ X)(\mathit{kl}\ ((\mathsf{el}\ C) \cap (\mathit{ext}\ B)))))$

Some lemmas using these definitions are given below. Firstly, by lemmas 4.6 and 4.5, we get:

LEMMA 4.28. $\forall A\, B\, a.\ \eta\, A\, (\mathtt{kl}\, a) \wedge \eta\, B\, (\mathtt{el}\, A) \rightarrow \exists\, X\, Y.\ \eta\, X\, (\mathtt{el}\, Y) \wedge \eta\, X\, (\mathtt{el}\, B) \wedge \eta\, Y\, a \wedge \eta\, Y\, (\mathtt{el}\, A)$

Secondly, unfolding definition 4.4 and using lemma 4.28, we have that each class is also a collection:

LEMMA 4.29. $\forall A\, a.\ \eta\, A\, (\mathtt{kl}\, a) \rightarrow \eta\, A\, (coll\ a)$

Thirdly, using lemma 4.12 and theorem 3.1, with $\phi\ A = ext\ A$ in the *ext* definition, we get that the relation *ext* built on top of η is irreflexive:

LEMMA 4.30. $\forall A.\ Individual A \rightarrow \neg\eta\, A\, (ext\ A)$

By theorem 3.1, with $\phi\, B = ext\, B$ and $\phi\, B = \mathtt{pt}\, B$, in the respective *ext* and pt definitions, from lemma 4.1 we get the left distributivity of *ext* over pt:

LEMMA 4.31. $\forall A\, B\, C.\ \eta\, A\, (ext\ B) \wedge \eta\, C\, (\mathtt{pt}\, B) \rightarrow \eta\, A\, (ext\ C)$

From theorem 3.1 with $\phi\ B = ext\ B$ in the *ext* definition we obtain that the relation *ext* built on top of η is symmetric:

LEMMA 4.32. $\forall A\, B.\ \eta\, A\, (ext\ B) \rightarrow \eta\, B\, (ext\ A)$

Using theorem 4.3, with $\phi\, B = ext\, B$ in the *ext* definition, and theorem A.5, we get:

LEMMA 4.33. $\forall A\, B.\ \eta\, A\, (\mathtt{el}\, B) \rightarrow \neg\eta\, A\, (ext\ B)$

Unfolding definition 4.4 and using lemma 4.12, we get:

LEMMA 4.34. $\forall A\, a.\ \eta\, A\, (coll\ a) \rightarrow \exists\, B.\ \eta\, B\ a \wedge \eta\, B\, (\mathtt{el}\, A)$

From definition 4.5, with $\phi\, B = subColl\, B$, theorems 3.6, 4.8 we get:

LEMMA 4.35. $\forall A\, B.\ \eta\, A\, (subColl\ B) \rightarrow \eta\, A\, (\mathtt{el}\, B)$

Using definition 4.5, with $\phi\ B = subColl\ B$, and theorems 3.2 and 4.1, we obtain

LEMMA 4.36. $\forall A\, B.\ \eta\, A\, (\mathtt{el}\, B) \rightarrow \eta\, A\, (subColl\, B)$

By theorems 4.35 and 4.36 we obtain that the relation *subColl* is equivalent to relation el:

LEMMA 4.37. $\forall A\, B.\ \eta\, A\, (\mathtt{el}\, B) \leftrightarrow \eta\, A\, (subColl\, B)$

By theorems 4.18 and 4.37 we get:

Lemma 4.38. $\forall A\, B.\ \eta\, A\ (\mathsf{pt}\, B) \rightarrow\ \eta\, A\ (\mathit{subColl}\ B)$

Applying lemmas 4.29, 4.3, 4.36 and 4.4 we have that any class is also the class of its collections:

Lemma 4.39. $\forall A\, a.\ \eta\, A\ (\mathsf{kl}\, a) \leftrightarrow\ \eta\, A\ (\mathsf{kl}(\mathit{coll}\ a))$

Using definition 4.2 together with lemmas 4.34, 4.23, 3.6, 4.3, 4.39, 4.3 and 4.25 we get that any collection is also an element of its corresponding class:

Lemma 4.40. $\forall A\, a.\ \eta\, A\ (\mathit{coll}\ a) \rightarrow\ \eta\, A\ (\mathsf{el}(\mathsf{kl}\, a))$

Unfolding *coll*'s definition and using lemmas 4.12, 4.26, 4.40, 4.34, 4.14, 4.25 and 4.23, we get:

Lemma 4.41. $\forall A\, B.\ \mathit{Individual}\ A\ \wedge\ (\forall\, C.\ \eta\, C\ (\mathsf{el}\, A) \rightarrow\ \exists\, X.$
$\eta\, X\ (\mathsf{el}\, C) \wedge \eta\, X\ (\mathsf{el}\, B)) \rightarrow \eta\, A\ (\mathsf{el}\, B)$

Unlike many works which formalize mereology without the ε operator, we show that there is no need in the present approach to introduce the so-called "weak supplementation principle" since it is a theorem. It says that if A is a (proper) part of a whole B, then there exists some other C which is the complement of A relative to B.

Theorem 4.6. $\forall A\, B.\ \eta\, A\ (\mathsf{pt}\, B) \rightarrow\ \exists\, C.\ \eta\, C\ (\mathit{relCompl}\ A\ B)$

Proof. Using lemmas 4.38, 4.19, unfolding definition 4.1 and lemmas 4.15, 4.17, 4.37 and 4.4. Then using 4.12, 4.41, 4.17 and usual first-order lemmas (contraposition, De Morgan, contradiction) and finally, unfolding definitions 4.3, 4.6 and 3.15 together with theorem 4.3. ⊣

The implemented version relies on a logical framework based on type theoretical foundations that characterize the logic *proof-theoretically*. The mereological system has been specified in Coq. A complete version of the resulting code (37 definitions, 416 lemmas and 22 theorems) is available in code_mereology:

Proposition 4.1 (decidability). *λ-MM in finite domains is decidable.*

Proof. A well-known result from Tarski states that the first-order theory $\mathcal{T}(B)$ of boolean algebras is decidable. Since mereology is isomorphic to a boolean algebra without the bottom element, and provided that it is axiomatized in monadic second-order logic in finite domains, then decidability is preserved. ⊣

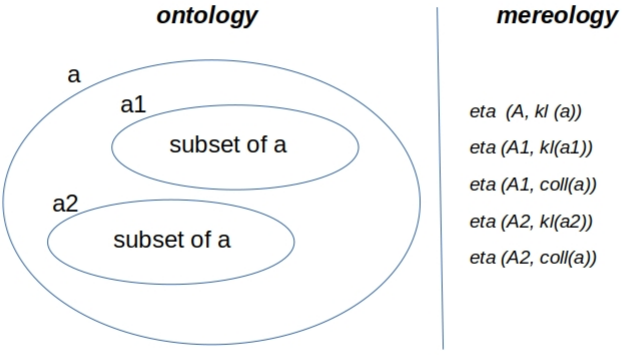

Figure 1. The concept of collection.

Assuming that functions are a crucial part of mereology gives the following benefits. Variable binding is simplified since functions can return names and LO offers primitives which operate on names (see subsection 2.1). A formula which is written: $a \subseteq b \rightarrow (coll\ a) \subseteq (coll\ b)$ would require three variables in a classical relational setting (a, b and A) such that, e.g., the second member becomes: $\forall A.\ coll(A, a) \rightarrow coll(A, b)$. Another benefit is that all name variables can be quantified without involving ontological commitments.

4.3. Expressing set theory in λ-MM

We will prove here that the role of collections corresponds to that of distributive sets. However, their relation with collective classes is more subtle. Notice that whereas a class refers to an individual name, a collection is not an individual name. Suppose that we have a name a such that $\eta\, A\ (\mathtt{kl}\, a)$. The class related to the name a contains all collections $coll\, a$ as stated in lemma 4.39. If we have another name b such that $b \subseteq a$, then a new class is introduced $\eta\, B(\mathtt{kl}\, b)$ such that this class is also a collection of a, i.e., $\eta\, B(coll\ \ a)$. In other words, the collection is the counterpart of the subset relation with collective classes (see figure 1). In summary, collections are sets and membership is addressed by η.

