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Comparing Calculi for First-Order Infinite-Valued Łukasiewicz Logic and First-Order Rational Pavelka Logic

Abstract. We consider first-order infinite-valued Łukasiewicz logic and its expansion, first-order rational Pavelka logic RPL∀. From the viewpoint of provability, we compare several Gentzen-type hypersequent calculi for these logics with each other and with Hájek's Hilbert-type calculi for the same logics. To facilitate comparing previously known calculi for the logics, we define two new analytic calculi for RPL∀ and include them in our comparison. The key part of the comparison is a density elimination proof that introduces no cuts for one of the hypersequent calculi considered.

Keywords: many-valued logic; mathematical fuzzy logic; first-order infinite-valued Łukasiewicz logic; first-order rational Pavelka logic; proof theory; Hilbert-type calculus; Gentzen-type hypersequent calculus; density elimination; conservative extension.

1. Introduction

Mathematical fuzzy logics provide formal foundations for approximate reasoning. Among the most important such logics are first-order infinite-valued Łukasiewicz logic Ł∀ and its expansion by rational truth constants, first-order rational Pavelka logic RPL∀; see [18] as well as [13, 14]. As for most fuzzy logics, the intended, or standard, semantics for Ł∀ and RPL∀ has the interval $[0, 1]$ of real numbers as the set of truth values; valid Ł∀- and RPL∀-formulas are those taking only the truth value 1.

The set of all valid Ł∀-formulas (over a sufficiently rich signature) is not recursively enumerable [26], more precisely, is Π_2-complete [25]; the

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same holds for RPL∀ [18, Section 6.3]. Therefore, for these two logics, finitary calculi (i.e., calculi with a recursive set of axioms and a finite number of recursive inference rules) have to be incomplete, but of course, must be sound. We consider only sound finitary calculi for Ł∀ and RPL∀ in the present article; as to infinitary calculi for the logics, one can find a brief overview and a recent result in [17].

There are equivalent Hilbert-type calculi for Ł∀ (resp. RPL∀), Hájek's calculus for Ł∀ (resp. RPL∀) from [18] being the standard one. Hájek's calculus for Ł∀ (resp. RPL∀) is complete with respect to a certain algebraic semantics over so-called MV-chains (resp. MV-chains contaning the rational unit interval); see [18]. It is proved in [19] that Hájek's calculus for RPL∀ is a conservative extension of the one for Ł∀.

Besides Hilbert-type calculi, the Gentzen-type calculi mentioned below are known for these logics.

For Ł∀, an analytic hypersequent calculus GŁ∀ with structural inference rules is presented in [2, 23], and it is shown in [2] that GŁ∀ extended with the cut rule proves exactly the same Ł∀-sentences as Hájek's calculus for Ł∀.

With the aim of developing proof search methods for Ł∀ and RPL∀, in [16, 17] we introduced the following calculi.

The structural rules of GŁ∀ create too high a degree of nondeterminism for bottom-up proof search. So in [16] we excluded them from GŁ∀ to obtain an analytic cumulative[1] hypersequent calculus[2] G^1RP∀ for RPL∀, and showed that all GŁ∀-provable sentences are G^1RP∀-provable. Also, in [16] we introduced a noncumulative variant G^2RP∀ of G^1RP∀; G^2RP∀ is suitable for bottom-up proof search for prenex RPL∀-sentences; and all G^2RP∀-provable sentences are G^1RP∀-provable.

However, from the viewpoint of bottom-up proof search (for arbitrary, not necessarily prenex RPL∀-sentences), a defect in the calculi G^1RP∀ and G^2RP∀ is that designations of multisets of formulas are repeated in the premises of some of the inference rules. The defect is at least an obvious reason for the inefficiency of bottom-up proof search, because each copy of a nonatomic formula from repeated multisets is generally to be decomposed.

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[1] We say that a hypersequent calculus is *cumulative* if all its rules are cumulative; and a hypersequent rule is *cumulative* if, for its every application, each premise includes the conclusion (cf. [27, item 3.5.11]).

[2] The calculi G^iRP∀ ($i = 1, 2, 3$) were denoted by G^iŁ∀ in [16, 17]; but now we change these designations for the sake of a more memorable notation.

We got rid of such repetitions in [17] by presenting an analytic noncumulative hypersequent calculus G^3RP∀ for RPL∀ without structural inference rules; this calculus is *repetition-free*, in the sense that designations of multisets of formulas are not repeated in any premise of its rules. As shown in [17], all the inference rules of G^3RP∀ are height-preserving invertible; G^3RP∀ is well suited to bottom-up proof search for arbitrary RPL∀-sentences; and all G^1RP∀-provable sentences (and so all GŁ∀-provable sentences) are G^3RP∀-provable.

The main goals of the present article are: (1) to find out whether G^3RP∀ is a conservative extension of GŁ∀, and (2) to compare G^3RP∀ with Hájek's calculus for RPL∀.

It turns out that, in order to to reach our goals, it is very helpful to introduce two auxiliary analytic hypersequent calculi for RPL∀: (1) a calculus G^0RP∀ whose rules are simpler than the ones of G^3RP∀ (and of G^1RP∀), and whose axioms are the same as those of G^3RP∀ (which are rather complicated and defined in nonsyntactic terms); and (2) a calculus GRP∀ whose axioms are quite simple and defined in nearly syntactic terms, and whose rules are essentially the ones of GŁ∀. Thus, we include these new calculi in our comparison. The key part of the comparison is a proof of the admissibility for G^0RP∀ of some variants of the density rule, which underlie some rules of G^3RP∀. The features of the proof are discussed in the concluding section, in the context of works related to the elimination of applications of the density rule from formal proofs.

This article is organized as follows. In Section 2 we describe the syntax and the standard semantics of the logics Ł∀ and RPL∀, then formulate the calculi GŁ∀, G^0RP∀, and G^3RP∀. In Section 3 we introduce the (so-called nearly syntactic) calculus GRP∀ for RPL∀, which turns out to be a conservative extension of GŁ∀ and complete (with respect to the standard semantics) for the quantifier-free fragment of RPL∀. In Section 4 we show that G^0RP∀ is a conservative extension of GRP∀, and that any G^0RP∀-provable sentence is G^3RP∀-provable. In Section 5 we establish the admissibility for G^0RP∀ of two variants of the density rule, and using this, show that G^3RP∀ and G^0RP∀ are equivalent; hence we conclude that G^3RP∀ is a conservative extension of GŁ∀. In Section 6 we formulate Hájek's Hilbert-type calculus HRP∀ for RPL∀; describe the algebraic semantics for RPL∀ over MV-chains contaning the rational unit interval; and using the semantics and our two auxiliary calculi, establish that G^3RP∀ extended with the cut rule proves exactly the same

RPL∀-sentences as HRP∀. Finally, in Section 7 we discuss our results and related works.

2. Preliminaries

First we describe the syntax and the standard semantics of the logics Ł∀ and RPL∀ (see [18]).

Given a signature, which may contain predicate and function symbols of any nonnegative arities, Ł∀- and RPL∀-formulas are defined as follows. The notion of a *term* is standard. *Atomic* Ł∀*-formulas* are the truth constant $\bar{0}$ and predicate symbols with terms as their arguments. *Atomic* RPL∀*-formulas* are atomic Ł∀-formulas and truth constants $\bar{r}$ for all positive rational numbers $r \leqslant 1$. Ł∀- and RPL∀*-formulas* are built up as usual from atomic Ł∀- and RPL∀-formulas, respectively, using the following *logical symbols*: the binary connective $\to$ and the quantifiers $\forall$ and $\exists$.

An *interpretation* $\langle \mathcal{D}, \mu \rangle$ of a given signature is defined as in classical logic, except that the map μ takes each n-ary predicate symbol P to a predicate $\mu(P) : \mathcal{D}^n \to [0, 1]$. Let $M = \langle \mathcal{D}, \mu \rangle$ be an interpretation. Then an M*-valuation* is a map of the set of all individual variables to $\mathcal{D}$. For an M-valuation ν, an individual variable x, and an element $d \in \mathcal{D}$, by $\nu[x \mapsto d]$ we denote the M-valuation that may differ from ν only on x and obeys the condition $\nu[x \mapsto d](x) = d$.

The *value* $|t|_{M,\nu}$ *of a term* t under an interpretation M and an M-valuation ν is defined in the standard manner. The *truth value* $|A|_{M,\nu}$ *of an* RPL∀*-formula* A under an interpretation $M = \langle \mathcal{D}, \mu \rangle$ and an M-valuation ν is defined as follows:

- $|\bar{r}|_{M,\nu} = r$;
- $|P(t_1, \ldots, t_n)|_{M,\nu} = \mu(P)(|t_1|_{M,\nu}, \ldots, |t_n|_{M,\nu})$ for an n-ary predicate symbol P and terms $t_1, \ldots, t_n$;
- $|B \to C|_{M,\nu} = \min(1,\ 1 - |B|_{M,\nu} + |C|_{M,\nu})$;
- $|\forall x B|_{M,\nu} = \inf_{d \in \mathcal{D}} |B|_{M,\nu[x \mapsto d]}$;
- $|\exists x B|_{M,\nu} = \sup_{d \in \mathcal{D}} |B|_{M,\nu[x \mapsto d]}$.

An RPL∀-formula (in particular, an Ł∀-formula) A is called *valid*, also written $\models A$, if $|A|_{M,\nu} = 1$ for every interpretation M and every M-valuation ν.

In what follows, unless otherwise indicated, we work with a fixed signature that includes a countably infinite set of nullary function symbols called *parameters*. Nullary predicate symbols are also called *propositional variables*. The result of substituting a term t for all free occurrences of an individual variable x in an RPL∀-formula A is denoted by $(A)^x_t$. The letters k, l, m, n (possibly with subscripts) stand for nonnegative integers. An expression $k..n$ denotes the set $\{k, k+1, \ldots, n\}$ if $k \leqslant n$, and the empty set otherwise.

Let us formulate the auxiliary hypersequent calculus $\mathrm{G}^0\mathrm{RP}\forall$ and define accompanying notions and notation common to several calculi considered.

We introduce two countably infinite, disjoint sets of new words and call such words *semipropositional variables of type* 0 and *of type* 1, respectively. An RPL∀-formula as well as a semipropositional variable (of any type) is said to be a *formula*.

An $\mathrm{RPL}\forall^1_0$-*sequent* (or simply a *sequent*) is written $\Gamma \Rightarrow \Delta$ and is an ordered pair of finite multisets Γ and Δ consisting of formulas. An $\mathrm{RPL}\forall^1_0$-*hypersequent* (a *hypersequent* for short) is a finite multiset of sequents and is written $\Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$ or $\left[\Gamma_i \Rightarrow \Delta_i\right]_{i \in 1..n}$.

A sequent and a hypersequent that do not contain logical symbols are called *atomic*. Suppose that $\mathcal{H}$ is a hypersequent; then by $\mathcal{H}_{at}$ we denote the (atomic) hypersequent obtained from $\mathcal{H}$ by removing all nonatomic sequents.

We define an *hs-interpretation* as an interpretation $\langle \mathcal{D}, \mu \rangle$ in which the map μ additionally takes each semipropositional variable of type 0 to a real number in $[0, +\infty)$ and each semipropositional variable of type 1 to a real number in $(-\infty, 1]$. For a semipropositional variable $\mathfrak{p}$, an hs-interpretation $M = \langle \mathcal{D}, \mu \rangle$, and an M-valuation ν, the value $\mu(\mathfrak{p})$ will also be written as $|\mathfrak{p}|_M$ or as $|\mathfrak{p}|_{M,\nu}$.

