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Topology of Modal Propositions Depicted by Peirce's Gamma Graphs: Line, Square, Cube, and Four-Dimensional Polyhedron

Abstract. This paper presents the topological arrangements in four geometrical figures of modal propositions and their derivative relations by means of Peirce's gamma graphs and their rules of transformation. The idea of arraying the gamma graphs in a geometric and symmetrical order comes from Peirce himself who in a manuscript drew two cubes in which he presented the derivative relations of some (but no all) gamma graphs. Therefore, Peirce's insights of a topological order of gamma graphs are extended here backwards from the cube to the line and the square; and then forwards from the cube to the four-dimensional polyhedron.

Keywords: Charles S. Peirce; existential graphs; gamma graphs; topology; modal logic

1. Introduction

> I do not think I ever reflect in words: I employ visual diagrams, firstly, because this way of thinking is my natural language of self-communion, and secondly, because I am convinced that it is the best system for the purpose. Peirce MS 619, 1909, *Studies in Meaning*

In a manuscript sheet written as a preparation for his *Lowell Lectures* (1903), Peirce arrayed in two cubes the modal propositions and their derivative relations, i.e. gamma graphs that contains a broken cut (see Figure 1). The manuscript sheet "was rediscovered in 1969 but which has never been studied before" [Ma and Pietarinen, 2018, p. 3630]. Ma and Pietarinen [2018] studied those two cubes and explained the implicative relations between the modal propositions. However, they explained

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those relations mainly using contemporary symbolism of modal logic, which loses some advantages of Peirce's topological approach. Besides, they limited their approach to Peirce's cubes, and they did not consider the limitations of Peirce's proposal. From my point of view, Peirce's cubes (tridimensional) can be understood more easily with a previous unidimensional and bidimensional approach (line and square), and can be extended to a four-dimensional figure. In fact, I believe that Peirce's cube II [so-called by Ma and Pietarinen, 2018] is incomplete and lacking some gamma graphs and transformations.

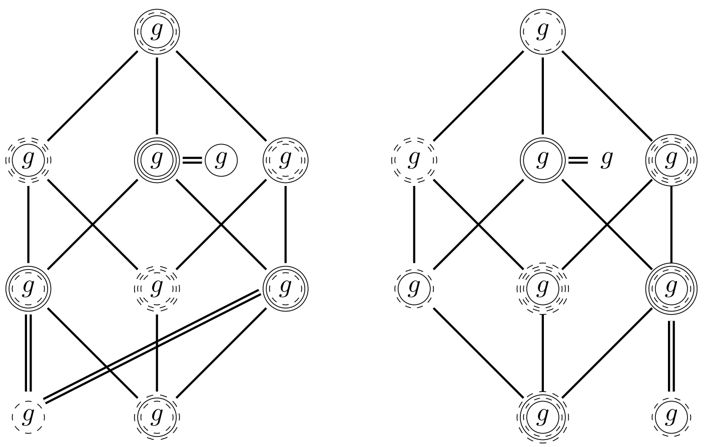


Figure 1. Cubes I and II

The new geometric figure that complements Peirce's cube II is, from my point of view, one of the hidden fruits or treasures of the gamma system that Peirce expected to find in this third part of his existential graphs [see Zeeman, 1964, p. 31; and Zalamea, 2011, p. 91]. It is not as great a secret as it would be to find the proof of pragmatism but at least it is an attempt for clearing up the whole spectrum of modal propositions that can be scribed and transformed with the broken cut. This paper, then, is an inquiry beyond the borders of what Peirce achieved in the manuscript S-1 in which he attempted a topological arrangement of gamma graphs in two cubes. The gamma system is particularly incomplete and Peirce himself put his trust in future scholars to develop its whole potential.

> The above are the conventions establishing the System of Existential Graphs in its present state. I should be greatly disappointed if this were

to be its final state. For it is at present far from the ideal perfection to which I hope some student may bring it.