Let us consider *Zermelo* set theory for example, and let us explain how its corresponding axioms also reside in λ-MM. The extensionality axiom is obviously assumed since it is the basis of Leśniewski's theory. Furthermore it has been set in definition 3.4. Other axioms such as extensionality, pairing, axiom of the union, of the power set or the empty set become lemmas in λ-MM. For details, see the Coq file code. We only provide here as an example, the pairing as a proved lemma.

LEMMA 4.42. $\forall\ A\ B\ a, \eta\ A\ (\textsf{coll}\ a) \wedge \eta\ B\ (\textsf{coll}\ a) \rightarrow$
$\exists\ C, \eta\ C\ (\textsf{coll}\ a) \rightarrow \forall\ X, \eta\ X\ (\textsf{el}\ A) \leftrightarrow \eta\ X\ A \vee \eta\ X\ B.$

Semantically, given two sets A and B, there exists a set C whose members are exactly the two given sets. Since A, B and C are names, they are constrained to be sets by the predicate $\eta\, A(coll\ a)$, while a denotes the space (a proper class) containing all its collections (sets).

5. The λ-MM version of Tarski's geometry of solids

The previous work can be directly applied for a new specification of Tarski's mereogeometry which has proposed a method for axiomatizing geometry using the notion of a *sphere* (a.k.a. a *balls*) as primitive instead of usual point-based theories (see Tarski, 1929, 1956a). For that purpose, he has added the sphere primitive to Leśniewski's mereology relying on the previously defined el primitive. He showed how to give a categorical axiomatization for the geometry of solids whose models are isomorphic to the structure of regular open sets of points which are defined in the Euclidean point-based geometry. However, Tarski's theory is not fully formalized and recent works have addressed this issue (see, e.g., Bennett, 2001; Gruszczynski and Pietruszczak, 2008, 2009; Borgo and Masolo, 2010) on the basis of set theory. One might wonder about the influence of Leśniewski's work on Tarski's foundations of the geometry of solids. A closer investigation of the 1956 version of Tarski's proposal sheds light on this question and reveals the explicit influence of Leśniewski's mereology (see, e.g., Betti and Loeb, 2012, for more details). A recent paper follows this line by proposing a model of Tarski's geometry of solids based on Leśniewski's mereology (see Clay, 2021). While his objective was to investigate to what extent Tarski's paper can be formalized, the present objective rather focus on proofs and their availability for applications such as formal ontologies. For that purpose, we provide here an application of the previous theory which follows Tarski's ideas and uses λ-MM as a library.

5.1. Tarski's definitions

We briefly summarize the informal definitions stated by Tarski in his 1956 paper.

DEFINITION 5.1. A sphere A is *externally tangent* to a sphere B if (i) A is disjoint from B; (ii) given two spheres X and Y containing as a part A and disjoint from B, at least one of them is part of the other.

DEFINITION 5.2. A sphere A is *internally tangent* to a sphere B if (i) A is a proper part of B; (ii) given two spheres X and Y containing A as a part and forming part of B, at least one of them is part of the other.

DEFINITION 5.3. Spheres A and B are *externally diametrical* to the sphere C if (i) each of A and B is externally tangent to C; (ii) given two spheres X and Y disjoint from the sphere C and such that A is part of X and B is part of Y, the sphere X is disjoint from Y.

DEFINITION 5.4. Spheres A and B are *internally diametrical* to the sphere C if (i) each of A and B is internally tangent to C; (ii) given two spheres X and Y disjoint from C and such that A is externally tangent to X and B to Y, the sphere X is disjoint from Y.

DEFINITION 5.5. A sphere A is *concentric* with a sphere B if one of the following conditions is satisfied: (i) A and B are identical; (ii) A is a proper part of B and besides, given two spheres X and Y externally diametrical to A and internally tangent to B, these spheres are internally diametrical to B; (iii) B is a proper part of A and besides, given two spheres X and Y externally diametrical to B and internally tangent to A, these spheres are internally diametrical to A.

DEFINITION 5.6. A *point* is a class of all spheres which are concentric with a given sphere.

DEFINITION 5.7. Points a and b are equidistant from the point c if there exists a sphere X which belongs as an element to c and which moreover satisfies the following condition: no sphere Y belonging to a or b is part of X or is disjoint from X.

DEFINITION 5.8. A *solid* is an arbitrary sum of spheres.

DEFINITION 5.9. A point a is an interior point of a solid B if there exists a sphere A which is at the same time an element of a and a part of B.

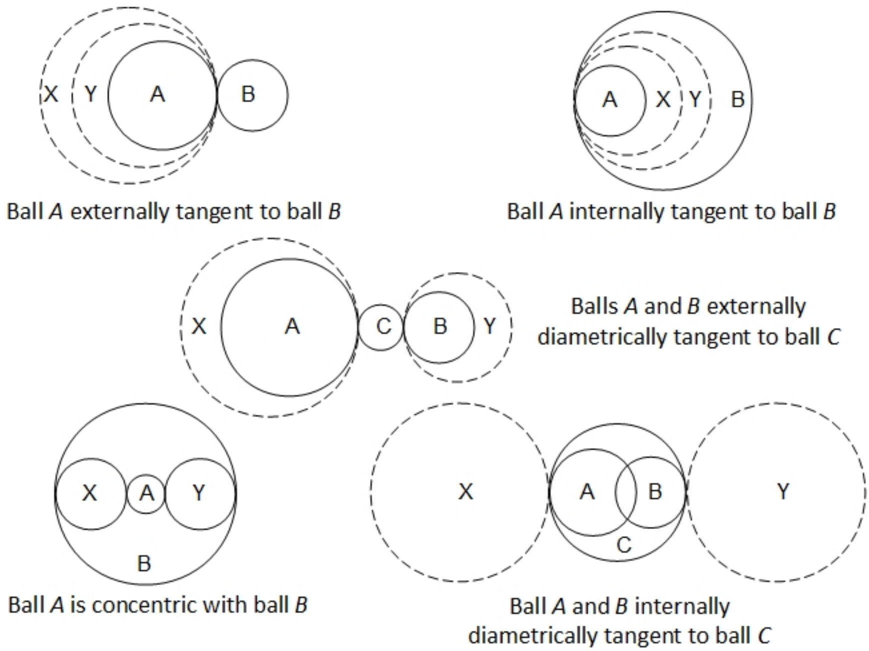

Figure 2. Tarski's primitive definitions.

These definitions which are extracted from (Tarski, 1956a) have been cited in many places and previously in (Gruszczynski and Pietruszczak, 2008, 2009; Dapoigny and Barlatier, 2015). The definitions have been illustrated schematically in their 2-D versions in figure 2. Notice that the word "disjoint" which occurs in these definitions is a synonym of exterior (*ext*) in λ-MM. In addition, the names *sphere* and *ball* will be considered as synonymous in the following.

5.2. Specification of Tarski's geometry of solids

A first version of Tarski's geometry of solids has been published in (Dapoigny and Barlatier, 2015) in which an embedding has been set forth. However, the current approach differs from it in many places. First, the axiomatization here relies on monadic second-order logic instead of expressing directly mereology in type theory. Second, the axiomatization relies on a boolean-valued model (without zero) which guarantees suitable properties of λ-MM. Third, we have formalized names as an inductive type instead of simple type with a strong use of characteristic functions. In addition, the present version offers a better way to ensure soundness since the current specification only requires a minimal set of axioms (two) for mereology. Finally, the monadic second-order version of mereology using an extended set of additional functions (i.e., relative complement, sums) is another aspect which renders the theory more expressive and departs from the first version. More precisely, the original contribution of the present section holds in the following specifications: (i) the geometrical universe is seen as a collective class of balls (i.e., a proper class) whose elements are individual balls or ball collections (sets), (ii) solids are built out of ball collections, from which we can infer that the geometrical universe and individual balls are also solids and (iii) points are collective classes of balls concentric with a given ball, which are constrained to be saturated sub-spheres of the geometrical universe.