For a finite multiset Γ of formulas, an hs-interpretation M, and an M-valuation ν, we put

$$\|\Gamma\|_{M,\nu} = \sum_{A \in \Gamma} (|A|_{M,\nu} - 1),$$

where the summation is performed taking multiplicities of multiset elements into account, and $\sum_{A \in \varnothing}(\ldots) = 0$. A sequent $\Gamma \Rightarrow \Delta$ is called *true* under an hs-interpretation M and an M-valuation ν if

$$\|\Gamma\|_{M,\nu} \leqslant \|\Delta\|_{M,\nu}.$$

Following [2, Definition 1], we say that a hypersequent $\mathcal{H}$ is *valid* (and write $\models \mathcal{H}$) if, for every hs-interpretation M and every M-valuation ν, some sequent in $\mathcal{H}$ is true under M and ν. Note that, for an RPL∀-formula A, $\vDash A$ iff $\vDash (\Rightarrow A)$. To denote that a hypersequent $\mathcal{G}$ is not valid, we write $\nvDash \mathcal{G}$.

Unless otherwise specified, below the letters A, B, and C denote any RPL∀-formulas, Γ, Δ, Π, and Σ any finite multisets of formulas, S any sequent, $\mathcal{G}$ and $\mathcal{H}$ any hypersequents, x any individual variable, t any closed term, a any parameter, and r and s any rational numbers such that $0 \leqslant r \leqslant 1$ and $0 \leqslant s \leqslant 1$; all these letters may have subscripts and superscripts. Also $\mathfrak{p}_i$ $(i = 0, 1)$ denotes any semipropositional variable of type i.

The language of the calculus $\mathrm{G}^0\mathrm{RP}\forall$ consists of all possible hypersequents. A hypersequent $\mathcal{H}$ is called an *axiom of* $\mathrm{G}^0\mathrm{RP}\forall$ if $\vDash \mathcal{H}_{at}$.

Remark 2.1. To determine whether or not a hypersequent is an axiom of $\mathrm{G}^0\mathrm{RP}\forall$, from the atomic sequents of the hypersequent one can construct a system of strict and nonstrict linear inequalities over real numbers with rational coefficients and check whether or not the system is inconsistent. The construction and the check can be performed by a polynomial time algorithm much as described in [16, Section 4.2] and, in more detail, in [15, Section 5].

The inference rules of $\mathrm{G}^0\mathrm{RP}\forall$ are:

$$\frac{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta} \;(\to\Rightarrow)^0,$$

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta \,|\, \Gamma \Rightarrow \Delta; \quad \mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta \,|\, \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta} \;(\Rightarrow\to)^0,$$

$$\frac{\mathcal{G} \,|\, \Gamma, \forall x A \Rightarrow \Delta \,|\, \Gamma, (A)^x_t \Rightarrow \Delta}{\mathcal{G} \,|\, \Gamma, \forall x A \Rightarrow \Delta} \;(\forall\Rightarrow)^0,$$

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta \,|\, \Gamma \Rightarrow (A)^x_a, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow \forall x A, \Delta} \;(\Rightarrow\forall)^0,$$

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow \exists x A, \Delta \,|\, \Gamma \Rightarrow (A)^x_t, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow \exists x A, \Delta} \;(\Rightarrow\exists)^0,$$

$$\frac{\mathcal{G} \,|\, \Gamma, \exists x A \Rightarrow \Delta \,|\, \Gamma, (A)^x_a \Rightarrow \Delta}{\mathcal{G} \,|\, \Gamma, \exists x A \Rightarrow \Delta} \;(\exists\Rightarrow)^0,$$

where a does not occur in the conclusion of $(\Rightarrow\forall)^0$ or $(\exists\Rightarrow)^0$.

Remark 2.2. The soundness of the calculus $\mathrm{G}^0\mathrm{RP}\forall$ can be easily proved now; but it will also follow from the facts that every $\mathrm{G}^0\mathrm{RP}\forall$-provable hypersequent is $\mathrm{G}^3\mathrm{RP}\forall$-provable (see Theorems 4.5 and 4.6 below), and that the calculus $\mathrm{G}^3\mathrm{RP}\forall$ is sound (see [17, Theorem 1]).

For convenience in comparing calculi, we also introduce a calculus $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$ that is very close to the calculus $\mathrm{G}^1\mathrm{RP}\forall$, defined in [16] and in the next paragraph. We get $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$ from $\mathrm{G}^0\mathrm{RP}\forall$ by replacing the inference rule $(\to\Rightarrow)^0$ with

$$\frac{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta\,|\,\Gamma, \mathfrak{p}_1 \Rightarrow \Delta\,|\,B \Rightarrow \mathfrak{p}_1, A}{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)^{\hat{1}},$$

where $\mathfrak{p}_1$ does not occur in the conclusion.

The calculus $\mathrm{G}^1\mathrm{RP}\forall$ [16] is obtained from $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$ by restricting the language of $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$ to hypersequents not containing semipropositional variables of type 0; such hypersequents are called $\mathrm{RPL}\forall^1$-*hypersequents*.

The rule of $\mathrm{G}^i\mathrm{RP}\forall$ ($i = 1, \hat{1}$) that corresponds to a rule of $\mathrm{G}^0\mathrm{RP}\forall$ is denoted just as the latter but with the superscript i instead of 0.

Remark 2.3. It is clear that, for an $\mathrm{RPL}\forall^1$-hypersequent $\mathcal{H}$, a $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$-proof of $\mathcal{H}$ is a $\mathrm{G}^1\mathrm{RP}\forall$-proof of $\mathcal{H}$, and conversely.

The calculus $\mathrm{G}^3\mathrm{RP}\forall$ [17] is obtained from $\mathrm{G}^0\mathrm{RP}\forall$ by replacing all the inference rules with the following ones:

$$\frac{\mathcal{G}\,|\,\Gamma, \mathfrak{p}_1 \Rightarrow \Delta\,|\,B \Rightarrow \mathfrak{p}_1, A}{\mathcal{G}\,|\,\Gamma, A \to B \Rightarrow \Delta}\ (\to\Rightarrow)^3,$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta; \quad \mathcal{G}\,|\,\Gamma, A \Rightarrow B, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow\to)^3,$$

$$\frac{\mathcal{G}\,|\,\Gamma, \mathfrak{p}_1 \Rightarrow \Delta\,|\,\forall x A \Rightarrow \mathfrak{p}_1\,|\,(A)^x_t \Rightarrow \mathfrak{p}_1}{\mathcal{G}\,|\,\Gamma, \forall x A \Rightarrow \Delta}\ (\forall\Rightarrow)^3, \quad \frac{\mathcal{G}\,|\,\Gamma \Rightarrow (A)^x_a, \Delta}{\mathcal{G}\,|\,\Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow\forall)^3,$$

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \mathfrak{p}_0, \Delta\,|\,\mathfrak{p}_0 \Rightarrow \exists x A\,|\,\mathfrak{p}_0 \Rightarrow (A)^x_t}{\mathcal{G}\,|\,\Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow\exists)^3, \quad \frac{\mathcal{G}\,|\,\Gamma, (A)^x_a \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \exists x A \Rightarrow \Delta}\ (\exists\Rightarrow)^3,$$

where $\mathfrak{p}_1$ does not occur in the conclusion of $(\to\Rightarrow)^3$ or $(\forall\Rightarrow)^3$, $\mathfrak{p}_0$ does not occur in the conclusion of $(\Rightarrow\exists)^3$, and a does not occur in the conclusion of $(\Rightarrow\forall)^3$ or $(\exists\Rightarrow)^3$.

For an application of an inference rule of $\mathrm{G}^i\mathrm{RP}\forall$ ($i = 0, 1, \hat{1}, 3$), the *principal* formula occurrence and the *principal* sequent occurrence are defined in essentially the same manner as in [20, § 49] and [27, items 3.1.1 and 3.5.1]. The notion of an *ancestor* of a sequent occurrence in a

$\mathrm{G}^i\mathrm{RP}\forall$-proof ($i = 0, 1, \hat{1}, 3$) is defined much as the notion of an ancestor of a formula occurrence is defined in [20, § 49]; see also [17, Section 3].

Now we formulate the calculus GŁ∀ [2, 23], using parameters instead of free individual variables, which are syntactically distinct from bound individual variables in [2, 23]. The language of GŁ∀ consists of all possible Ł∀-*hypersequents*, i.e., hypersequents that contain neither semipropositional variables nor truth constants $\bar{r}$ with $r > 0$.

The axiom schemes of GŁ∀ are:

$$A \Rightarrow A \ \ (\mathrm{id})^{\text{Ł}}, \qquad \Rightarrow \ \ (\Lambda)^{\text{Ł}}, \qquad \bar{0} \Rightarrow A \ \ (\bar{0} \Rightarrow)^{\text{Ł}},$$

where A is an Ł∀-formula.

The inference rules of GŁ∀ are:

$$\frac{\mathcal{G}}{\mathcal{G} \mid S}\ (\mathrm{ew})^{\text{Ł}}, \quad \frac{\mathcal{G} \mid S \mid S}{\mathcal{G} \mid S}\ (\mathrm{ec})^{\text{Ł}}, \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, C \Rightarrow \Delta}\ (\mathrm{wl})^{\text{Ł}},$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\mathrm{split})^{\text{Ł}}, \quad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1; \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{mix})^{\text{Ł}},$$

$$\frac{\mathcal{G} \mid \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \mid \Gamma, A \to B \Rightarrow \Delta}\ (\to \Rightarrow)^{\text{Ł}}, \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta; \quad \mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \to B, \Delta}\ (\Rightarrow \to)^{\text{Ł}},$$

$$\frac{\mathcal{G} \mid \Gamma, (A)^x_t \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta}\ (\forall \Rightarrow)^{\text{Ł}}, \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow (A)^x_a, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \forall x A, \Delta}\ (\Rightarrow \forall)^{\text{Ł}},$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow (A)^x_t, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \exists x A, \Delta}\ (\Rightarrow \exists)^{\text{Ł}}, \quad \frac{\mathcal{G} \mid \Gamma, (A)^x_a \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \exists x A \Rightarrow \Delta}\ (\exists \Rightarrow)^{\text{Ł}},$$

where all the premises and conclusions are Ł∀-hypersequents, and a does not occur in the conclusion of $(\Rightarrow \forall)^{\text{Ł}}$ or $(\exists \Rightarrow)^{\text{Ł}}$. The first five of these rules are called *structural*; the others, *logical*.

For each calculus formulated above, its every one-premise rule in whose premise a, t, or $\mathfrak{p}_i$ ($i = 0, 1$) is distinguished (i.e., shown explicitly in the premise scheme, such as a in $\mathcal{G} \mid \Gamma \Rightarrow (A)^x_a, \Delta$), and for any application of the rule, the a, t, or $\mathfrak{p}_i$ is called, respectively, the *proper* parameter, *proper* term, or *proper* semipropositional variable of the application.