Ms. 295, c. 1906, p. 53; see also [Zalamea, 2011, pp. 93, 105]

2. Topology of gamma graphs

In order to develop this new topological approach, it is necessary to understand only three basic elements of Peirce's existential graphs: the sheet of assertion, the continuous cut, and the broken cut. The sheet of assertion is the bidimensional space in which any graph that is scribed must be considered as true. The continuous cut is a continuous line that meets itself to create an enclosed space scribed in the sheet of assertion. It creates a different region or area than that of the sheet of assertion. Thus, the area inside the continuous cut must be considered as false. The broken cut is a discontinuous or intermittent line in which the area of assertion and denial are not so determined and precise because the area of the actual and the area of the possible permeate each other. If the area inside a continuous cut is interpreted as not true, the area inside a broken cut is interpreted as *possibly not* true. The broken cut, therefore, allows Peirce to explore the semantic of modal logic.[1]

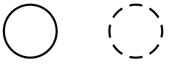
Furthermore, it is necessary to know three basic rules of transformation of those graphs: double continuous cut (DN; CP 4.567), opening of a continuous cut (Ta) and closing a broken cut (Tb). The rule of the double continuous cut is the rule of the double negation in classical logic. A double continuous cut may be drawn around any graph or inside any graph as long as it does not cross or intersect other cuts or graphs, and as long as there is nothing scribed between the two cuts. The other two rules are known as the rules of the broken cut, and they indicate when a continuous cut can become broken (Ta: Opening a continuous cut in an even area), and when a broken one can become continuous (Tb: Closing a broken cut in an odd area).[2]

The number of possible modal propositions depends on the combinations of those two types of cuts in a sheet of assertion [see Ma and

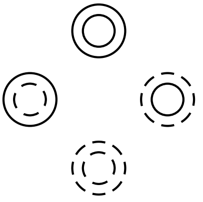
[1] For a complete presentation of the system of existential graphs see [Roberts, 1963; Shin, 2002; Zeeman, 1964]. For a discussion of the contribution of the gamma system to logic see [Oostra, 2012; Pietarinen, 2006].

[2] See R 467 Peirce 1903b; R 478 Peirce 1903e; see also [Pietarinen, 2006, p. 349], [Oostra, 2012, p. 29] [Ma and Pietarinen, 2018, p. 3628–9] and [Zeeman, 1964, p. 16].

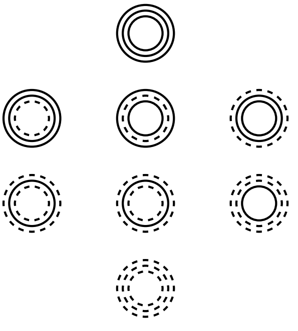
Pietarinen, 2017, p. 92; Zeeman, 1964, p. 6]. Hence, the number of graphs that can be scribed with only one cut is two ($2^1 = 2$): one graph with a continuous cut, one with a broken cut.



The number of graphs that contain two cuts is four ($2^2 = 4$): one graph with two continuous cuts; two graphs that contain at least one broken cut; one graph with two broken cuts.



The number of gamma graphs that can be drawn using three cuts is eight ($2^3 = 8$): one graph with three continuous cuts; three graphs whit one broken cut and two continuous cuts; three graphs with two broken cuts and one continuous cut; and one graph with three broken cuts.



Finally, the number of gamma graphs containing four cuts is sixteen ($2^4 = 16$): one graph with four continuous cuts; four graphs with one broken cut and three continuous cuts; six graphs with two broken cuts and two continuous cuts; four graphs with three broken cuts and one continuous cut; and one graph with four broken cuts (see Figure 2).

The number of possible cuts around a graph is unlimited. The plasticity of the existential graphs and the unlimitedness of the sheet of assertion allows as many cuts to be drawn as needed. Nonetheless, I am not going to explore the possibilities beyond four cuts. A key reason

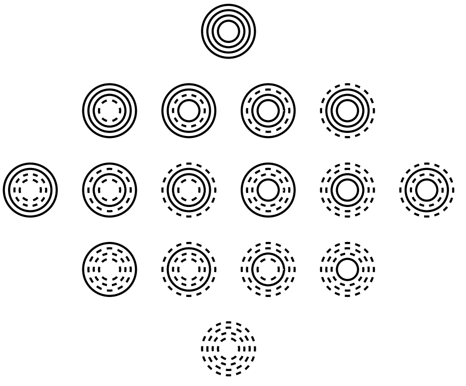
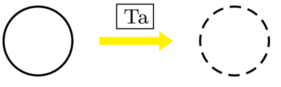


Figure 2.

is that graphs with five cuts and beyond creates some graphs that are equivalent to graphs already drawn with four or fewer cuts. In fact, some graphs with four or fewer cuts are semantically equivalents.