In mereology, any (collective) class or collection is built from names (see definitions 4.2 and 4.4) and provided that the foundational primitives such as $\mathtt{el}$ or $\mathtt{pt}$ only accept singular names as arguments, then collections and classes follow the same rule. The universe of mereogeometrical names admits a single primitive name, the (ontological) plural name *balls*. Then, each singular name refers (e.g., $\eta(B, balls)$) to the distributive set *balls*. In the present model of mereogeometry, all quantifications occur on singular names, leaving room for mere first-order logic reasoning. The whole space of mereogeometry is formalized as a collective class of balls based on the single primitive ($balls$):

DEFINITION 5.10. $Gspace : N := \mathsf{Caract}\ (\mathtt{fun}\ A{:}object \Rightarrow (\eta\ (\iota\ A)\ (\mathtt{kl}\ balls)))$

Notice that this assumption is similar to that of (Gruszczynski and Pietruszczak, 2008, 2009) in which the universe of discourse of mereogeometry coincides with arbitrary "mereological sums[6] of balls". By theorem 4.3, unfolding definition 5.10, then with lemma 4.10 we get that any individual ball A is an element of the class *Gspace*:[7]

LEMMA 5.1. $\forall\, A.\ \eta\ A\ balls\ \rightarrow \eta\, A\ (\mathtt{el}\, Gspace)$

Unfolding definitions 3.16 and 5.10, by lemma 4.22, we have that the existence of the geometric space assumes that there exists at least a ball:

THEOREM 5.1. $exists\_at\_least\ \ balls \leftrightarrow\ \exists\, E,\ \eta\, E\ Gspace$

[6] Mereological sums are the set-based counterpart of collective classes, i.e., $\mathtt{kl}$.

[7] We will see in the following part, that *Gspace* also involves solids and points, themselves including balls.

Since Tarski assumes that balls, as a primitive, is an existing name it follows that individual balls are also existing names:

AXIOM 5.1. *exists_at_least balls*

Unfolding definitions 3.17 and 5.10 and applying lemma 4.4 we have that there exists at most one geometric space:

THEOREM 5.2. *exists_at_most Gspace*

Using theorems 5.1 and 5.2, it results that the geometric space is unique. Using these properties, the first postulates of Pieri can be fulfilled by proving (i) the existence of at most one geometric space and (ii) that there exists at least a ball in that space. All Tarski's definitions are introduced following the same scheme. The first definition addresses the external tangency of two balls. We define an externally tangent ball as a name (seen as a set), and specify additional constraints on its elements, i.e., they are totally ordered:

DEFINITION 5.11. $ET\ (B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow$
$(\mathtt{In}\ balls\ X \wedge \eta\ B\ balls \wedge \eta\ (\iota\ X)(ext\ B) \wedge (\forall\ x\ y, \eta\ x\ balls \wedge$
$\eta\ y\ balls \wedge \eta\ (\iota\ X)\ (\mathtt{el}\ x) \wedge \eta\ x\ (ext\ B) \wedge \eta\ (\iota\ X)(\mathtt{el}\ y) \wedge$
$\eta\ y\ (ext\ B) \rightarrow (\eta\ x\ (\mathtt{el}\ y) \vee \eta\ y\ (\mathtt{el}\ x))))).$

For instance, in a term having the form $\eta\, A(ET\ B)$, the name A collects all singletons $(\iota\, X)$ for which the formula under the scope of Caract is true. The specification of internally tangent balls is similar:

DEFINITION 5.12. $IT\ (B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ A{:}object \Rightarrow$
$(\mathtt{In}\ balls\ A \wedge \eta\ B\ balls \wedge \eta\ (\iota\ A)(pt\ B) \wedge (\forall\ x\ y, \eta\ x\ balls \wedge$
$\eta\ y\ balls \wedge \eta\ (\iota\ A)(\mathtt{el}\ x) \wedge \eta\ x\ (\mathtt{el}\ B) \wedge \eta\ (\iota\ A)(\mathtt{el}\ y) \wedge$
$\eta\ y\ (\mathtt{el}\ B) \rightarrow (\eta\ x\ (\mathtt{el}\ y) \vee \eta\ y\ (\mathtt{el}\ x))))).$

For the definitions of diametrically tangent spheres, we introduce ternary relations in the Leśniewski's style (see, e.g., definition 4.6). For example, "A and B are externally diametrical to C" is read as "A is externally diametrically tangent to B relative to C".

DEFINITION 5.13. $ED\ (C\ B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow$
$(\eta\ (\iota\ X)(ET\ C) \wedge \eta\ B\ (ET\ C) \wedge (\forall\ P\ Q, \eta\ P\ balls \wedge \eta$
$Q\ balls \wedge \eta\ (\iota\ X)(\mathtt{el}\ P) \wedge \eta\ P\ (ext\ C) \wedge \eta\ B\ (\mathtt{el}\ Q) \wedge$
$\eta\ Q\ (ext\ C) \rightarrow \eta\ P\ (ext\ Q)))).$

A number of lemmas can be inferred using definitions 5.13 and 5.11 and applying lemma 4.7, such as the fact that externally diametrically tangent balls are always external:

LEMMA 5.2. $\forall\, A\, B\, C.\ \eta\, A\ (ED\ C\ B)\ \rightarrow\ \eta\, A\ (ext\ B)$

The definition of *internally diametrical* follows the same rule, however we rather adopt the view of (Clay, 2021) since it appears as more useful from the computational perspective.

DEFINITION 5.14. $ID\ (C\ B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow$
$(\eta\ (\iota\ X)(\mathtt{el}\ C) \wedge \eta\ B\ (\mathtt{el}\ C) \wedge (\exists\ P\ Q\ R, \eta\ P\ balls\ \wedge$
$\eta\ Q\ balls \wedge \eta\ R\ balls \wedge \eta\ P\ (ED\ (\iota\ X)\ Q) \wedge \eta\ P\ (ED\ B\ R)\ \wedge$
$\eta\ Q\ (ED\ C\ R))))$.

Then we introduce the formalization of concentric spheres (A is concentric with B) according to (Clay, 2021) for the same motivation ($\leftrightarrow$ denotes singular equality).

DEFINITION 5.15. $Concent\ (B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow$
$(In\ balls\ X \wedge \eta\ B\ balls \wedge ((\iota\ X) \leftrightarrow B\ \vee$
$(\eta\ (\iota\ X)\ (pt\ B) \wedge \exists\ P\ Q\ R\ S, \eta\ P\ (ED\ (\iota\ X)\ Q)\ \wedge$
$\eta\ P\ (ID\ B\ Q) \wedge \eta\ R\ (ED\ (\iota\ X)\ S) \wedge \eta\ R\ (ID\ B\ S)\ \wedge$
$\neg(P \leftrightarrow R) \wedge \neg(P \leftrightarrow S)) \vee (\eta\ B\ (pt\ (\iota\ X)) \wedge \exists\ P\ Q\ R\ S,$
$\eta\ P\ (ED\ B\ Q) \wedge \eta\ P\ (ID\ (\iota\ X)\ Q) \wedge \eta\ R\ (ED\ B\ S)\ \wedge$
$\eta\ R\ (ID\ (\iota\ X)\ S) \wedge \neg(P \leftrightarrow R) \wedge \neg(P \leftrightarrow S)))))$.

From this definition, by unfolding definition 5.15, from lemma 3.7, reflexivity and symmetry follow:

LEMMA 5.3. $\forall\, A.\ \eta\, A\ balls\ \rightarrow \eta\, A\ (Concent\ A)$

Using definition 5.15 and the symmetry of set_eq, we get:

LEMMA 5.4. $\forall\, A\, B.\ \eta\, A\ (Concent\ B)\ \rightarrow \eta\, B\ (Concent\ A)$

Transitivity requires a supplementary axiom:

AXIOM 5.2. $\forall\, A\, B\, C.\ \eta\, A\ (Concent\ B)\ \wedge\ \eta\, B\ (Concent\ C)\ \rightarrow$
$\eta\, A\ (Concent\ C)$

The definition of a solid is coherent with previous work (Dapoigny and Barlatier, 2015). We assert that a solid is a collection of balls, the meaning of collections being previously explained in figure 1.

DEFINITION 5.16. *Solid* : N :=
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow (\eta\ (\iota\ X)\ (coll\ balls)))$.

Using definitions 5.16, 4.4, 3.9 and lemma 4.12, we can prove that (i) any ball is also a solid (while the converse is not true), (ii) the geometric space is a solid and (iii) any solid is an element of the geometric space.