The provability (resp. unprovability) of an object α in a calculus $\mathfrak{C}$ is written $\vdash_{\mathfrak{C}} \alpha$ (resp. $\nvdash_{\mathfrak{C}} \alpha$). By a proof in a calculus, we mean a proof tree. In depicting a proof tree D, if we place a designation over a node N of D and do not separate the designation from N by a horizontal line, then we regard this designation as the one for the proof tree whose root is N and that is a subtree of D. A *proof search tree* is defined as a proof tree, but its leaves are not required to be axioms of the calculus under consideration.

A *proof of* (*for*) *an* RPL∀-*formula* A in a hypersequent calculus given in this article is a proof of the hypersequent $\Rightarrow A$ in this calculus.

3. The nearly syntactic hypersequent calculus GRP∀ for RPL∀

In this section we extend the calculus GŁ∀ to obtain the analytic hypersequent calculus GRP∀ for RPL∀ with rather simple axioms defined in nearly syntactic terms. Because of the simplicity of its axioms, the calculus GRP∀ will be very helpful in comparing our calculus $\mathrm{G^3RP}\forall$ with Hájek's Hilbert-type one for RPL∀.

The language of GRP∀ consists of all hypersequents not containing semipropositional variables; such hypersequents are called RPL∀-*hypersequents*.

The axiom schemes of GRP∀ are:

$$A \Rightarrow A \ \ (\mathrm{id})^{\mathrm{P}} \qquad \text{and} \qquad \bar{r}_1, \dots, \bar{r}_l \Rightarrow \bar{s}_1, \dots, \bar{s}_m, A_1, \dots, A_n \ \ (\mathrm{le})^{\mathrm{P}},$$

where

$$\sum_{i=1}^{l}(r_i - 1) \leqslant \sum_{j=1}^{m}(s_j - 1) - n, \ \text{ or equivalently } \ m + n + \sum_{i=1}^{l} r_i \leqslant l + \sum_{j=1}^{m} s_j.$$

(Recall that l, m, n are any nonnegative integers, by our convention in Section 2.)

Remark 3.1. It is readily seen that any axiom of GŁ∀ is an axiom of GRP∀.

The inference rules of GRP∀ are those of GŁ∀ but with RPL∀-hypersequents in place of Ł∀-hypersequents. We denote the rules of GRP∀ as the ones of GŁ∀ but with the superscript P: $(\mathrm{ew})^{\mathrm{P}}$, $(\mathrm{ec})^{\mathrm{P}}$, etc.

PROPOSITION 3.1. GRP∀ *is a conservative extension of* GŁ∀; *i.e., for any* Ł∀-*hypersequent* $\mathcal{H}$, $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$ *iff* $\vdash_{\mathrm{GŁ}\forall} \mathcal{H}$.

PROOF. Let $\mathcal{H}$ be an Ł∀-hypersequent.

In view of Remark 3.1, $\vdash_{\mathrm{GŁ}\forall} \mathcal{H}$ implies $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$.

Conversely, suppose that $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$. To obtain $\vdash_{\mathrm{GŁ}\forall} \mathcal{H}$, it suffices to show that any Ł∀-hypersequent $\mathcal{G}$ that is an instance of the axiom scheme $(\mathrm{le})^{\mathrm{P}}$ of GRP∀ is GŁ∀-provable. Such an Ł∀-hypersequent $\mathcal{G}$ is of the form

$$\bar{r}_1, \dots, \bar{r}_l \Rightarrow \bar{s}_1, \dots, \bar{s}_m, A_1, \dots, A_n,$$

where $r_i = 0$ for all i, $s_j = 0$ for all j, A_k is an Ł∀-formula for all k, and $m + n \leqslant l$. We can construct a GŁ∀-proof of $\mathcal{G}$ by applying (zero or more times) the rules $(\mathrm{mix})^{\text{Ł}}$ and $(\mathrm{wl})^{\text{Ł}}$ backwards and getting GŁ∀-axioms $\bar{0} \Rightarrow \bar{0}$, $\bar{0} \Rightarrow A_k$, or $\Rightarrow$. ⊣

PROPOSITION 3.2 (soundness of GRP∀). *Let $\mathcal{H}$ be an* RPL∀*-hypersequent. If $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$, then $\vDash \mathcal{H}$.*

PROOF. All the axioms of GRP∀ are clearly valid. The soundness of the inference rules of GŁ∀ is verified in [2, 23]; and this verification carries over to GRP∀. ⊣

PROPOSITION 3.3 (completeness of GRP∀ for quantifier-free RPL∀-hypersequents). *Let $\mathcal{H}$ be a quantifier-free* RPL∀*-hypersequent. If $\vDash \mathcal{H}$, then $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$.*

Our proof of Proposition 3.3 extends the proof of the analogous claim for Ł∀ and GŁ∀, namely the proof of Theorem 6.24 in [23], and employs the following Lemmas 3.4 and 3.5.

LEMMA 3.4 (Lemmas 6.22 and 6.23 in [23]).
(a) *Consider the rules*

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma, B \Rightarrow A, \Delta}{\mathcal{G} \,|\, \Gamma, A \to B \Rightarrow \Delta} \quad \textit{and} \quad \frac{\mathcal{G} \,|\, \Gamma \Rightarrow \Delta; \quad \mathcal{G} \,|\, \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow A \to B, \Delta}$$

whose premises and conclusions are quantifier-free Ł∀*-hypersequents. Each of these rules is derivable in* GŁ∀ *and is such that, for its every application, the conclusion is valid iff so are all the premises.*
(b) *Every quantifier-free* Ł∀*-hypersequent can be obtained by these two rules from finitely many atomic* Ł∀*-hypersequents.*

By $\ell(\mathcal{G})$ we denote the number of distinct nonconstant atomic RPL∀-formulas occurring in the antecedents of the sequents in an atomic RPL∀-hypersequent $\mathcal{G}$.

The next lemma is in fact established in the proof of Theorem 6.24 in [23].

Lemma 3.5. *Let $\mathcal{G}$ be a valid atomic Ł∀-hypersequent with $\ell(\mathcal{G}) > 0$. Then $\mathcal{G}$ is GŁ∀-provable from a valid atomic Ł∀-hypersequent $\mathcal{H}$ with $\ell(\mathcal{H}) < \ell(\mathcal{G})$.*

Remark 3.2. Lemmas 3.4 and 3.5 readily carry over to RPL∀ and GRP∀ (instead of Ł∀ and GŁ∀, respectively).

Proof of Proposition 3.3. By Lemma 3.4 together with Remark 3.2, it is sufficient to show that $\vdash_{\mathrm{GRP}\forall} \mathcal{G}$, where $\mathcal{G}$ is a valid atomic RPL∀-hypersequent. We proceed by induction on $\ell(\mathcal{G})$.

1. Suppose that $\ell(\mathcal{G}) = 0$. Then each sequent in $\mathcal{G}$ is of the form

$$\bar{r}_1, \dots, \bar{r}_l \Rightarrow \bar{s}_1, \dots, \bar{s}_m, A_1, \dots, A_n,$$

where $A_1, \dots, A_n$ are nonconstant atomic RPL∀-formulas. In $\mathcal{G}$ there exists a sequent S for which

$$\sum_{i=1}^{l}(r_i - 1) \leqslant \sum_{j=1}^{m}(s_j - 1) - n;$$

as otherwise, in $\mathcal{G}$ there is no true sequent under some hs-interpretation M and some M-valuation ν such that $|A_k|_{M,\nu} = 0$ for all k. Thus, S is an instance of the axiom scheme $(\mathrm{le})^{\mathrm{P}}$ of GRP∀; and $\mathcal{G}$ can be obtained from S by (zero or more) applications of the rule $(\mathrm{ew})^{\mathrm{P}}$.

2. In the case when $\ell(\mathcal{G}) > 0$, we apply Lemma 3.5 together with Remark 3.2 and then use the induction hypothesis. ⊣

In the sequel we need the following lemma, which is a direct consequence of Proposition 3.3.

Lemma 3.6. *Suppose that $\mathcal{G}$ is an* RPL∀*-hypersequent* (*over the signature we work with*)*; A_1, …, A_n are* RPL∀*-formulas* (*over the same signature*)*; $p_1, \dots, p_n$ are distinct propositional variables; $\mathcal{H}$ is a valid quantifier-free* RPL∀*-hypersequent over a signature containing $p_1, \dots, p_n$; $\mathcal{G}$ comes from $\mathcal{H}$ by simultaneously replacing all occurrences of p_1, …, p_n with A_1, …, A_n, respectively. Then $\vdash_{\mathrm{GRP}\forall} \mathcal{G}$.*

Proof. By Proposition 3.3, there is a GRP∀-proof D of $\mathcal{H}$. Simultaneously replacing all occurrences of $p_1, \dots, p_n$ in D with $A_1, \dots, A_n$, respectively, yields the desired GRP∀-proof of $\mathcal{G}$. ⊣

4. Initial relationships between hypersequent calculi for RPL∀

In this section we show that the calculus $\mathrm{G^0RP\forall}$ is a conservative extension of the calculus GRP∀; and that, for any hypersequent $\mathcal{H}$, we have: $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}$ implies $\vdash_{\mathrm{G^{\hat{1}}RP\forall}} \mathcal{H}$, which in turn implies $\vdash_{\mathrm{G^3RP\forall}} \mathcal{H}$.

LEMMA 4.1. *Let $\mathcal{H}$ be an* RPL∀*-hypersequent. Then $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}$ implies $\vdash_{\mathrm{GRP\forall}} \mathcal{H}$. Moreover, all the rules of* $\mathrm{G^0RP\forall}$ *are derivable in* GRP∀ *if their premises and conclusions are restricted to* RPL∀*-hypersequents.*

PROOF. If $\mathcal{H}$ is an axiom of $\mathrm{G^0RP\forall}$, then $\vDash \mathcal{H}_{at}$; so by Proposition 3.3, we get $\vdash_{\mathrm{GRP\forall}} \mathcal{H}_{at}$, whence $\vdash_{\mathrm{GRP\forall}} \mathcal{H}$ by the rule $(\mathrm{ew})^{\mathrm{P}}$.

To finish the proof, it is sufficient to show that all the rules of $\mathrm{G^0RP\forall}$ are derivable in GRP∀ if their premises and conclusions are restricted to RPL∀-hypersequents. For the rule $(\to\Rightarrow)^0$, we have:

$$\dfrac{\dfrac{\dfrac{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma\Rightarrow\Delta\,|\,\Gamma, B\Rightarrow A,\Delta}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma\Rightarrow\Delta\,|\,\Gamma, A\to B\Rightarrow\Delta}\,(\to\Rightarrow)^{\mathrm{P}}}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma, A\to B\Rightarrow\Delta}\,(\mathrm{wl})^{\mathrm{P}}}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta}\,(\mathrm{ec})^{\mathrm{P}}\times 2.$$

For the rule $(\forall\Rightarrow)^0$, we have:

$$\dfrac{\dfrac{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,(A)^x_t\Rightarrow\Delta}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta\,|\,\Gamma,\forall xA\Rightarrow\Delta}\,(\forall\Rightarrow)^{\mathrm{P}}}{\mathcal{G}\,|\,\Gamma,\forall xA\Rightarrow\Delta}\,(\mathrm{ec})^{\mathrm{P}}.$$

The other rules of $\mathrm{G^0RP\forall}$ are treated similarly to $(\forall\Rightarrow)^0$. ⊣

To show that $\vdash_{\mathrm{GRP\forall}} \mathcal{H}$ implies $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}$, and for later use, we introduce the following rules. For each rule $\mathcal{R}^{\text{Ł}}$ of GŁ∀, let $\mathcal{R}^*$ be the rule like $\mathcal{R}$ but with ($\mathrm{RPL\forall^1_0}$-)hypersequents in place of Ł∀-hypersequents; thus we have the rules $(\mathrm{ew})^*$, $(\mathrm{ec})^*$, etc.