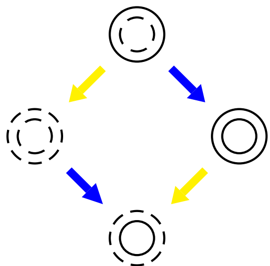
As the reader may have noticed, the number of possible cuts around a graph is the threshold of the topology for modal propositions. Since there are only two possible graphs with one cut, then they can be arrayed in a line. Graphs with two cuts have four different forms, and so they can be arrayed in a square depicted in a bidimensional surface. Since three different cuts can create up to eight different gamma graphs, they can be arrayed in a cube (tridimensional). That is exactly what Peirce did in his so-called Cube I (see Figure 1). Finally, four different cuts can create up to sixteen different gamma graphs. Peirce included those graphs with four cuts in Cube II (see Figure 1), but he neglected some of those graphs. Cube II then must be corrected adding those neglected graphs and, therefore, they must be arrayed in a four-dimensional polyhedron with sixteen vertices.

The line. Given a continuous cut around a graph, the rule of opening (Ta) permits its transformation into a broken cut. That means that if a graph is false, it implies the possibility of its falsehood.



Semantically, the line indicates that a false proposition implies that that same proposition *possibly not* true.

The square. As we said, there are four and only four possible gamma graphs with two cuts. The rules of transformation permit the transformation of those four graphs into each other. The following square must be read from top to bottom following the arrows that indicate the transformations.



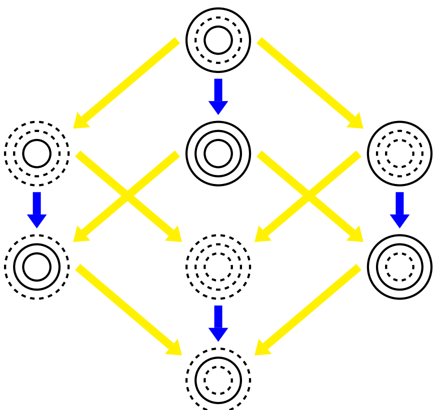
The graph from the top, "it is necessary that", permits two transformations or implications. The graphs from the middle permit only one transformation. The graph from the bottom cannot be transformed in any manner. The transformations to the left are all applications of the rule of opening the continuous cut in even area (in yellow). The transformations that occur to the right are applications of the rule of closing a broken cut in odd area (in blue).

One benefit of this topological approach is that it allows us to see all possible transformations. When Ma and Pietarinen [2018, p. 3629], for instance, explained the rules of transformation of the broken cut, they instantiated them with three inferences, when in reality there are four possible transformations. They gave

$$(t)\, g \Rightarrow g \qquad (t_1)\, g \Rightarrow g \qquad (t_2)\, g \Rightarrow g$$

They missed the transition from the double broken cut to the possibility graph, namely, from the left graph to the bottom graph of the square. To be fair, what Ma and Pietarinen wanted to show is that the two rules of transformation found by Peirce allow the gamma graph be equivalent to the standard modal logic in which the axiom T holds: the necessity implies truth, truth implies possibility. Nonetheless, the rules of transformation lead to other implications or inferences: from double broken cut (possibly necessary) to possibility. Although "there is not much utility in a *double broken cut*" (R 467 Peirce 1903b), as Peirce recognized, a comprehensive account of all possible modal propositions must include it. The square contains only four transformations and four modal propositions, but in the next figures, the implications are more abundant.