LEMMA 5.5. $\forall A.\ \eta\, A\ balls\ \rightarrow \eta\, A\ Solid$

By unfolding definitions 5.16, 5.10, from lemma 4.29 we get:

LEMMA 5.6. $\forall A.\ \eta\, A\ Gspace\ \rightarrow \eta\, A\ Solid$

Unfolding definitions 5.16 and 5.10, by lemmas 4.40, 3.13 and 4.10, we obtain:

LEMMA 5.7. $\forall A.\ \eta\, A\ Solid\ \rightarrow \eta\, A\ (\mathtt{el}\ Gspace)$

The definition of a point deserves some attention. We depart from the common view which describes a point as a non-finite set of concentric spheres, because quantification over points would lead outside first-order logic. The idea is to consider a point as an individual name. We first recall the following definition inspired from (Jaśkowski, 1948). A ball A is a saturated sub-sphere of a solid B iff (i) it is a part of B, (ii) it is a ball, and (iii) given an arbitrary ball X, if A is part of X and X is part of B, then X is equal to A. More formally:

DEFINITION 5.17. *sat_subsphere* $(B{:}N)$: N :=
$$\begin{aligned}&\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow\\ &\quad (\ \eta\ (\iota\ X)\ balls \wedge \eta\ B\ Solid \wedge \eta\ (\iota\ X)\ (\mathtt{el}\ B)\ \wedge\\ &\quad (\forall\ A, \eta\ A\ balls \wedge \eta\ (\iota\ X)(\mathtt{el}\ A) \rightarrow \eta\ (\iota\ X)\ A)))\end{aligned}$$

Second, let us consider the following lemma which states that for any given ball Y, if there exists another ball X of which it is a proper part, which is concentric with Y and maximal in the geometric space, then X is the class of the set of all concentric balls with X. Indeed, unfolding definitions 5.17 and 4.1, applying lemmas 4.16 and 4.11, we get:

LEMMA 5.8. $\forall Y.\ \eta\, Y\ balls\ \rightarrow\ \exists X.\ \eta\, X\ (sat\_subsphere\ Gspace) \wedge$
$\eta\, Y\ (\mathtt{pt}\, X)\ \wedge\ \eta\, Y\ (Concent\ X)\ \rightarrow \eta\, X\ (kl(Concent\ X))$

According to these results, the definition of a point A involves (i) the concept of a maximal sub-sphere in the geometric space and (ii) the class of all spheres concentric with A:

DEFINITION 5.18. $Point : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow$
$(\eta\ (\iota\ X)\ (sat\_subsphere\ Gspace) \wedge$
$\eta\ (\iota\ X)\ (klass\ (Concent\ (\iota\ X))))).$

With such a definition, a point is seen as an individual, i.e., a sphere that encompasses all spheres concentric with it. It gives rise to the following lemmas. Firstly, by definitions 5.18, 5.17 and 4.2, we get:

LEMMA 5.9. $\forall A.\ \eta\, A\ Point \wedge \eta\, B\ Point \wedge \eta\, A\ (Concent\ B) \rightarrow \eta\, A\ B$

Secondly, applying lemmas 4.24 and 5.3, we have:

LEMMA 5.10. $\forall A.\ \eta\, A\ balls \rightarrow \eta\, A\ (\mathsf{el}(\mathsf{kl}(Concent\ A)))$

Thirdly, by applying 4.27, 5.10 and definition 5.18, we get:

LEMMA 5.11. $\forall X.\ \eta\, X\ balls \rightarrow \exists Y.\ \eta\, Y\ (sat\_subsphere\ Gspace) \wedge \eta\, Y\ (\mathsf{kl}(Concent\ X)) \rightarrow \eta\, Y\ Point$

Fourthly, from definitions 5.18 and 5.17, we have:

LEMMA 5.12. $\forall A.\ \eta\, A\ Point \rightarrow \eta\, A\ (\mathsf{el}\, Gspace)$

Fifth, using definitions 5.18 and 5.17, and lemma 5.3, we get:

LEMMA 5.13. $\forall A.\ \eta\, A\ Point \rightarrow \exists B.\ \eta\, B\ (sat\_subsphere\ Gspace) \wedge \eta\, A\ (Concent\ B)$

Finally, using definition 5.15 and lemma 5.13, we obtain:

LEMMA 5.14. $\forall A.\ \eta\, A\ Point \rightarrow \eta\, A\ balls$

Among others, lemma 5.9 demonstrates that two concentric points refer to the same point, while lemma 5.11 proves that every ball gives rise to a point, 5.12 shows that any point is an element of the geometric space and lemma 5.14 confirms that points are convertibles to spheres.

Now we come to the specification of equidistant points. The underlying idea is to divide the equidistant definition into sub-parts that can be more easily worked with. Some previous works have already investigate such an approach (e.g., the centered on the boundary relation in (Bennett, 2001)). Besides, a similar suggestion which has been detailed in (Jaśkowski, 1948), will be the basis of the following definition:

DEFINITION 5.19. $\textit{on\_surface}\ (A : N) : N :=$
$$\begin{aligned}&\mathsf{Caract}\ (\mathtt{fun}\ X{:}\textit{object} \Rightarrow\\ &\quad (\eta\ A\ \textit{balls} \wedge \eta\ (\iota\ X)\ \textit{Point} \wedge (\forall\ B, \eta\ B\ \textit{balls} \wedge\\ &\quad \eta\ B\ (\textit{Concent}\ (\iota\ X)) \to \neg\eta\ B\ (\mathtt{el}\ A) \wedge \neg\eta\ B\ (\textit{ext}\ A)))).\end{aligned}$$

It means that a point B is situated on the surface of a ball A if for any concentric element X of B, X is neither external nor internal to A (see Jaśkowski, 1948, p. 301). From definition 5.19 and lemmas 5.3, 4.8 and 3.7, we can infer that a point cannot be on its own surface:

LEMMA 5.15. $\forall\, A.\ \neg\,\eta\, A\ (\textit{on\_surface}\ A)$

Using previous specifications, we are now able to express that points B and C are equidistant from ball A:

DEFINITION 5.20. $\textit{Equid}\ (B\ C{:}N) : N :=$
$$\begin{aligned}&\mathsf{Caract}\ (\mathtt{fun}\ X{:}\textit{object} \Rightarrow (\eta\ B\ \textit{Point} \wedge \eta\ C\ \textit{Point} \wedge\\ &\quad \eta\ B\ (\textit{on\_surface}\ (\iota\ X)) \wedge \eta\ C\ (\textit{on\_surface}\ (\iota\ X)))).\end{aligned}$$

Using lemma 5.15 after unfolding definition 5.20, we will first prove that the case corresponding to an equidistance relation involving a single point must be ruled out:

LEMMA 5.16. $\forall\, A.\ \neg\,\eta\, A\ (\textit{Equid}\ A\ A)$

This lemma avoids the contradiction highlighted in (Gruszczynski and Pietruszczak, 2008) meaning that there is no need to modify the definition of *Equid*. The following lemmas respectively express reflexivity on the two first arguments (third axiom of Pieri), symmetry and transitivity of the equidistance relation. Firstly, by unfolding definitions 5.19 and 5.20:

LEMMA 5.17. $\forall\, A\, B.\ \eta\, B\ (\textit{on\_surface}\ A)\ \to\ \eta\, A\ (\textit{Equid}\ B\ B)$

Secondly, by simply unfolding definition 5.20:

LEMMA 5.18. $\forall\, A\, B\, C.\ \eta\, A\ (\textit{Equid}\ B\ C)\ \to\ \eta\, A\ (\textit{Equid}\ C\ B)$

Thirdly, from unfolding definition 5.20:

LEMMA 5.19. $\forall\, A\, B\, C\, D.\ \eta\, A\ \textit{Point}\ \to\ \exists\, A'.\ \eta\, A'\ \textit{balls} \wedge$
$$\eta\, A'\ (\textit{Equid}\ B\ C) \wedge \eta\, A'\ (\textit{Equid}\ C\ D)\ \to \eta\, A'\ (\textit{Equid}\ B\ D)$$

Finally, the last definition involves the formalization of Tarski's Definition 9 of interior points within a solid. It correspond to the simple well-formed definition:

DEFINITION 5.21. $InteriorPoint\ (B : N) : N :=$
$\mathsf{Caract}\ (\texttt{fun}\ X{:}object \Rightarrow (\eta\ B\ Solid \land \eta\ (\iota\ X)\ Point \land$
$\exists\ C, \eta\ C\ (Concent\ (\iota\ X)) \land \eta\ C\ (\mathsf{el}\ B))).$

In the paper (Tarski, 1956a), apart from axioms resulting from regular open sets, two alternative axioms are said to be equivalent. We will prove here that this assumption is true.

AXIOM 5.3. *If A is a solid and B a part of A, then B is also a solid.*

AXIOM 5.4. *If A is a sphere and B a part of A, there exists a sphere C which is a part of B.*

From the two following lemmas, we demonstrate that they are inferentially equivalent.