LEMMA 4.2. *The rules* $(\mathrm{ew})^*$, $(\mathrm{ec})^*$, $(\mathrm{wl})^*$, $(\mathrm{split})^*$, $(\mathrm{mix})^*$, $(\to\Rightarrow)^*$, $(\Rightarrow\to)^*$, $(\forall\Rightarrow)^*$, $(\Rightarrow\forall)^*$, $(\Rightarrow\exists)^*$, *and* $(\exists\Rightarrow)^*$ *are admissible for* $\mathrm{G^0RP\forall}$. *Moreover, the rules* $(\mathrm{ew})^*$, $(\mathrm{ec})^*$, *and* $(\mathrm{split})^*$ *are height-preserving admissible, or briefly hp-admissible, for* $\mathrm{G^0RP\forall}$.

PROOF. 1. It is clear that $(\mathrm{ew})^*$ is hp-admissible for $\mathrm{G^0RP\forall}$.

2. Since all the rules of $\mathrm{G^0RP\forall}$ are cumulative, it follows easily that $(\mathrm{ec})^*$ is hp-admissible for $\mathrm{G^0RP\forall}$ (cf., e.g, [27, item 3.5.11] and [16, Lemma 5]).

3. To prove that

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, C \Rightarrow \Delta}\ (\mathrm{wl})^*$$

is admissible for $\mathrm{G}^0\mathrm{RP}\forall$, we use induction on the number of logical symbol occurrences in the RPL∀-formula C. Let

$$\mathcal{H}_1 = (\mathcal{G}\,|\,\Gamma \Rightarrow \Delta) \quad \text{and} \quad \mathcal{H}_2 = (\mathcal{G}\,|\,\Gamma, C \Rightarrow \Delta).$$

We can assume that there is a ($\mathrm{G}^0\mathrm{RP}\forall$-)proof D_1 for $\mathcal{H}_1$ such that no proper parameter from D_1 occurs in C.

3.1. Suppose that C is atomic or is of the form $(A \to B)$. From D_1 we construct a proof search tree D_2^0 for $\mathcal{H}_2$ as follows. For each occurrence $\mathcal{S}$ of a sequent of the form $\Pi \Rightarrow \Sigma$, if $\mathcal{S}$ is an ancestor of the distinguished occurrence of the sequent $\Gamma \Rightarrow \Delta$ in the root of D_1, then we replace $\mathcal{S}$ by an occurrence $\mathcal{S}'$ of the sequent $\Pi, C \Rightarrow \Sigma$. We also mark $\mathcal{S}'$ if $\mathcal{S}$ is an atomic sequent occurrence in a leaf of D_1.

If C is atomic, then D_2^0 is a proof for $\mathcal{H}_2$. Indeed, when the atomic RPL∀-formula C is added to the antecedents of some sequents in a hypersequent that is an axiom (of $\mathrm{G}^0\mathrm{RP}\forall$), the hypersequent remains an axiom, since for every atomic sequent $\Pi \Rightarrow \Sigma$, hs-interpretation M, and M-valuation ν, the sequent $\Pi, C \Rightarrow \Sigma$ is atomic too, and $\|\Pi\|_{M,\nu} \leqslant \|\Sigma\|_{M,\nu}$ implies $\|\Pi, C\|_{M,\nu} \leqslant \|\Sigma\|_{M,\nu}$.

Now suppose that C is of the form $(A \to B)$, and $\mathcal{S}_0, \dots, \mathcal{S}_{l-1}$ are all distinct marked sequent occurrences in D_2^0.

We expand D_2^0 by performing the following for each $i = 0, \dots, l-1$: on the only branch $\mathcal{B}_i$ of D_2^i containing $\mathcal{S}_i$, apply the rule $(\to\Rightarrow)^0$ backwards to the ancestor of $\mathcal{S}_i$ in the leaf on $\mathcal{B}_i$, and denote by D_2^{i+1} the tree obtained as a result of this backward application.

Note that, if $\mathcal{S}_i$ is an occurrence of a sequent of the form $\Pi_i, C \Rightarrow \Sigma_i$, then the atomic sequent $\Pi_i \Rightarrow \Sigma_i$ is on the continuation of the branch $\mathcal{B}_i$ in D_2^{i+1}. Therefore, it is easy to see that D_2^l is a proof for $\mathcal{H}_2$.

3.2. Suppose that C is of the form $\mathsf{Q}xA$, where Q is a quantifier. By the induction hypothesis, there is a proof for $\mathcal{H} = (\mathcal{H}_2\,|\,\Gamma, (A)^x_a \Rightarrow \Delta)$, where a is a parameter not occurring in $\mathcal{H}_2$. By applying the rule $(\mathsf{Q}\Rightarrow)^0$ to the distinguished occurrence of $(A)^x_a$ in $\mathcal{H}$, we get a proof for $\mathcal{H}_2$.

4. Given the hp-admissibility of $(\mathrm{ec})^*$ for $\mathrm{G}^0\mathrm{RP}\forall$ (see item 2), the proof of the hp-admissibility of $(\mathrm{split})^*$ for $\mathrm{G}^0\mathrm{RP}\forall$ is very similar to the proof of Lemma 7 in [16], where the admissibility (in fact, hp-admissibility) of the same rule for $\mathrm{G}^1\mathrm{RP}\forall$ is demonstrated.

5. The proof of the admissibility of $(\mathrm{mix})^*$ for $\mathrm{G^0RP\forall}$ can be obtained from the proof of Lemma 8 in [16] (where the admissibility of the same rule for $\mathrm{G^1RP\forall}$ is shown) by identifying the notion of a completable ancestor of a sequent occurrence with the notion of an ancestor of a sequent occurrence (the former notion is used in [16]).

6. Since the rule $(\mathrm{ew})^*$ is admissible for $\mathrm{G^0RP\forall}$, it follows easily that the rules $(\to\Rightarrow)^*$, $(\Rightarrow\to)^*$, $(\forall\Rightarrow)^*$, $(\Rightarrow\forall)^*$, $(\Rightarrow\exists)^*$, and $(\exists\Rightarrow)^*$ are admissible for $\mathrm{G^0RP\forall}$. ⊣

LEMMA 4.3. *Every axiom of* $\mathrm{GRP\forall}$ *is* $\mathrm{G^0RP\forall}$*-provable.*

PROOF. *Case* $(\mathrm{id})^{\mathrm{P}}$. We show that a $\mathrm{GRP\forall}$-axiom $A \Rightarrow A$ is $\mathrm{G^0RP\forall}$-provable by induction on the number of logical symbol occurrences in A.

If A is atomic, then $A \Rightarrow A$ is an axiom of $\mathrm{G^0RP\forall}$.

Otherwise, we obtain $A \Rightarrow A$ as follows, according as A has the form $B \to C$, or $\forall x B$, or $\exists x B$:

$$(\mathrm{wl})^* \dfrac{\dfrac{\Rightarrow}{B \to C \Rightarrow;} \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B; \quad C \Rightarrow C}{B, C \Rightarrow B, C}(\mathrm{mix})^*}{B \to C, B \Rightarrow C}(\to\Rightarrow)^*}{}}{B \to C \Rightarrow B \to C}(\Rightarrow\to)^*, \qquad \dfrac{\dfrac{(B)^x_a \Rightarrow (B)^x_a}{\forall x B \Rightarrow (B)^x_a}(\forall\Rightarrow)^*}{\forall x B \Rightarrow \forall x B}(\Rightarrow\forall)^*,$$

and similarly for $\exists x B \Rightarrow \exists x B$, with a not occurring in B. The rules used here are admissible for $\mathrm{G^0RP\forall}$ by Lemma 4.2; the hypersequent $\Rightarrow$ is an axiom of $\mathrm{G^0RP\forall}$. By the induction hypothesis applied to $B \Rightarrow B$, $C \Rightarrow C$, and $(B)^x_a \Rightarrow (B)^x_a$, we are done with *case* $(\mathrm{id})^{\mathrm{P}}$.

Case $(\mathrm{le})^{\mathrm{P}}$. Now consider a $\mathrm{GRP\forall}$-axiom S of the form

$$\bar{r}_1, \dots, \bar{r}_l \Rightarrow \bar{s}_1, \dots, \bar{s}_m, A_1, \dots, A_n, \text{ where } \sum_{i=1}^{l}(r_i - 1) \leqslant \sum_{j=1}^{m}(s_j - 1) - n.$$

To show that S is $\mathrm{G^0RP\forall}$-provable, we employ induction on the number of logical symbol occurrences in S.

If S is atomic, then S is an axiom of $\mathrm{G^0RP\forall}$.

Otherwise, let us assume for definiteness that A_n contains a logical symbol, and write S as $\Gamma \Rightarrow \Delta, A_n$. Then we obtain S as follows, according as A_n has the form $B \to C$ or $\mathsf{Q} x B$, where Q is a quantifier:

$$\dfrac{\Gamma \Rightarrow \Delta; \qquad \dfrac{\Gamma \Rightarrow \Delta, C}{\Gamma, B \Rightarrow \Delta, C}(\mathrm{wl})^*}{\Gamma \Rightarrow \Delta, B \to C}(\Rightarrow\to)^*, \qquad \dfrac{\Gamma \Rightarrow \Delta, (B)^x_a}{\Gamma \Rightarrow \Delta, \mathsf{Q} x B}(\Rightarrow \mathsf{Q})^*,$$

with a not occurring in S. The three upper sequents are instances of the axiom scheme $(\mathrm{le})^{\mathrm{P}}$ of GRP∀, because the sequent $\Gamma \Rightarrow \Delta$ has the form (which is to be compared with the above form of S)

$$\bar{r}_1, \ldots, \bar{r}_l \Rightarrow \bar{s}_1, \ldots, \bar{s}_m, A_1, \ldots, A_{n-1}, \quad \text{where}$$
$$\sum_{i=1}^{l}(r_i - 1) \leqslant \sum_{j=1}^{m}(s_j - 1) - n \leqslant \sum_{j=1}^{m}(s_j - 1) - (n-1),$$

and the other two sequents have the same form as the above form of S. Finally, we apply the induction hypothesis to each of the three sequents. ⊣

Theorem 4.4. $\mathrm{G^0RP}\forall$ *is a conservative extension of* GRP∀; *i.e., for any* RPL∀*-hypersequent* $\mathcal{H}$, $\vdash_{\mathrm{G^0RP}\forall} \mathcal{H}$ *iff* $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$.

Proof. Lemma 4.1 gives us the left-to-right direction. For the right-to-left direction, observe that, by Lemma 4.2, all the rules of GRP∀ are admissible for $\mathrm{G^0RP}\forall$; and by Lemma 4.3, all the axioms of GRP∀ are $\mathrm{G^0RP}\forall$-provable. ⊣

Theorem 4.5. *If* $\vdash_{\mathrm{G^0RP}\forall} \mathcal{H}$, *then* $\vdash_{\mathrm{G^{\hat{1}}RP}\forall} \mathcal{H}$.