The Cube. Cube I is an invention by Peirce. From my point of view, he arrayed the gamma graphs that contain three cuts in a cube because it has eight vertices. The array is not arbitrary or due just to the number of possible graphs, but it also reflects the rules of transformation. Therefore, the place of each graph permits to appreciate the implicative relations among them. It is a perfect and symmetrical way of visualizing all possible implications of the modal propositions depicted in gamma graphs. The cube must be also read from top to bottom. The first graph in the top can undergo three types of transformations. The other graphs can be transformed in two ways. The last graph at the bottom cannot be transformed in any manner. Every vertical relation identified with a vertical arrow is an application of the rule of closing a broken cut in odd areas (in blue); every other diagonal relation, either to the right or to the left, is an application of the rule of opening a continuous cut in even areas (in yellow).

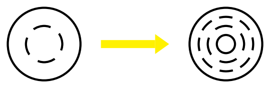


In consequence the cube depicts all possible transformations or implications of gamma graphs with three cuts. Each arrow represents a particular transformation; therefore, there are exactly twelve implications in the cube. Ma and Pietarinen [2018, p. 3630] analyzed only the eight individual transformations effected by Peirce in S-1 before he drew the cubes. The fact that Peirce only showed eight transformations before depicting the cubes, does not mean that there are not more transformations. As I just mentioned, Cube I (S-1) represents twelve transformations, and Cube II (S-1) another twelve.

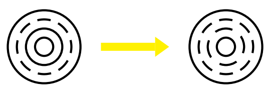
According to Ma and Pietarinen [2018, p. 3630] "It is easy to see that Cube I can be obtained from Cube II by replacing g by g". It is true that one of the graphs of Cube I can be transformed by means of the rule of double cut (double negation) into g, and also that one of the graphs in Cube II can be transformed by the rule of double negation into g. However, I do not think that the actual purpose of those cubes is to reach primary graphs such as g or g. I believe the cubes were designed to arrange topologically all possible modal graphs and the implications among them. I am proposing an alternative reading of those cubes different to that proposed by Ma and Pietarinen [2018]. Those cubes contain other applications of the rule of double cut as well that also leads to the derivation of a primary graphs such as g. Likewise, Cube II also contains two graphs that are equivalent to g, due also to the same reason. My interpretation for those cubes by Peirce is that they were intended to arrange topologically the derivative relations of modal propositions. If there is a primordial graph in those cubes, they are not g and g, but the top graph at each cube; the top graph in cube I is the graph of impossibility and the derived graphs below it shows all possible derivations from 'impossibility' reaching falsehood, for instance, g. The top graph in Cube II is the graph of 'necessity' and the derived graphs below it show all possible derivations from 'necessity', for instance it can be derived 'actuality' or 'truth', g.

Nonetheless, Ma and Pietarinen's interpretation is an interesting one because it makes us see that each dimension is grounded in the previous dimension. Thus, a point is required for the line; the line for the square, the square for the cube, and the cube for the four-dimensional polyhedron. In other words, the unidimensional is the base for the bidimensional, the bidimensional is the base for the tridimensional; and the latter for the four-dimensional. Thus, a graph such as g, which has no cut at all is equivalent to a graph with a double continuous cut g. From this graph with two cuts, the square is built. Likewise, from the graph g , if we employ the rule of double negation, we can obtain an equivalent graph with three continuous cuts. The graph with one cut leads to the line; the graph with three cuts leads to the cube. How the square and the cube led to a type of four-dimensional polyhedron, I will show in the next section.

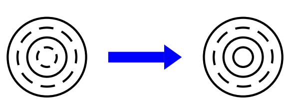
The four-dimensional polyhedron. This new figure is constructed around Cube II created by Peirce. Cube II contains graphs with two cuts and graphs with four cuts (see Figure 1). Those graphs with two cuts are logically equivalent to graphs with four cuts in which a double continuous cut has been erased. Actually, the four graphs with two cuts that I just organized in a square constitute one of the faces of Cube II. Although Peirce's procedure of scribing the simplest graph is correct (erasing unnecessary double continuous cuts), the reading of Cube II is hard to follow. For instance, the graph at the top with a continuous cut around a broken cut (necessity) is transformed to the right with the rule of opening into a continuous cut that contains two broken cuts and a continuous cut, as follows:



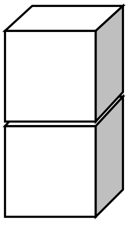
At first sight, it is not clear how the rule of opening is applied here. However, if it is observed that the original graph at left is a graph with four cuts, the transformation is evident: the third continuous cut that is an even area is opened (Rule of opening):



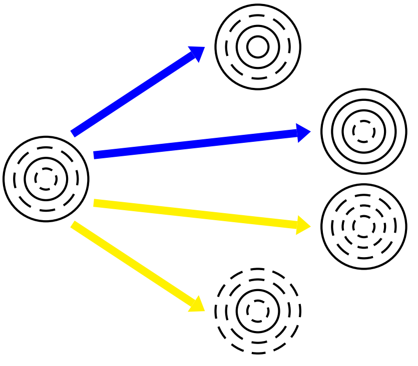
As a result, for the sake of clarity I am going to employ the graphs with four cuts, although they include unnecessary double continuous cuts. Therefore, the top graph in Cube II must be  and not . Another difficulty of Cube II is that it contains only eight gamma graphs with four cuts. It is lacking the remaining eight graphs with four cuts. The first thing to do in order to locate the remaining eight graphs is to look for possible transformations, i.e. implications. The graph at the bottom of Cube II, , does not allow any further transformations, namely it does not permit the closure of any of its broken cuts because it does not contain broken cuts in odd areas; and it does not permit the opening of a continuous cut, because its continuous cuts are not in even areas. Therefore, Cube II cannot be extended downwards. On the contrary, the four graphs at the top square of Cube II may be the result of a transformation of a square above it. For instance, the graph that we just mentioned at the beginning of this paragraph, as the top graph in Cube II, is a graph that may be the result of a transformation of a previous graph by means of the rule of closing a broken cut in an even area, a graph that is not included in Cube II:



Similarly, all four graphs from the top face of Cube II may be implications of previous graphs that have not been included in Cube II. Hence, Cube II has to be extended upwards with another square above it. So, at first, Cube II becomes into a rectangular prism with twelve vertices.

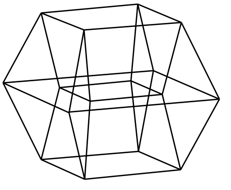


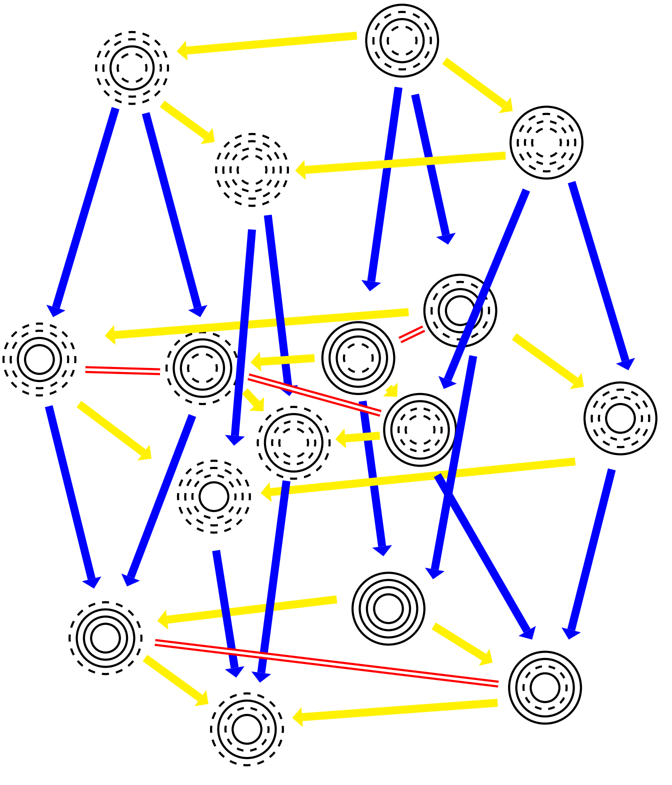
This rectangular prism, however, contains only twelves vertices, and therefore, only twelve gamma graphs with four cuts. There are still four more gamma graphs with four cuts that are not represented in this polyhedron. At the top of the rectangular prism is located the graph: , which means double necessity: "necessarily necessary". As the reader can see, its cuts cannot be the result of any closing or opening of a previous graph, and so it is impossible to keep extending the rectangular prism upwards. However, the graph  has two continuous cuts in even areas, and two broken cuts in odd areas, and it can therefore undergo four transformations, not only three.