LEMMA 5.20. $(\forall\, A\, B.\ \eta\, A\ balls\ \land\ \eta\, B\ (\mathsf{el}\, A) \to\ \exists\, C.\ \eta\, C\ balls \land$
$\eta\, C\ (\mathsf{el}\, B)\ \to\ \forall\, A\, B.\ \eta\, A\ Solid \land \eta\, B\ (\mathsf{el}\, A)\ \to \eta\ B\ Solid$

PROOF. By first unfolding definitions 5.16, 3.9, 4.5 and applying lemma 4.36, then rewriting multiple times definition 4.4 and using lemmas 4.12 and theorem 4.1 (three times), together with lemmas about singletons, the result follows. ⊣

By lemma 5.5 and definitions 5.16 and 4.4, we get:

LEMMA 5.21. $(\forall\, A\, B.\ \eta\, A\ Solid\ \land\ \eta\, B\ (\mathsf{el}\, A)\ \to\ \eta\, B\ Solid)\ \to\ \forall\, A\, B.$
$\eta\, A\ balls\ \land\ \eta\, B\ (\mathsf{el}\, A) \to\ \exists\, C.\ \eta\, C\ balls\ \land \eta\, C\ (\mathsf{el}\, B)$

These two lemmas are converse implications which give rise to a single theorem establishing the equivalence. We can deduce two consequences: (i) the assertion of Tarski was correct and (ii) the present conceptualization of a solid as mereological collection of balls is coherent with these axioms and their equivalence.

5.3. Extension of geometry of solids

The previous model has been extended by several authors (see, e.g., Borgo et al., 1996; Dugat et al., 1999; Bennett, 2001, to cite a few). If one wants to specify a geometrical model, mereogeometry should be extended by introducing a *between* relation to concentricity and equidistance. We revisit here the idea of (Clay, 2021), which consists in two steps, (i) to introduce a relation between balls and (ii) to derive a relation between points. However, our approach using mereological expressions for solids and points relies only on singular names to preserve first-order quantification.

DEFINITION 5.22. $Btw\ (B\ C{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow (\mathtt{In}\ balls\ X \wedge \exists\ D\ E\ F, \eta\ D\ (Concent\ B) \wedge \eta\ E\ (Concent\ C) \wedge \eta\ F\ (Concent\ (\iota\ X)) \wedge \eta\ D\ (ED\ F\ E)))$

Using this definition, the two lemmas referred as T4 and T5 in (Clay, 2021) are easily proved. Firstly, using contraposition, unfolding definition 5.22, lemmas 5.2, 5.4, 5.2, definition 5.15, we get:

LEMMA 5.22. $\forall\ A\ B\ D.\ \eta\ D\ (Btw\ A\ B) \rightarrow \neg\eta\ A\ (Concent\ B)$

Secondly, by unfolding definition 5.22, rewriting definition 5.15, then using multiple times lemmas 5.4 and 5.2, we get:

LEMMA 5.23. $\forall\ A\ B\ D\ P\ Q\ R.\ \eta\ R\ (Concent\ D) \wedge \eta\ P\ (Concent\ A) \wedge \eta\ Q\ (Concent\ B) \rightarrow (\eta\ D\ (Btw\ A\ B) \leftrightarrow \eta\ D\ (Btw\ A\ B))$

We can now express the relation between involving points as follows:

DEFINITION 5.23. $btw\ (A\ B{:}N) : N :=$
$\mathsf{Caract}\ (\mathtt{fun}\ X{:}object \Rightarrow (\eta\ A\ Point \wedge \eta\ B\ Point \wedge \exists\ A'\ B', \eta\ A'\ balls \wedge \eta\ B'\ balls \wedge \eta\ A'\ (Concent\ A) \wedge \eta\ B'\ (Concent\ B) \wedge \eta\ (\iota\ X)\ (Btw\ A'\ B'))).$

From which, first unfolding definitions 5.22, 5.23 and 3.12, then using lemma 5.2 and finally, unfolding again def. 3.12 and applying lemma 5.3, we can derive that all balls concentric with a given ball lying between points A and B, are included into all balls that are between A and B.

LEMMA 5.24. $\forall\ A\ B\ D.\ \eta\ D\ (btw\ A\ B) \leftrightarrow (\eta\ D\ balls \wedge (Concent\ B) \subseteq (btw\ A\ B))$

The code for mereogeometry is available in code_mereogeometry and has required 37 definitions, 416 lemmas and 22 theorems.

6. Conclusion

We have proposed in the first part, an interpretation of Leśniewski's mereology in monadic second-order logic whose purpose is to provide an expressive theory. Leśniewski's axioms[8] become theorems under the

[8] Except for the initial axiom M.

λ-MM translation, and hence it follows that the translation of every theorem of Leśniewski's first-order theory of ontology is also a theorem of our system. In addition, the weak supplementation principle has also been proved as a theorem. Further, most axioms of set theory become provable using collections as sets and collective classes as proper classes. In the future, we also plan a more thorough investigation about a mereological version of Morse's class-theory in the same spirit as pointed out in early works (see, e.g., Pietruszczak, 1996; Welch and Horsten, 2016) but with quite distinct assumptions e.g., the use of collections.

The library has been expressed in the Coq theorem prover (see Bertot and Castéran, 2004))using *CoqHammer* (see Czajka and Kaliszyk, 2018), an automated reasoning hammer tool for Coq, but other provers are possible (e.g., Isabelle/HOL Wenzel et al., 2008). The choice of Coq has been motivated by an accessible reading of proofs by contrast with other theorem provers. CoqHammer[9] is an extension of Coq which delegates automated proving to different first-order theorem provers. It is a translation from the Calculus of Inductive Constructions, with certain extensions introduced by Coq, to untyped first-order logic. It involves some automated reasoners such as *Z3*, *CVC4*, *Eprover* and *Vampire*. Whenever automated reasoning fails with CoqHammer, then interactive theorem proving is manually achieved in Coq. The output is a library having the form of an executable file that can be incorporated in any Coq-based application, such as Tarski's geometry of solids.

In a second part, we have suggested a mereology-based specification of Tarski's geometry of solids relying on the previous library. Many definitions are original such as that of geometric space (*Gspace*), the concept of *Point* specified from saturated sub-spheres, and that of *Equidistance*. We have shown that they are able to verify important properties of points such as the equidistance between them. Finally, we prove the inferential equivalence between the two axioms set forth by Tarski. Adding the definition of betweenness we are able to revisit the specification of Clay. The present formalization is a basis of a long-term project to formalize a region-based theory of space based on Tarski's geometry of solids. In addition, it is intended to serve as a theoretical basis in formal ontologies by explaining how to exploit monadic second-order mereology in order to design specific part-of relations. Applications in formal ontologies will be also considered as another investigation domain.

[9] Its main limitations are higher-order features and type classes.

References

Bennett, B., 2001, "A categorical axiomatisation of region-based geometry", *Fundamenta Informaticae*, 46 (1–2): 145–158.

Bertot, Y., and P. Castéran, 2004, *Interactive Theorem Proving and Program Development. Coq'Art: The Calculus of Inductive Constructions*, Texts in Theoretical Computer Science, An EATCS series, Springer Verlag.

Betti, A. and I. Loeb, 2012, "On Tarski's foundations of the geometry of solids", *The Bulletin of Symbolic Logic*, 18 (2): 230–260. DOI: 10.2178/bsl/1333560806

Borgo, S., N. Guarino and C. Masolo, 1996, "A pointless theory of space based on strong connection and congruence", pages 220–229 in L. Carlucci Aiello et al. (eds.), *Procs. of the International Conference on Principles of Knowledge Representation and Reasoning (KR 1996)*, Morgan Kaufmann.

Borgo, S., and C. Masolo, 2010, "Full mereogeometries", *Review of Symbolic Logic*, 3 (4): 521–567. DOI: 10.1017/S1755020310000110

Casati, R., and A. C. Varzi, 1999, *Parts and Places: The Structures of Spatial Representation*, Cambridge (Ma): MIT Press.

Church, A., 1940, "A formulation of the simple theory of types", *Journal of Symbolic Logic*, 5: 56–68.

Clay, R. E., 1969, "Sole axioms for partially ordered sets", *Logique et Analyse*, 12 (48): 361–375.