Proof. Every axiom of $\mathrm{G^0RP}\forall$ is an axiom of $\mathrm{G^{\hat{1}}RP}\forall$. Every rule of $\mathrm{G^0RP}\forall$, except for the rule $(\rightarrow\Rightarrow)^0$, is a rule of $\mathrm{G^{\hat{1}}RP}\forall$. Hence, it suffices to prove that $(\rightarrow\Rightarrow)^0$ is admissible for $\mathrm{G^{\hat{1}}RP}\forall$.

To do this, we use the rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \mathfrak{p}_1 \Rightarrow \mathfrak{p}_1, \Delta}\ (\mathrm{sp_1}{\Rightarrow}\mathrm{sp_1})^* \qquad \text{and} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \mathfrak{p}_1 \Rightarrow \Delta}\ (\mathrm{wl})^*_{\mathrm{sp_1}},$$

whose hp-admissibility for $\mathrm{G^{\hat{1}}RP}\forall$ is obvious. We also use the rules $(\mathrm{ec})^*$ and $(\mathrm{split})^*$, noticing that the proofs of their hp-admissibility for $\mathrm{G^{\hat{1}}RP}\forall$ are entirely analogous to the proofs of Lemmas 5 and 7 in [16], respectively, where these rules are shown to be hp-admissible for $\mathrm{G^1RP}\forall$.

The conclusion of the rule $(\rightarrow\Rightarrow)^0$ can be obtained from its premise by rules that are admissible for $\mathrm{G^{\hat{1}}RP}\forall$ as displayed in Figure 1, where $\mathfrak{p}_1$ does not occur in $\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta$. Thus, $(\rightarrow\Rightarrow)^0$ is admissible for $\mathrm{G^{\hat{1}}RP}\forall$. ⊣

Theorem 4.6. *If* $\vdash_{\mathrm{G^{\hat{1}}RP}\forall} \mathcal{H}$, *then* $\vdash_{\mathrm{G^3RP}\forall} \mathcal{H}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma\Rightarrow\Delta\,|\,\Gamma, B\Rightarrow A,\Delta}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma\Rightarrow\Delta\,|\,\Gamma, B,\mathfrak{p}_1\Rightarrow\mathfrak{p}_1, A,\Delta}\,(\mathrm{sp}_1{\Rightarrow}\mathrm{sp}_1)^*}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,B\Rightarrow\mathfrak{p}_1, A}\,(\mathrm{split})^*}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,B\Rightarrow\mathfrak{p}_1, A}\,(\mathrm{wl})^*_{\mathrm{sp}_1}}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,B\Rightarrow\mathfrak{p}_1, A}\,(\mathrm{ec})^*}{\mathcal{G}\,|\,\Gamma, A\to B\Rightarrow\Delta}\,(\to\Rightarrow)^{\hat{1}},$$

Figure 1. Obtaining the conclusion of the rule $(\to\Rightarrow)^0$ from its premise by rules that are admissible for $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$.

PROOF. This proof comes from the proofs of Lemma 6 and Theorem 2 in [17] (where it is shown that $\vdash_{\mathrm{G}^1\mathrm{RP}\forall}\mathcal{H}$ implies $\vdash_{\mathrm{G}^3\mathrm{RP}\forall}\mathcal{H}$) by substituting the superscript $\hat{1}$ for the superscript 1 (in "$\mathrm{G}^1\mathrm{RP}\forall$" and in the designations of the rules of $\mathrm{G}^1\mathrm{RP}\forall$). ⊣

5. The admissibility for $\mathbf{G^0RP\forall}$ of variants of the density rule and further relationships between hypersequent calculi for RPL∀

The primary goal of this section is to show that the calculi $\mathrm{G}^0\mathrm{RP}\forall$ and $\mathrm{G}^3\mathrm{RP}\forall$ are equivalent, i.e., they prove exactly the same hypersequents. In view of Theorems 4.5 and 4.6, it is enough to demonstrate that all $\mathrm{G}^3\mathrm{RP}\forall$-provable hypersequents are $\mathrm{G}^0\mathrm{RP}\forall$-provable. For this, we establish that all the rules of $\mathrm{G}^3\mathrm{RP}\forall$ are admissible for $\mathrm{G}^0\mathrm{RP}\forall$.

As we show in the proof of the following Lemma 5.1 (cf. also [17, Section 3]), the rules $(\to\Rightarrow)^3$, $(\forall\Rightarrow)^3$, and $(\Rightarrow\exists)^3$ of $\mathrm{G}^3\mathrm{RP}\forall$ are based on the rules

$$\frac{\mathcal{G}\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,C\Rightarrow\mathfrak{p}_1}{\mathcal{G}\,|\,\Gamma, C\Rightarrow\Delta}\,(\mathrm{den}_1)\quad\text{and}\quad\frac{\mathcal{G}\,|\,\Gamma\Rightarrow\mathfrak{p}_0,\Delta\,|\,\mathfrak{p}_0\Rightarrow C}{\mathcal{G}\,|\,\Gamma\Rightarrow C,\Delta}\,(\mathrm{den}_0),$$

where $\mathfrak{p}_i$ does not occur in the conclusion of (den_i), $i = 0, 1$. The last two rules can be characterized as nonstandard variants of the density rule, cf. [23, Section 4.5].

Remark 5.1. The (standard) *density rule* in the hypersequent formulation is:

$$\frac{\mathcal{G}\,|\,\Gamma,\mathfrak{p}\Rightarrow\Delta\,|\,\Pi\Rightarrow\mathfrak{p},\Sigma}{\mathcal{G}\,|\,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\,(\mathrm{den}),$$

where $\mathfrak{p}$ is a propositional variable not occurring in the conclusion; see [23, Section 4.5]. Given our definition of the validity of a hypersequent, it is not hard to check that (den) is unsound, but becomes sound if we expand the notion of a hypersequent by adding new-type semipropositional variables interpreted by any real numbers, and require $\mathfrak{p}$ to be such a variable not occurring in the conclusion. We will refer to this modified rule (den) as the *nonstandard density rule*.

Lemma 5.1. *If the rules* (den_1) *and* (den_0) *are admissible for* $\text{G}^0\text{RP}\forall$, *then* $\vdash_{\text{G}^3\text{RP}\forall} \mathcal{H}$ *implies* $\vdash_{\text{G}^0\text{RP}\forall} \mathcal{H}$.

Proof. Any axiom of $\text{G}^3\text{RP}\forall$ is an axiom of $\text{G}^0\text{RP}\forall$. Assuming that ($\text{den}_1$) and ($\text{den}_0$) are admissible for $\text{G}^0\text{RP}\forall$, we then show that all the rules of $\text{G}^3\text{RP}\forall$ are admissible for $\text{G}^0\text{RP}\forall$. The conclusion of the rule $(\rightarrow\Rightarrow)^3$ is obtained from its premise as follows:

$$\cfrac{\cfrac{\cfrac{\mathcal{G}\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,B\Rightarrow\mathfrak{p}_1,A}{\mathcal{G}\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,B\Rightarrow\mathfrak{p}_1,A\,|\,\Rightarrow\mathfrak{p}_1\,|\,A\rightarrow B\Rightarrow\mathfrak{p}_1}\,(\text{ew})^*\times 2}{\mathcal{G}\,|\,\Gamma,\mathfrak{p}_1\Rightarrow\Delta\,|\,A\rightarrow B\Rightarrow\mathfrak{p}_1}\,(\rightarrow\Rightarrow)^0}{\mathcal{G}\,|\,\Gamma,A\rightarrow B\Rightarrow\Delta}\,(\text{den}_1),$$

$(\text{ew})^*$ being admissible for $\text{G}^0\text{RP}\forall$ by Lemma 4.2. The conclusion of the rule $(\Rightarrow\exists)^3$ is obtained from its premise thus:

$$\cfrac{\cfrac{\mathcal{G}\,|\,\Gamma\Rightarrow\mathfrak{p}_0,\Delta\,|\,\mathfrak{p}_0\Rightarrow\exists xA\,|\,\mathfrak{p}_0\Rightarrow(A)^x_t}{\mathcal{G}\,|\,\Gamma\Rightarrow\mathfrak{p}_0,\Delta\,|\,\mathfrak{p}_0\Rightarrow\exists xA}\,(\Rightarrow\exists)^0}{\mathcal{G}\,|\,\Gamma\Rightarrow\exists xA,\Delta}\,(\text{den}_0).$$

The rule $(\forall\Rightarrow)^3$ is treated similarly to $(\Rightarrow\exists)^3$, but with an application of (den_1). Finally, the admissibility for $\text{G}^0\text{RP}\forall$ of the rules $(\Rightarrow\rightarrow)^3$, $(\Rightarrow\forall)^3$, and $(\exists\Rightarrow)^3$ follows easily from the admissibility of $(\text{ew})^*$ for $\text{G}^0\text{RP}\forall$. ⊣

Lemmas 5.3 and 5.8 below ensure that the rules (den_1) and (den_0) are admissible for $\text{G}^0\text{RP}\forall$. The idea of how we proceed is as follows.

Suppose that we have a $\text{G}^0\text{RP}\forall$-proof D supplemented with an application of (den_1) to the bottom hypersequent of D, e.g., as displayed in Figure 2; and we want to show that the conclusion of this application is $\text{G}^0\text{RP}\forall$-provable. We try to lift the application of (den_1) up in D, preserving at the bottom the original conclusion of this application. But

$$\dfrac{\dfrac{\begin{array}{c} D_1 \\ \mathcal{G} \,|\, \Gamma, A \to B, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, B, \mathfrak{p}_1 \Rightarrow A, \Delta \\ |\, \forall x C' \Rightarrow \mathfrak{p}_1 \,|\, (C')^x_t \Rightarrow \mathfrak{p}_1 \end{array}}{\mathcal{G} \,|\, \Gamma, A \to B, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, B, \mathfrak{p}_1 \Rightarrow A, \Delta \,|\, \overbrace{\forall x C'}^{C} \Rightarrow \mathfrak{p}_1}\,(\forall \Rightarrow)^0}{\dfrac{\mathcal{G} \,|\, \Gamma, A \to B, \mathfrak{p}_1 \Rightarrow \Delta \,|\, C \Rightarrow \mathfrak{p}_1}{\mathcal{G} \,|\, \Gamma, A \to B, C \Rightarrow \Delta}\,(\mathrm{den}_1) \text{ or } (\mathrm{gden}_1)}\,(\to \Rightarrow)^0$$

Figure 2. An example $\mathrm{G}^0\mathrm{RP}\forall$-proof D supplemented with an application of the rule (den_1).