In a cube or at the top of a rectangular prism, every vertex only joins three edges; therefore, those polyhedra are not good figures for arranging topologically the sixteen graphs with four cuts. Moreover, the graphs at the top face of the rectangular prism can undergo also more transformations than those depicted in a cube or in the rectangular prism. Therefore, it is necessary that the figure depicting the sixteen gamma graphs with four cuts includes four lines from top to bottom. The best figure that depicts those graphs and their implicative relations is the four-dimensional polyhedron whose image can be seen below. This new figure must be read also from top to bottom. Every blue arrow indicates an application of the rule of closing a broken cut in an odd area; blue arrows always move vertically. Every yellow arrow indicates an application of the rule of opening a continuous cut in an even area; yellow arrows move horizontally. Double red lines indicate equivalence of the graphs due to application of double negations.

Source [Hollasch, 1991]





3. Conclusions and challenges

The four geometrical figures here presented (line, square, cube, and four-dimensional polyhedron) are perfect and symmetrical ways of depicting the derivative relations of modal propositions. The number of compound cuts indicates the number of dimensions of the figure. Thus, since there are only two basic cuts (continuous and broken cut) and for every positive integer n, there are 2^n compound cuts, it can be established that 2^1 cuts produce a unidimensional figure of 2 vertices, 2^2 cuts produce a bidimensional figure of 4 vertices, 2^3 a three-dimensional figure of 8 vertices, and 2^4 a four-dimensional figure of 16 vertices (see Table below).

Each figure is ruled by a predominant modal proposition (gamma graph) that permits more transformations than the others in that figure. The line is dominated by a continuous cut (negation) that can be transformed once. The square is dominated by the graph of necessity that permits two transformations. The cube is dominated by the graph of impossibility that can undergo three transformations. And the four-dimensional polyhedron is dominated by the graph of double necessity (necessarily necessary) that permits four transformations.

These conclusions allow me to foresee some further developments in the topology of gamma graphs. If we want to extend our inquiry to graphs of five, six or more cuts, we can expect that the dominating (graph at the top) must be a graph whose cuts determine the number of possible transformations and the number of dimensions of the figure (see Table 1 below). The development of further geometrical figures is a challenge for a future inquiry or other researchers who want to explore those paths. These final remarks indicate some routes they can take in order to look for those figures.

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Ma, M., and A. V. Pietarinen, 2018, "Gamma graph calculi for modal logics", *Synthese* 195: 3621–3650. DOI: 10.1007/s11229-017-1390-3

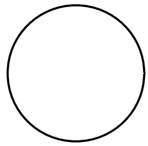
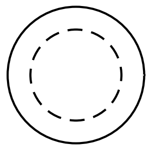
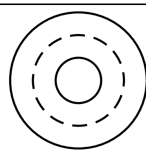
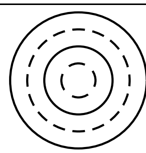
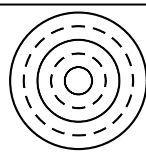
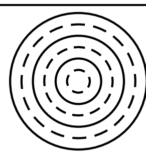
Number of cuts and transformations	*Top graph*	*Topological figure*	*Number of graphs and vertices*
One continuous cut that can be opened		Line	$2^1 = 2$
Graph of necessity whose internal broken cut can be closed and external continuous cut can be broken		Square	$2^2 = 4$
Graph of impossibility whose three cuts can be transformed		Cube	$2^3 = 8$
Graph of double necessity whose four cuts can be transformed		Four-dimensional polyhedron	$2^4 = 16$
Five-cut graph that permits five transformations:		Five-dimensional figure	$2^5 = 32$
Six-cut graph that permits six transformations:		Six-dimensional figure	$2^6 = 64$
$\vdots$	$\vdots$	$\vdots$	2^n

Table 1.

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