Clay, R. E., 1974, "Relation of Leśniewski's mereology to Boolean algebra", *The Journal of Symbolic Logic*, 39 (4): 638–648. DOI: 10.2307/2272847

Clay, R. E., 2021, "Leśniewski, Tarski and the geometry of solids", *Logique et Analyse*, 254: 131–148. DOI: 10.2143/LEA.254.0.3289654

Cocchiarella, N. B., 2001, "A conceptualist interpretation of Leśniewski's ontology", *History and Philosophy of Logic*, 22 (1): 29–43. DOI: 10.1080/01445340110113381

Cotnoir, A. J. and A. Bacon, 2012, "Non-wellfounded mereology", *Review of Symbolic Logic*, 5 (2): 187–204. DOI: 10.1017/S1755020311000293

Cotnoir, A. J., 2021, "Is weak supplementation analytic?", *Synthese*, 198 (18–S): 4229–4245. DOI: 10.1007/s11229-018-02066-9

Czajka, Ł., and C. Kaliszyk, 2018, "Hammer for Coq: Automation for dependent type theory", *Journal of Automated Reasoning*, 61, 423–453. DOI: 10.1007/s10817-018-9458-4

Dapoigny, R., and P. Barlatier, 2015, "A Coq-based axiomatization of Tarski's mereogeometry", pages 108–129 in S. Fabrikant et al. (eds,), *Spatial Information Theory*, COSIT 2015, vol. 9368 of Lecture Notes in Computer Science, Springer. DOI: 10.1007/978-3-319-23374-1_6

Dugat, V., P. Gambarotto and Y. Larvor, 1999, "Qualitative theory of shape and orientation", pages 45–53 in R. V. Rodriguez (ed.), *Procs. of the Workshop on Hot Topics in Spatial and Temporal Reasoning*, International Joint Conference on Artificial Intelligence (IJCAI 1999).

Gruszczyński, R., and A. Pietruszczak, 2008, "Full development of Tarski's geometry of solids", *Bulletin of Symbolic Logic*, 14 (4): 481–540. DOI: 10.2178/bsl/1231081462

Gruszczyński, R., and A. Pietruszczak, 2009, "Space, points and mereology. On foundations of point-free Euclidean geometry", *Logic and Logical Philosophy*, 18 (2): 145–188. DOI: 10.12775/LLP.2009.009

Hamkins, J. D., and M. Kikuchi, 2016, "Set-theoretic mereology", *Logic and Logical Philosophy*, 25 (3): 285–308. DOI: 10.12775/LLP.2016.007

Henkin, L., 1963, "A theory of propositional types", *Fundamenta Mathematicae*, 52: 323–334.

Jaśkowski, S., 1948, "Une modification des définitions fondamentales de la géométrie des corps de M. A. Tarski", Annales de la Société Polonaise de Mathématique, 21, 298–301.

Kaminski, M. and G. Smolka, 2008, "A minimal propositional type theory", Technical Report, Saarland University.

Lejewski, C., 1963, "A note on a problem concerning the axiomatic foundations of mereology", *Notre Dame Journal of Formal Logic*, 4 (2): 135–139. DOI: 10.1305/ndjfl/1093957503

Lejewski, C., 1967, "A single axiom for the mereological notion of proper part", *Notre Dame Journal of Formal Logic*, 8 (4): 279–285. DOI: 10.1305/ndjfl/1094068839

Leonard, H. S., and N. Goodman, 1940, "The calculus of individuals and its uses", *The Journal of Symbolic Logic*, 5 (2): 45–55.

Leśniewski, S., 1916, "Podstawy ogólnej teoryi mnogosci.", Moskow: Prace Polskiego Kola Naukowego w Moskwie, Sekcya matematyczno-przyrodnicza, (English translation 1992, by D. I. Barnett: "Foundations of the general theory of sets. I", pages 129–173 in S. Leśniewski, *Collected Works*, S. J. Surma, J. Srzednicki, D. I. Barnett, and F. V. Rickey (eds.), volume 44 of Nijhoff International Philosophy Series, Dordrecht: Kluwer, 1992.

Lewis, D., 1991, *Parts of Classes*, Blackwell. DOI: 10.2307/2219902

Makowsky, J. A., 2004, "Algorithmic uses of the Feferman-Vaught theorem", *Annals of Pure and Applied Logic*, 126 (1–3): 159–213. DOI: 10.1016/j.apal.2003.11.002

Pietruszczak, A., 1996, "Mereological sets of distributive classes", *Logic and Logical Philosophy*, 4: 105–122. DOI: 10.12775/LLP.1996.005

Pietruszczak, A., 2005, "Pieces of mereology", *Logic and Logical Philosophy*, 14 (2): 211–234. DOI: 10.12775/LLP.2005.014

Pietruszczak, A., 2015, "Classical mereology is not elementarily axiomatizable", *Logic and Logical Philosophy*, 24 (4): 485–498. DOI: 10.12775/LLP.2015.017

Pietruszczak, A., 2020, *Foundations of the Theory of Parthood. A Study of Mereology*, Trends in Logic, vol. 54, Springer International Publishing. DOI: 10.1007/978-3-030-36533-2

Simons, P., 1987, *Parts: A Study in Ontology*, Second Edition, Clarendon Press, Oxford. DOI: 10.1093/acprof:oso/9780199241460.001.0001

Sinisi, V. F., 1983, "Leśniewski's foundations of mathematics", *Topoi*, 2:3–52. DOI: 10.1007/BF00139700

Smirnov, V. A., 1987, "Strict embedding of the elementary ontology into the monadic second-order calculus of predicates admitting the empty individual domain", *Studia Logica*, 46 (1): 1–15. DOI: 10.1007/BF00396902

Sobociński, B., 1984, "Studies in Leśniewski's mereology", pages 217–227 in V. F. Rickey, J. T. Srzednicki (eds.), *Leśniewski's Systems: Ontology and Mereology*, vol. 13 of Nijhoff International Philosophy Series, Martinus Nijhoff Publishers, Kluwer Academic Publishers Group. DOI: 10.1007/978-94-009-6089-3

Tarski, A., 1929, "Les fondements de la géométrie des corps", pages 29–33 in *Annales de la société Polonaise de Mathématiques*, Kraków.

Tarski, A., 1956a, "Fundations of the geometry of solids", pages 24–29 in *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, J. H. Woodger (ed.), Clarendon Press: Oxford.

Tarski, A., 1956b, "On the foundation of Boolean algebra", pages 320–341 in *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, J. H. Woodger (ed.), Clarendon Press: Oxford.

Tsai, H., 2013, "A comprehensive picture of the decidability of mereological theories", *Studia Logica*, 101(5): 987–1012. DOI: 10.1007/s11225-012-9405-z

Tsai, H., 2015, "Notes on models of first-order mereological theories", *Logic and Logical Philosophy*, 24 (4): 469–482. DOI: 10.12775/LLP.2015.009

Varzi, A. C., 1996, "Parts, wholes, and part-whole relations: The prospects of mereotopology", *Data and Knowledge Engineering*, 20 (3): 259–286. DOI: 10.1016/S0169-023X(96)00017-1

Welch, P. D., and L. F. M. Horsten, 2016, "Reflecting on absolute infinity", *Journal of Philosophy*, 113 (2): 89–111. DOI: 10.5840/jphil201611325

Wenzel, M., L. C. Paulson and T. Nipkow, 2008, "The Isabelle Framework", pages 33–38 in *Procs. of Theorem Proving in Higher Order Logics*, 21st International Conference, TPHOLs 2008, Montreal, Canada. DOI: 10.1007/978-3-540-71067-7_7

A. Appendix: The Boolean model of mereology

We introduce the le_o relation such that: $le\_o : object \to object \to bool$. For more clarity, we write $X \leqslant Y$ instead of $le\_o(X, Y)$. Tarski has expressed a set of three axioms for a Boolean algebra on a set of objects and it is shown that these three axioms are inferentially equivalent to a single axiom (see Clay, 1969). Clay has proved that axiom A.1 is inferentially equivalent to a pair $<W, ZD>$ in which $<W>$ stand for a formula which is independent whether the lowest element exists or not, and ZD is a formula expressing that zero is deleted. Axiom A.1 stands for a complete boolean algebra with zero deleted (V denotes the universe of objects):