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} D_1 \\ \mathcal{G} \,|\, \Gamma, A \to B, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, B, \mathfrak{p}_1 \Rightarrow A, \Delta \\ |\, \forall x C' \Rightarrow \mathfrak{p}_1 \,|\, (C')^x_t \Rightarrow \mathfrak{p}_1 \end{array}}{\mathcal{G} \,|\, \Gamma, A \to B, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, \mathfrak{p}_1 \Rightarrow \Delta \,|\, \Gamma, B, \mathfrak{p}_1 \Rightarrow A, \Delta \,|\, \overbrace{\forall x C'}^{C} \Rightarrow \mathfrak{p}_1}\,(\forall \Rightarrow)^0}{\mathcal{G} \,|\, \Gamma, A \to B, C \Rightarrow \Delta \,|\, \Gamma, C \Rightarrow \Delta \,|\, \Gamma, B, C \Rightarrow A, \Delta}\,(\mathrm{gden}_1)}{\mathcal{G} \,|\, \Gamma, A \to B, C \Rightarrow \Delta}\,(\to \Rightarrow)^0$$

Figure 3. Lifting the application of (gden_1) in the example $\mathrm{G}^0\mathrm{RP}\forall$-proof D.

we see that we actually need to lift up applications of a more general version of (den_1), such as

$$\frac{\mathcal{G} \,\big|\, \big[\Gamma_i, \mathfrak{p}_1 \Rightarrow \Delta_i\big]_{i \in 1..m} \,\big|\, \big[\Pi_j \Rightarrow \mathfrak{p}_1, \Sigma_j\big]_{j \in 1..n}}{\mathcal{G} \,\big|\, \big[\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\big]^{i \in 1..m}_{j \in 1..n}}\,(\mathrm{gden}_1),$$

where $m \geqslant 1$, $n \geqslant 1$, the premise contains a sequent of the form $C \Rightarrow \mathfrak{p}_1$, and $\mathfrak{p}_1$ does not occur in the conclusion.

The condition that the premise of the generalized version (gden_1) of (den_1) contains $C \Rightarrow \mathfrak{p}_1$ is in accordance with that the premise of (den_1) contains $C \Rightarrow \mathfrak{p}_1$ and that $\mathrm{G}^0\mathrm{RP}\forall$ is cumulative (so each hypersequent of a $\mathrm{G}^0\mathrm{RP}\forall$-proof for a hypersequent containing the sequent $C \Rightarrow \mathfrak{p}_1$ contains this sequent too). We make use of the condition in treating the base case where the premise of (gden_1) is a $\mathrm{G}^0\mathrm{RP}\forall$-axiom in order to show that the conclusion is $\mathrm{G}^0\mathrm{RP}\forall$-provable.

Now suppose that the application of (gden_1) in Figure 2 is lifted one level up in D so that all arising applications of (gden_1) are recursively lifted up to the axioms of D; and thus the conclusion of the application of (gden_1) in Figure 3 is shown to be $\mathrm{G}^0\mathrm{RP}\forall$-provable. Then from this conclusion of (gden_1), we obtain the desired hypersequent by the rule $(\to\Rightarrow)^0$ (in more complicated cases, by some rules that are admissible for $\mathrm{G}^0\mathrm{RP}\forall$).

In proving Lemma 5.3 (on the admissibility of (gden_1) for $\mathrm{G}^0\mathrm{RP}\forall$), we are going to preprocess a $\mathrm{G}^0\mathrm{RP}\forall$-proof of a hypersequent containing a sequent of the form $C \Rightarrow \mathfrak{p}_1$, using the following lemma.

LEMMA 5.2. *Suppose that* $\mathcal{H} = (\mathcal{G} \mid C \Rightarrow \mathfrak{p}_1)$ *is an axiom of* $\mathrm{G}^0\mathrm{RP}\forall$. *Then a* $\mathrm{G}^0\mathrm{RP}\forall$*-proof of* $\mathcal{H}$ *can be constructed in which each leaf hypersequent* $\mathcal{L}$ *contains a sequent of the form* $C_\mathcal{L} \Rightarrow \mathfrak{p}_1$ *or* $\Rightarrow \mathfrak{p}_1$, *where* $C_\mathcal{L}$ *is an atomic* RPL∀*-formula.*

PROOF. The RPL∀-formula C has the form

$$\mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n C' \quad \text{or} \quad \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n (A \to B),$$

where $\mathsf{Q}_1, \ldots, \mathsf{Q}_n$ are quantifiers and C' is an atomic RPL∀-formula. The desired $\mathrm{G}^0\mathrm{RP}\forall$-proof can be obtained from $\mathcal{H}$ by n backward applications of the rules $(\mathsf{Q}_1 \Rightarrow)^0$, …, $(\mathsf{Q}_n \Rightarrow)^0$, respectively, with any n new parameters as the proper terms or the proper parameters of these rule applications; and by one more backward application of the rule $(\to\Rightarrow)^0$ if $C = \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n (A \to B)$. ⊣

LEMMA 5.3 (admissibility of the generalization (gden_1) of (den_1) for $\mathrm{G}^0\mathrm{RP}\forall$). *Suppose that* $m \geqslant 1$, $n \geqslant 1$,

$$\mathcal{H} = \Big(\mathcal{G} \mid [\Gamma_i, \mathfrak{p}_1 \Rightarrow \Delta_i]_{i\in 1..m} \mid [\Pi_j \Rightarrow \mathfrak{p}_1, \Sigma_j]_{j\in 1..n}\Big),$$
$$\mathcal{H}' = \Big(\mathcal{G} \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]^{i\in 1..m}_{j\in 1..n}\Big),$$

$\mathfrak{p}_1$ *does not occur in* $\mathcal{H}'$, $\mathcal{H}$ *contains a sequent of the form* $C \Rightarrow \mathfrak{p}_1$, *and* $\vdash_{\mathrm{G}^0\mathrm{RP}\forall} \mathcal{H}$. *Then* $\vdash_{\mathrm{G}^0\mathrm{RP}\forall} \mathcal{H}'$.

PROOF. By Lemma 5.2, there exists a ($\mathrm{G}^0\mathrm{RP}\forall$-)proof D of $\mathcal{H}$ in which each leaf hypersequent $\mathcal{L}$ contains a sequent of the form $C_\mathcal{L} \Rightarrow \mathfrak{p}_1$ or $\Rightarrow \mathfrak{p}_1$, where $C_\mathcal{L}$ is an atomic RPL∀-formula. We transform D into a proof of $\mathcal{H}'$ using induction on the height of D.

1. Suppose that $\mathcal{H}$ is an axiom (of $\mathrm{G}^0\mathrm{RP}\forall$); i.e., $\vDash \mathcal{H}_{at}$. Without loss of generality we assume that

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \mid [\Gamma_i, \mathfrak{p}_1 \Rightarrow \Delta_i]_{i\in 1..k} \mid [\Pi_j \Rightarrow \mathfrak{p}_1, \Sigma_j]_{j\in 1..l}\Big),$$

where $0 \leqslant k \leqslant m$, $0 < l \leqslant n$, and the sequent $\Pi_1 \Rightarrow \mathfrak{p}_1, \Sigma_1$ has the form $C_1 \Rightarrow \mathfrak{p}_1$ or $\Rightarrow \mathfrak{p}_1$. Let $\mathcal{H}'_{at} = (\mathcal{H}')_{at}$.

1.1. Consider the case where $k \neq 0$. We have

$$\mathcal{H}'_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]^{i \in 1..k}_{j \in 1..l} \Big)$$

and want to show that $\vDash \mathcal{H}'_{at}$.

Suppose otherwise; i.e., for some hs-interpretation M and some M-valuation ν, there is no true sequent in $\mathcal{G}_{at}$, and for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

By the density of the set $\mathbb{R}$ of all real numbers, there exists $\xi \in \mathbb{R}$ such that, for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \xi - 1 < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

In particular, $\xi < \|\Pi_1\|_{M,\nu} - \|\Sigma_1\|_{M,\nu} + 1 = \|\Pi_1\|_{M,\nu} + 1 \leqslant 1$.

Define an hs-interpretation M_1 to be like M but set $|\mathfrak{p}_1|_{M_1} = \xi$. Since $\mathfrak{p}_1$ does not occur in $\mathcal{G}_{at}$, Γ_i, Δ_i $(i \in 1..k)$, Π_j, Σ_j $(j \in 1..l)$, we see that no sequent in $\mathcal{H}_{at}$ is true under the hs-interpretation M_1 and M_1-valuation ν. Hence $\nvDash \mathcal{H}_{at}$, a contradiction.

Therefore $\vDash \mathcal{H}'_{at}$, and so $\mathcal{H}'$ is an axiom.

1.2. Now consider the case where $k = 0$. Then

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Pi_j \Rightarrow \mathfrak{p}_1, \Sigma_j]_{j \in 1..l} \Big)$$

and $\mathcal{H}'_{at} = \mathcal{G}_{at}$. Since $\mathfrak{p}_1$ does not occur in $\mathcal{G}_{at}$, Π_j, Σ_j $(j \in 1..l)$, and hs-interpretations can take $\mathfrak{p}_1$ to negative real numbers whose absolute values are arbitrarily large, we conclude that $\vDash \mathcal{H}_{at}$ implies $\vDash \mathcal{G}_{at}$. Thus $\vDash \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

2. Suppose that the root hypersequent $\mathcal{H}$ in D is the conclusion of an application R of a rule $\mathcal{R}$, and $\mathcal{S}$ is the principal sequent occurrence in R.

2.1. If $\mathcal{S}$ is in the distinguished occurrence of $\mathcal{G}$ in $\mathcal{H}$, then we apply the induction hypothesis to the proof of each premise of R, and next we get a proof of $\mathcal{H}'$ by $\mathcal{R}$.

2.2. Now suppose that $\mathcal{S}$ is not in the distinguished occurrence of $\mathcal{G}$ in $\mathcal{H}$, and for definiteness assume that $\mathcal{S}$ is the distinguished occurrence of $\Gamma_1, \mathfrak{p}_1 \Rightarrow \Delta_1$ in $\mathcal{H}$.

2.2.1. If $\mathcal{R}$ is the rule $(\rightarrow\Rightarrow)^0$, then $\Gamma_1 = (\Gamma'_1, A \rightarrow B)$ for some Γ'_1, and the proof D has the form:

$$\frac{\begin{array}{c} D_1 \\ \mathcal{H} \mid \Gamma_1', \mathfrak{p}_1 \Rightarrow \Delta_1 \mid \Gamma_1', B, \mathfrak{p}_1 \Rightarrow A, \Delta_1 \end{array}}{\mathcal{H}} \, (\rightarrow\Rightarrow)^0,$$

where $\mathcal{H}$ is

$$\mathcal{G} \mid \Gamma_1', A \rightarrow B,\, \mathfrak{p}_1 \Rightarrow \Delta_1 \mid [\Gamma_i, \mathfrak{p}_1 \Rightarrow \Delta_i]_{i\in 2..m} \mid [\Pi_j \Rightarrow \mathfrak{p}_1, \Sigma_j]_{j\in 1..n}.$$

By the induction hypothesis, we transform D_1 into a proof of

$$\mathcal{H}' \mid [\Gamma_1', \Pi_j \Rightarrow \Delta_1, \Sigma_j]_{j\in 1..n} \mid [\Gamma_1', B, \Pi_j \Rightarrow A, \Delta_1, \Sigma_j]_{j\in 1..n},$$

whence we obtain a proof for $\mathcal{H}'$ by n applications of $(\rightarrow\Rightarrow)^0$.

2.2.2. The rules $(\forall\Rightarrow)^0$ and $(\Rightarrow\exists)^0$ are treated similarly to the rule $(\rightarrow\Rightarrow)^0$ in item 2.2.1.