AXIOM A.1. $\forall XY.\, X \leqslant Y \leftrightarrow (\texttt{In}\, V\, X \wedge \texttt{In}\, V\, Y \wedge (Y \leqslant Y \to$
$(\forall\, \beta\, \alpha.\, incl\, \alpha\, V \wedge incl\, \beta\, V \wedge \texttt{In}\, \alpha\, Y \wedge (\forall\, Z.\, \texttt{In}\, \beta\, Z \leftrightarrow$
$((\forall W.\, \texttt{In}\, \alpha\, W \to W \leqslant Z) \wedge$
$(\forall\, W.\, W \leqslant Z \to \exists\, ST.\, \texttt{In}\, \alpha\, S \wedge T \leqslant W \wedge T \leqslant S))) \to$
$\exists L.\, set\_eq\, \beta\, (\iota\, L) \wedge X \leqslant L)))$

From the above axiom we can derive a simpler version denoted A.1′ in which the terms $X \in V$ and $Y \in V$ are removed. In the case of a boolean algebra without zero, the resulting system would not have a minimal element leaving out the case of a boolean algebra having precisely two elements. Therefore, we assume that there exists at least two distinct objects in the boolean algebra:

AXIOM A.2. $\exists\, XY.\, \neg(X = Y)$

The central idea is to prove, from axioms A.1 and A.2 together with the supremum and lowerbound definitions, that $\leqslant$ is a partial order, i.e., it is reflexive, transitive and antisymmetric. We will express $<W, ZD>$ as a pair of theorems. It requires first to introduce the supremum definition. Reflexivity is easily proved by axiom A.1:[10]

LEMMA A.1. $\forall X.\ X \leqslant X$

We introduce the *supremum* of a set a, $sup(a)$, by the definition:

DEFINITION A.1. $sup(a : N) : N := \mathsf{Caract}\ (\texttt{fun}\ X{:}object \Rightarrow$
$((\forall\ Y, \texttt{In}\ a\ Y \rightarrow Y \leqslant X) \wedge$
$\forall\ Y{:}object,\ Y \leqslant X \rightarrow \exists\ S\ T,$
$\texttt{In}\ a\ S \wedge T \leqslant Y \wedge T \leqslant S))$

From which we can infer the following lemmas. Firstly, by definitions A.1 and 3.2, then applying propositional extensionality, we get:

LEMMA A.2. $\forall X a.\ \texttt{In}\ (sup\ a)\ X \leftrightarrow (\forall Y.\ \texttt{In}\ a\ Y \rightarrow Y \leqslant X) \wedge$
$(\forall Y.\ Y \leqslant X \rightarrow \exists ST.\ \texttt{In}\ a\ S \wedge T \leqslant Y \wedge T \leqslant S)$

Secondly, by lemmas A.1, 3.4, A.2 and axiom A.1, we obtain:

LEMMA A.3. $\forall X a.\ \texttt{In}\ a\ X \rightarrow \exists Z.\ set\_eq\ (sup\ a)(\iota\, Z)$

Thirdly, by A.2 and lemmas about singletons, we get:

LEMMA A.4. $\forall X Z.\ (\forall Y.\ Z \leqslant Y) \rightarrow \texttt{In}\ (sup\ (\iota\ Z))X$

Fourthly, by lemmas A.4, A.3 and lemmas on singletons, we get:

LEMMA A.5. $\forall X Y Z.\ (\forall Y.\ Z \leqslant Y) \rightarrow X = Y$

Fifth, by lemma A.5, contraposition and negation of the existential, we get:

LEMMA A.6. $\forall X Y Z.\ \neg(X = Y) \rightarrow (\exists W.\ \neg(Z \leqslant W))$

Now, by lemma A.6, we can prove *ZD* as follows:

LEMMA A.7. $(\exists X Y.\ \neg(X = Y)) \rightarrow (\forall Z.\ \exists W.\ \neg(Z \leqslant W))$

From lemma A.7, we get:

[10] In fact using its simplified version.

LEMMA A.8. $(\exists\ X\ Y{:}object,\ \neg(X = Y)) \to \forall\ Z{:}object,\ (\forall\ \beta\ \alpha,$ $(\mathtt{In}\ \alpha\ Z \wedge (\forall\ X,\ \mathtt{In}\ \beta\ X \leftrightarrow ((\forall\ Y,\ \mathtt{In}\ \alpha\ Y \to Y \leqslant X) \wedge$ $(\forall\ S\ T,\ S \leqslant X \wedge \neg(S \leqslant T) \to \exists\ P\ Q\ R,\ \mathtt{In}\ \alpha\ P \wedge$ $Q \leqslant S \wedge Q \leqslant P \wedge \neg(Q \leqslant R))))) \leftrightarrow (\mathtt{In}\ \alpha\ Z \wedge (\forall\ X,$ $In\ \beta\ X \leftrightarrow ((\forall\ Y,\ \mathtt{In}\ \alpha\ Y \to Y \leqslant X) \wedge (\forall\ Y,\ Y \leqslant X \to$ $\exists\ S\ T,\ \mathtt{In}\ \alpha\ S \wedge T \leqslant Y \wedge T \leqslant S))))).$

Moreover, applying four times lemma A.8, we obtain:

LEMMA A.9. $(\exists\ X\ Y{:}object,\ \neg(X = Y)) \to (\forall\ X\ Y{:}object,\ X \leqslant Y \leftrightarrow$ $(Y \leqslant Y \to (\forall\ \beta\ \alpha,\ incl\ \alpha\ V \wedge incl\ \beta\ V \wedge \mathtt{In}\ \alpha\ Y \wedge$ $(\forall\ Z,\ \mathtt{In}\ \beta\ Z \leftrightarrow ((\forall\ W,\ \mathtt{In}\ \alpha\ W \to W \leqslant Z) \wedge$ $(\forall\ W,\ W \leqslant Z \to \exists\ S\ T,\ \mathtt{In}\ \alpha\ S \wedge T \leqslant W \wedge T \leqslant S))) \to$ $\exists\ L,\ set\_eq\ \beta\ (\iota\ L) \wedge X \leqslant L))) \leftrightarrow$ $\forall\ X\ Y{:}object,\ X \leqslant Y \leftrightarrow (Y \leqslant Y \to (\forall\ \beta\ \alpha,\ incl\ \alpha\ V \wedge$ $incl\ \beta\ V \wedge \mathtt{In}\ \alpha\ Y \wedge (\forall\ Z,\ \mathtt{In}\ \beta\ Z \leftrightarrow ((\forall\ W,\ \mathtt{In}\ \alpha\ W \to$ $W \leqslant Z) \wedge (\forall\ W\ T,\ W \leqslant Z \wedge \neg(W \leqslant T) \to$ $\exists\ P\ Q\ R,\ \mathtt{In}\ \alpha\ P \wedge Q \leqslant W \wedge Q \leqslant P \wedge \neg(Q \leqslant R)))) \to$ $\exists\ L,\ set\_eq\ \beta\ (\iota\ L) \wedge X \leqslant L)).$

Using axiom A.2, lemmas A.9 and A.1 (the simplified version), we can now prove W:

THEOREM A.1. $\forall XY.\ X \leqslant Y \leftrightarrow (Y \leqslant Y \to (\forall\,\beta\,\alpha.incl\,\alpha\,V) \wedge$ $incl\,\beta\,V) \wedge \mathtt{In}\,\alpha\,Y \wedge (\forall Z.\ \mathtt{In}\,\beta\,Z \leftrightarrow$ $((\forall W.\ \mathtt{In}\,\alpha\,W \to W \leqslant Z) \wedge$ $(\forall WS.\ W \leqslant Z \wedge \neg(W \leqslant S) \to \exists\,PQR.\ \mathtt{In}\,\alpha\,P \wedge$ $Q \leqslant W \wedge Q \leqslant P \wedge \neg(Q \leqslant R)))) \to$ $\exists L.\ set\_eq\beta\,(\iota\ L) \wedge X \leqslant L))$

Using axiom A.2 and lemmas A.2 and A.7, we get:

LEMMA A.10. $\forall Xa.\ \mathtt{In}\,(sup\,a)\ X \leftrightarrow$ $((\forall Y.\ \mathtt{In}\,a\,Y \to Y \leqslant X) \wedge \forall WZ.\ W \leqslant X \wedge \neg(W \leqslant Z) \to$ $\exists\,PQR.\ \mathtt{In}\,a\,P \wedge Q \leqslant W \wedge Q \leqslant P \wedge \neg(Q \leqslant R))$

By lemma 3.7 and axiom A.1, we obtain:

LEMMA A.11. $\forall XY.\ (\forall Z.\ Y \leqslant Z \to X \leqslant Z) \to X \leqslant Y$

From theorem A.1, lemmas A.1 and A.10, we have:

LEMMA A.12. $\forall XYa.X \leqslant Y \wedge \mathtt{In}\ a\ X \to \exists L.set\_eq(sup\ a)(\iota\,L) \wedge X \leqslant L$

From lemmas A.1 and A.12, we get:

Lemma A.13. $\forall Aa.\ \texttt{In}\ a\ A \to \exists L. set\_eq(sup\ a)(\iota\ L)$

In order to complete the proof, we define the characteristic function of the *lowerBound* assuming that the lower bound of a given object Y is the set *lowerBound* Y of all X such that $X \leqslant Y$ is true:

Definition A.2. $lowerBound\ (Y{:}object) : N :=$
$$\mathsf{Caract}\ (\texttt{fun}\ X{:}object \Rightarrow (X \leqslant Y)).$$

Using definition A.1 and propositional extensionality, we get:

Lemma A.14. $\forall XY.\ \texttt{In}\ (lowerBound\ Y)\ X\ \leftrightarrow X \leqslant Y$

Using definition A.1, and lemmas A.10, A.14 and A.1, we get:

Lemma A.15. $\forall X.\texttt{In}\ (sup\ (lowerBound\ X))\ X$

Using first lemmas A.14 and A.1, then lemmas A.13, symmetry of *set_eq*, lemma A.15 and finally lemmas about singletons, we obtain:

Lemma A.16. $\forall X.\ set\_eq\ (\iota\, X)\ (sup(lowerBound\ X))$

Finally, using lemmas A.11, A.14, A.12, A.15, A.16, set extensionality for singletons (from lemmas 3.4 and 3.3), definitions 3.5 and 3.4, transitivity and asymmetry follow:

Theorem A.2. $\forall ABC.\ A \leqslant B\ \wedge\ B \leqslant C\ \to A \leqslant C$

From theorem A.2, definition 3.4, lemmas A.14, A.16, A.2 and using propositional extensionality, we get:

Theorem A.3. $\forall AB.\ A \leqslant B\ \wedge\ B \leqslant A\ \to\ A = B$

According to theorems A.1, A.2 and A.3, relation $\leqslant$ is a partial order relation. The introduction of the part-of relation referred to as el, relies on the relation $\leqslant$ as shown by definition 3.18. In this definition, the constructor provides the property which is defined in the right member. Subsets of names $el(a) \subseteq V$ are built from all values of X which satisfy the right member. From theorem 3.1, it can be rewritten as $\eta(\iota(X), el(a))$. Some additional lemmas are required to state that this minimal model satisfies the structure of a boolean algebra with zero deleted. We only recall important lemmas for the sake of simplicity. Firstly, unfolding definitions 3.9, 3.18, 3.5, 3.3, 3.2 and by lemmas about singletons, we get:

LEMMA A.17. $\forall\, AB.\ \eta\, B\,(\mathtt{el}\, A) \leftrightarrow (\mathit{Individual}\ B\ \wedge \mathit{Individual}\ A\ \wedge$
$\exists\, XY.\, \mathtt{In}\, B\, X \wedge \mathtt{In}\, A\, Y\ \wedge X \leqslant Y)$

Secondly, unfolding definition 3.18, 3.4, applying lemma A.17 and lemmas about singletons, we obtain:

LEMMA A.18. $\forall XY.\, X \leqslant Y \leftrightarrow\ \eta(\iota X)(\mathtt{el}(\iota Y))$

THEOREM A.4. $\forall\ X\ Y\ {:}\mathit{object},\ \eta\ (\iota\ X)\ (\mathtt{el}\ (\iota\ Y)) \leftrightarrow$
$(\mathtt{In}\ V\ X \wedge \mathtt{In}\ V\ Y \wedge (\eta\ (\iota\ Y)\ (\mathtt{el}\ (\iota\ Y)) \rightarrow$
$\forall\ \mathrm{b}\ \mathrm{a}\ {:}N,\ (\eta\ (\iota\ Y)\ \mathrm{a} \wedge (\forall\ Z : \mathit{object},\ \eta\ (\iota\ Z)\ \mathrm{b} \leftrightarrow$
$\mathtt{In}\ V\ Z \wedge (\forall\ W\ {:}\mathit{object},\ \eta\ (\iota\ W)\ \mathrm{a} \rightarrow \eta\ (\iota\ W)\ (\mathtt{el}\ (\iota\ Z)))\wedge$
$\forall\ W\ {:}\mathit{object},\ \eta\ (\iota\ W)\ (\mathtt{el}\ (\iota\ Z)) \rightarrow \exists\ S\ T{:}\mathit{object},$
$\eta\ (\iota\ S)\ \mathrm{a} \wedge \eta\ (\iota\ T)(\mathtt{el}\ (\iota\ W)) \wedge \eta\ (\iota\ T)(\mathtt{el}\ (\iota\ S)))$
$\rightarrow \exists\ L\ {:}\mathit{object},\ \mathit{set\_eq}\ \mathrm{b}\ (\iota\ L) \wedge \eta\ (\iota\ X)(\mathtt{el}\ (\iota\ L))))))$

PROOF. The first implication requires axiom A.1, lemmas A.18 (twice), 3.4, 3.10 (twice) and lemmas about singletons. The second implication is proved by applying several times lemmas A.18 and 3.10, then using definition 3.9 and lemmas about singletons. ⊣

Unfolding definitions 3.4, 3.3, 3.9 and 3.5, then by lemmas A.18, 3.2 and lemmas about singletons, we get:

LEMMA A.19. $\forall ab.(\exists AB.\ \eta\, A\, a \wedge \eta\, B\,(\mathtt{el}\, b) \wedge \eta\, B\,(\mathtt{el}\, A))$
$\rightarrow \exists XY.\ \eta(\iota\ X)\ a\ \wedge\ \eta(\iota\ Y)\ (\mathtt{el}\, b)\ \wedge \eta(\iota\ Y)\ (\mathtt{el}(\iota\ X))$

Using lemma A.19 together with lemmas about singletons, we obtain:

LEMMA A.20. $\forall ab.\ (\exists\, cd.\ \eta\ \ c\, b \wedge \eta\ \ d\,(\mathtt{el}\, a) \wedge \eta\ \ d\,(\mathtt{el}\, c)) \leftrightarrow$
$\exists EF.\ \eta(\iota\ E)\, b\ \wedge\ \eta(\iota\ F)\ (\mathtt{el}\, a)\ \wedge \eta(\iota\ F)\ (\mathtt{el}(\iota\ E))$

We finally obtain the expected result:

THEOREM A.5. $\forall AB,\ \eta\ A\ (\mathtt{el}\ B) \leftrightarrow (\mathit{eta}\ A\ A \wedge \eta\ B\ B\ \wedge$
$(\eta\ B\ (\mathtt{el}\ B) \rightarrow (\forall\ C\ a,\ \eta\ B\ a \wedge (\forall\ D,\ \eta\ D\ C \leftrightarrow$
$(\forall\ E\ {:}N,\ \eta\ E\ a \rightarrow \eta\ E\ (\mathtt{el}\ D)) \wedge (\forall\ E,\ \eta\ E\ (\mathtt{el}\ D) \rightarrow$
$\exists\ F\ G,\ \eta\ F\ a \wedge \eta\ G\ (\mathtt{el}\ E) \wedge \eta\ G\ (\mathtt{el}\ F))) \rightarrow \eta\ A(\mathtt{el} C))))$

PROOF. The first implication is proved unfolding definitions 3.9, 3.5, then using lemmas 3.6, 3.7, A.17, A.18, A.20, A.1, theorem A.4 and lemmas about singletons. The second implication, needs definitions 3.4, 3.5, 3.18, then lemmas 3.6, 3.7, 3.4, A.17, A.18, theorem A.4, symmetry of *set_eq* and lemmas about singletons. ⊣

Quantification occurs here over names, i.e., second-order variables. In such a way, theorem A.5 states that Mereology based on the $\mathtt{el}$ function has the structure of a complete boolean algebra without zero.

THEOREM A.6. *The structures of mereology and those of complete boolean algebra with zero deleted are identical.*

PROOF. The first implication is proved according to Clay's work. The second implication has been proved in λ-MM as detailed above. It follows that the equivalence holds. ⊣

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