2.2.3. If $\mathcal{R}$ is $(\Rightarrow\rightarrow)^0$, then $\Delta_1 = (A \rightarrow B,\, \Delta_1')$ for some Δ_1', and the proof D looks like this:

$$\frac{\begin{array}{c} D_1 \\ \mathcal{H} \mid \Gamma_1, \mathfrak{p}_1 \Rightarrow \Delta_1'; \end{array} \quad \begin{array}{c} D_2 \\ \mathcal{H} \mid \Gamma_1, A, \mathfrak{p}_1 \Rightarrow B, \Delta_1' \end{array}}{\mathcal{H}} \, (\Rightarrow\rightarrow)^0.$$

By the induction hypothesis applied to the proofs D_1 and D_2, we construct proofs of

$$\mathcal{H}' \mid [\Gamma_1, \Pi_j \Rightarrow \Delta_1', \Sigma_j]_{j\in 1..n} \quad \text{and} \quad \mathcal{H}' \mid [\Gamma_1, A, \Pi_j \Rightarrow B, \Delta_1', \Sigma_j]_{j\in 1..n},$$

respectively; whence we get a proof of $\mathcal{H}'$ by Lemma 5.4 below.

2.2.4. If $\mathcal{R}$ is $(\Rightarrow\forall)^0$, then $\Delta_1 = (\forall x A,\, \Delta_1')$ for some Δ_1', and the proof D has the form:

$$\frac{\begin{array}{c} D_1 \\ \mathcal{H} \mid \Gamma_1, \mathfrak{p}_1 \Rightarrow (A)^x_a, \Delta_1' \end{array}}{\mathcal{H}} \, (\Rightarrow\forall)^0,$$

where a does not occur in $\mathcal{H}$ (and hence, a does not occur in $\mathcal{H}'$). Using the induction hypothesis, we transform D_1 into a proof of

$$\mathcal{H}' \mid [\Gamma_1, \Pi_j \Rightarrow (A)^x_a, \Delta_1', \Sigma_j]_{j\in 1..n},$$

whence we obtain a proof of $\mathcal{H}'$ by Lemma 5.6.

2.2.5. The rule $(\exists\Rightarrow)^0$ is treated similarly to the rule $(\Rightarrow\forall)^0$ in item 2.2.4, using Lemma 5.7. ⊣

LEMMA 5.4. *Suppose that* $n \geqslant 1$,

$$\mathcal{H}'_n = \Bigl(\mathcal{G} \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i\in 1..n}\Bigr), \quad \mathcal{H}''_n = \Bigl(\mathcal{G} \,|\, [\Gamma_i, A \Rightarrow B, \Delta_i]_{i\in 1..n}\Bigr),$$

$\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}'_n$, $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}''_n$, *and* $[\Gamma_i \Rightarrow A \to B, \Delta_i]_{i\in 1..n} \subseteq \mathcal{G}$.
Then $\vdash_{\mathrm{G^0RP\forall}} \mathcal{G}$.

PROOF. We proceed by induction on n. If $n = 1$, then $\mathcal{G}$ is the conclusion of the rule $(\Rightarrow\to)^0$ applied to $\mathcal{H}'_1$ and $\mathcal{H}''_1$.

Now suppose that $n \geqslant 2$. By Lemma 5.5 below, from $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}'_n$ and $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}''_n$ it follows that the hypersequent

$$\mathcal{H}_n = \Bigl(\mathcal{G} \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i\in 1..(n-1)} \,|\, \Gamma_n, A \Rightarrow B, \Delta_n\Bigr)$$

is $\mathrm{G^0RP\forall}$-provable. Applying the rule $(\Rightarrow\to)^0$ to $\mathcal{H}'_n$ and $\mathcal{H}_n$ gives

$$\mathcal{H}'_{n-1} = \Bigl(\mathcal{G} \,|\, [\Gamma_i \Rightarrow \Delta_i]_{i\in 1..(n-1)}\Bigr).$$

Likewise we arrive at the $\mathrm{G^0RP\forall}$-provable hypersequent

$$\mathcal{H}''_{n-1} = \Bigl(\mathcal{G} \,|\, [\Gamma_i, A \Rightarrow B, \Delta_i]_{i\in 1..(n-1)}\Bigr).$$

Finally, by applying the induction hypothesis to $\mathcal{H}'_{n-1}$ and $\mathcal{H}''_{n-1}$, we get $\vdash_{\mathrm{G^0RP\forall}} \mathcal{G}$. ⊣

LEMMA 5.5. *Suppose that* $n \geqslant 2$,

$$\mathcal{H}' = \Bigl(\mathcal{G} \,|\, [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i\in 1..n}\Bigr), \quad \mathcal{H}'' = \Bigl(\mathcal{G} \,|\, [\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i]_{i\in 1..n}\Bigr),$$

$\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}'$, *and* $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}''$. *Then*

$$\vdash_{\mathrm{G^0RP\forall}} \Bigl(\mathcal{G} \,|\, [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i\in 1..(n-1)} \,|\, \Gamma_n, \Pi'' \Rightarrow \Sigma'', \Delta_n\Bigr).$$

PROOF. For each $k \in 1..n$, we put

$$\mathcal{H}_k = \Bigl(\mathcal{G} \,|\, [\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i]_{i\in 1..(n-1)} \,|\, [\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i]_{i\in k..n}\Bigr).$$

We can get $\mathcal{H}_1$ from $\mathcal{H}''$ by the rule $(\mathrm{ew})^*$. For each $k \in 1..(n-1)$, Figure 4 shows how to obtain $\mathcal{H}_{k+1}$ from $\mathcal{H}'$ and $\mathcal{H}_k$ using the rules $(\mathrm{ew})^*$, $(\mathrm{ec})^*$, $(\mathrm{split})^*$, and $(\mathrm{mix})^*$. These four rules are admissible for $\mathrm{G^0RP\forall}$ by Lemma 4.2. So $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}_n$ as required. ⊣

$$
\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{H}'}{\mathcal{G} \,\big|\, \big[\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i\big]_{i\in 1..n} \,\big|\, \big[\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i\big]_{i\in (k+1)..n}}\,{\scriptstyle (\text{ew})^*\times (n-k)} ; \qquad \mathcal{H}_k}{\begin{array}{c}\mathcal{G} \,\big|\, \big[\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i\big]_{i\in 1..(n-1)} \,\big|\, \big[\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i\big]_{i\in (k+1)..n} \\ \big|\, \Gamma_n, \Pi', \Gamma_k, \Pi'' \Rightarrow \Sigma', \Delta_n, \Sigma'', \Delta_k\end{array}}\,(\text{mix})^*}{\begin{array}{c}\mathcal{G} \,\big|\, \big[\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i\big]_{i\in 1..(n-1)} \,\big|\, \big[\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i\big]_{i\in (k+1)..n} \\ \big|\, \Gamma_k, \Pi' \Rightarrow \Sigma', \Delta_k \,\big|\, \Gamma_n, \Pi'' \Rightarrow \Sigma'', \Delta_n\end{array}}\,(\text{split})^*}{\mathcal{G} \,\big|\, \big[\Gamma_i, \Pi' \Rightarrow \Sigma', \Delta_i\big]_{i\in 1..(n-1)} \,\big|\, \big[\Gamma_i, \Pi'' \Rightarrow \Sigma'', \Delta_i\big]_{i\in (k+1)..n}}\,(\text{ec})^*\times 2
$$

Figure 4. Obtaining the bottom hypersequent $\mathcal{H}_{k+1}$ from $\mathcal{H}'$ and $\mathcal{H}_k$.

Lemma 5.6. *Suppose that* $n \geqslant 1$,

$$\vdash_{\mathrm{G^0RP\forall}} \Big(\mathcal{G} \,\big|\, \big[\Gamma_i \Rightarrow (A)^x_a, \Delta_i\big]_{i\in 1..n}\Big),$$

$\big[\Gamma_i \Rightarrow \forall x A, \Delta_i\big]_{i\in 1..n} \subseteq \mathcal{G}$, *and the parameter* a *does not occur in* $\mathcal{G}$. *Then* $\vdash_{\mathrm{G^0RP\forall}} \mathcal{G}$.

Proof. We can obtain $\mathcal{G}$ from $\mathcal{G} \,\big|\, \big[\Gamma_i \Rightarrow (A)^x_{a_i}, \Delta_i\big]_{i\in 1..n}$ by n applications of the rule $(\Rightarrow \forall)^0$, provided that the parameters $a_1, \ldots, a_n$ are distinct and none of them occurs in $\mathcal{G}$.

Therefore, it suffices to prove the following claim for every $n \geqslant 1$: *suppose that*

$$\mathcal{H}(a) = \Big(\mathcal{G}_0 \,\big|\, \big[\Gamma_i \Rightarrow (A)^x_a, \Delta_i\big]_{i\in 1..n}\Big),$$

$\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}(a)$, *and the parameters* $a, a_1, \ldots, a_n$ *are distinct and none of them occurs in* $\mathcal{G}_0$, A, Γ_i, Δ_i $(i \in 1..n)$*; then*

$$\vdash_{\mathrm{G^0RP\forall}} \Big(\mathcal{G}_0 \,\big|\, \big[\Gamma_i \Rightarrow (A)^x_{a_i}, \Delta_i\big]_{i\in 1..n}\Big).$$

We use induction on n. In the case $n = 1$, the claim is obvious.

Suppose that $n \geqslant 2$. Clearly, $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}(a)$ implies $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}(a_n)$. By Lemma 5.5, from $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}(a)$ and $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}(a_n)$ it follows that

$$\vdash_{\mathrm{G^0RP\forall}} \Big(\mathcal{G}_0 \,\big|\, \big[\Gamma_i \Rightarrow (A)^x_a, \Delta_i\big]_{i\in 1..(n-1)} \,\big|\, \Gamma_n \Rightarrow (A)^x_{a_n}, \Delta_n\Big),$$

whence by the induction hypothesis, we get what is required. ⊣

LEMMA 5.7. *Suppose that* $n \geqslant 1$,

$$\vdash_{\mathrm{G^0RP\forall}} \Big(\mathcal{G} \,|\, \big[\Gamma_i, (A)^x_a \Rightarrow \Delta_i\big]_{i\in 1..n}\Big),$$

$\big[\Gamma_i, \exists x A \Rightarrow \Delta_i\big]_{i\in 1..n} \subseteq \mathcal{G}$, *and the parameter* a *does not occur in* $\mathcal{G}$. *Then* $\vdash_{\mathrm{G^0RP\forall}} \mathcal{G}$.

PROOF. This proof is similar to that of Lemma 5.6. ⊣

For a finite multiset Δ, by $\#(\Delta)$ we denote the number of its elements, taking their multiplicities into account.

LEMMA 5.8 (admissibility of a generalization of (den_0) for $\mathrm{G^0RP\forall}$). *Suppose that* $m \geqslant 1$, $n \geqslant 1$,

$$\mathcal{H} = \Big(\mathcal{G} \,|\, \big[\Gamma_i, \mathfrak{p}_0 \Rightarrow \Delta_i\big]_{i\in 1..m} \,|\, \big[\Pi_j \Rightarrow \mathfrak{p}_0, \Sigma_j\big]_{j\in 1..n}\Big),$$
$$\mathcal{H}' = \Big(\mathcal{G} \,|\, \big[\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\big]^{i\in 1..m}_{j\in 1..n}\Big),$$

$\mathfrak{p}_0$ *does not occur in* $\mathcal{H}'$, $\mathcal{H}$ *contains a sequent of the form* $\mathfrak{p}_0 \Rightarrow C$, *and* $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}$. *Then* $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}'$.

PROOF. Using Lemma 5.9 below, we find a ($\mathrm{G^0RP\forall}$-)proof D of $\mathcal{H}$ in which each leaf hypersequent $\mathcal{L}$ contains an atomic sequent of the form $\Gamma_{\mathcal{L}}, \mathfrak{p}_0 \Rightarrow \Delta_{\mathcal{L}}$, where $\#(\Delta_{\mathcal{L}}) \leqslant 1$ and no semipropositional variable occurs in $\Gamma_{\mathcal{L}}$ or $\Delta_{\mathcal{L}}$. We show that $\vdash_{\mathrm{G^0RP\forall}} \mathcal{H}'$ by induction on the height of D.

1. Suppose that $\mathcal{H}$ is an axiom (of $\mathrm{G^0RP\forall}$); i.e., $\vDash \mathcal{H}_{at}$. We can harmlessly assume that

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,|\, \big[\Gamma_i, \mathfrak{p}_0 \Rightarrow \Delta_i\big]_{i\in 1..k} \,|\, \big[\Pi_j \Rightarrow \mathfrak{p}_0, \Sigma_j\big]_{j\in 1..l}\Big),$$

where $0 < k \leqslant m$, $0 \leqslant l \leqslant n$ and $\#(\Delta_1) \leqslant 1$. Let $\mathcal{H}'_{at} = (\mathcal{H}')_{at}$.

1.1. Consider the case where $l \neq 0$. We have

$$\mathcal{H}'_{at} = \Big(\mathcal{G}_{at} \,|\, \big[\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\big]^{i\in 1..k}_{j\in 1..l}\Big).$$

Suppose for a contradiction that $\nvDash \mathcal{H}'_{at}$; i.e., for some hs-interpretation M and some M-valuation ν, there is no true sequent in $\mathcal{G}_{at}$, and for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

Hence, for some real number ξ and for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \xi - 1 < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

In particular, $\xi > \|\Delta_1\|_{M,\nu} - \|\Gamma_1\|_{M,\nu} + 1 \geqslant \|\Delta_1\|_{M,\nu} + 1 \geqslant 0$.

Define an hs-interpretation M_0 to be the same as M except that $|\mathfrak{p}_0|_{M_0} = \xi$. Since $\mathfrak{p}_0$ does not occur in $\mathcal{G}_{at}$, Γ_i, Δ_i $(i \in 1..k)$, Π_j, Σ_j $(j \in 1..l)$, it follows that $\mathcal{H}_{at}$ has no true sequent under the hs-interpretation M_0 and M_0-valuation ν. So $\nvDash \mathcal{H}_{at}$, a contradiction.

Thus $\vDash \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

1.2. Now consider the case where $l = 0$. Then

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Gamma_i, \mathfrak{p}_0 \Rightarrow \Delta_i]_{i \in 1..k}\Big)$$

and $\mathcal{H}'_{at} = \mathcal{G}_{at}$. Since $\mathfrak{p}_0$ does not occur in $\mathcal{G}_{at}$, Γ_i, Δ_i $(i \in 1..k)$, and $\mathfrak{p}_0$ can assume arbitrarily large values under hs-interpretations, we see that $\vDash \mathcal{H}_{at}$ implies $\vDash \mathcal{G}_{at}$. So $\vDash \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

2. It remains to consider the case where the root hypersequent $\mathcal{H}$ in D is the conclusion of a rule application. But the argument for this case can be obtained from item 2 of the proof of Lemma 5.3 by replacing $\mathfrak{p}_1$ with $\mathfrak{p}_0$. ⊣

LEMMA 5.9. *Suppose that* $\mathcal{H} = (\mathcal{G} \,|\, \Gamma, \mathfrak{p}_0 \Rightarrow \Delta)$ *is an axiom of* $\mathrm{G^0RP}\forall$, $\#(\Delta) \leqslant 1$, *and no semipropositional variable occurs in* Γ *or* Δ. *Then a* $\mathrm{G^0RP}\forall$*-proof of* $\mathcal{H}$ *can be constructed in which each leaf hypersequent* $\mathcal{L}$ *contains an atomic sequent of the form* $\Gamma_{\mathcal{L}}, \mathfrak{p}_0 \Rightarrow \Delta_{\mathcal{L}}$, *where* $\#(\Delta_{\mathcal{L}}) \leqslant 1$ *and no semipropositional variable occurs in* $\Gamma_{\mathcal{L}}$ *or* $\Delta_{\mathcal{L}}$.

PROOF. We proceed by induction on the number of logical symbol occurrences in the sequent $S = (\Gamma, \mathfrak{p}_0 \Rightarrow \Delta)$. If S is atomic, then $\mathcal{H}$ is the desired ($\mathrm{G^0RP}\forall$-)proof. Otherwise, S has one of the forms given in items 1–4 below.

1. Suppose that $S = (\Gamma', A \to B, \mathfrak{p}_0 \Rightarrow \Delta)$. By applying the rule $(\to\Rightarrow)^0$ backwards to the distinguished occurrence of $A \to B$ in $\mathcal{H}$, we get the ($\mathrm{G^0RP}\forall$-)axiom $\mathcal{H}_1 = (\mathcal{G} \,|\, S \,|\, \Gamma', \mathfrak{p}_0 \Rightarrow \Delta \,|\, \Gamma', B, \mathfrak{p}_0 \Rightarrow A, \Delta)$. By the induction hypothesis applied to $\mathcal{H}_1$ with $(\Gamma', \mathfrak{p}_0 \Rightarrow \Delta)$ as S, we obtain the desired proof of $\mathcal{H}$.

2. Suppose that $S = (\Gamma, \mathfrak{p}_0 \Rightarrow A \to B)$. Applying the rule $(\Rightarrow\to)^0$ backwards to the distinguished occurrence of $A \to B$ in $\mathcal{H}$ yields the axioms $(\mathcal{G} \,|\, S \,|\, \Gamma, \mathfrak{p}_0 \Rightarrow)$ and $(\mathcal{G} \,|\, S \,|\, \Gamma, A, \mathfrak{p}_0 \Rightarrow B)$, to each of which the induction hypothesis applies.

3. Suppose that $S = (\Gamma, \mathfrak{p}_0 \Rightarrow \mathsf{Q}xA)$, where Q is a quantifier. We apply the rule $(\Rightarrow \forall)^0$ or $(\Rightarrow \exists)^0$ backwards to the distinguished occurrence of $\mathsf{Q}xA$ in $\mathcal{H}$, with a new parameter a as the proper parameter or the proper term, respectively. Thus we get the axiom $(\mathcal{G} \,|\, S \,|\, \Gamma, \mathfrak{p}_0 \Rightarrow (A)^x_a)$ and then use the induction hypothesis.

4. The case where $S = (\Gamma', \mathsf{Q}xA, \mathfrak{p}_0 \Rightarrow \Delta)$, with Q being a quantifier, is treated similarly to case 3. ⊣

Remark 5.2. The proofs of Lemmas 5.3 and 5.8 can be easily combined to establish the admissibility of the nonstandard density rule (given in Remark 5.1 on p. 284) for $\mathrm{G}^0\mathrm{RP}\forall$ (with the notion of a hypersequent expanded as mentioned in Remark 5.1).

THEOREM 5.10 (equivalence of $\mathrm{G}^0\mathrm{RP}\forall$, $\mathrm{G}^{\hat{1}}\mathrm{RP}\forall$, and $\mathrm{G}^3\mathrm{RP}\forall$). *The following are equivalent:* (a) $\vdash_{\mathrm{G}^0\mathrm{RP}\forall} \mathcal{H}$; (b) $\vdash_{\mathrm{G}^{\hat{1}}\mathrm{RP}\forall} \mathcal{H}$; (c) $\vdash_{\mathrm{G}^3\mathrm{RP}\forall} \mathcal{H}$.

PROOF. (a) implies (b) by Theorem 4.5. If (b), then (c) by Theorem 4.6. Finally, (c) implies (a) by Lemmas 5.1, 5.3, and 5.8. ⊣

COROLLARY 5.11. *For each* $i = 0, \hat{1}, 3$, *the calculus* $\mathrm{G}^i\mathrm{RP}\forall$ *is a conservative extension of* $\mathrm{G}^1\mathrm{RP}\forall$; *i.e., for any* $\mathrm{RPL}\forall^1$*-hypersequent* $\mathcal{H}$, $\vdash_{\mathrm{G}^i\mathrm{RP}\forall} \mathcal{H}$ *iff* $\vdash_{\mathrm{G}^1\mathrm{RP}\forall} \mathcal{H}$.

PROOF. Immediate from Theorem 5.10 and Remark 2.3. ⊣

COROLLARY 5.12.
(a) *For each* $i = 0, 1, \hat{1}, 3$, *the calculus* $\mathrm{G}^i\mathrm{RP}\forall$ *is a conservative extension of the calculus* $\mathrm{GRP}\forall$; *i.e., for any* $\mathrm{RPL}\forall$*-hypersequent* $\mathcal{H}$, $\vdash_{\mathrm{G}^i\mathrm{RP}\forall} \mathcal{H}$ *iff* $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$.
(b) *The calculus* $\mathrm{GRP}\forall$ *is a conservative extension of the calculus* $\mathrm{GŁ}\forall$; *i.e., for any* $\mathrm{Ł}\forall$*-hypersequent* $\mathcal{H}$, $\vdash_{\mathrm{GRP}\forall} \mathcal{H}$ *iff* $\vdash_{\mathrm{GŁ}\forall} \mathcal{H}$.

PROOF. (a) follows from Corollary 5.11 and Theorem 4.4; (b) is just (a reminder of) Proposition 3.1. ⊣

6. Comparing hypersequent calculi for RPL∀ with Hájek's calculus for RPL∀

In addition to Gentzen-type calculi for the logics RPL∀ and Ł∀, now we consider the calculi HRP∀ and HŁ∀, which are some of Hájek's variants of Hilbert-type calculi for RPL∀ and Ł∀, respectively (cf. [18]). In this section we give previously known relationships between HRP∀, HŁ∀, and GŁ∀; then we establish that G^3RP∀ (as well as G^0RP∀ and GRP∀) extended with the cut rule proves exactly the same RPL∀-sentences as HRP∀.

First let us formulate the calculi HRP∀ and HŁ∀.

The axiom schemes of HRP∀ are:

(Ł1) $A \to (B \to A)$;
(Ł2) $(A \to B) \to ((B \to C) \to (A \to C))$;
(Ł3) $(\neg A \to \neg B) \to (B \to A)$, where $\neg C$ is short for $(C \to \bar{0})$;
(Ł4) $((A \to B) \to B) \to ((B \to A) \to A)$;
(tc1) $(\bar{r}_1 \to \bar{r}_2) \to \bar{r}$, where $r = \min(1 - r_1 + r_2, 1)$;
(tc2) $\bar{r} \to (\bar{r}_1 \to \bar{r}_2)$, where $r = \min(1 - r_1 + r_2, 1)$;
(∀1) $\forall x A \to (A)^x_t$, where t is a term that is not necessarily closed and is free for x in A;
(∀2) $\forall x(A \to B) \to (A \to \forall x B)$, where x does not occur free in A;
(∃1) $(A)^x_t \to \exists x A$, where t is a term that is not necessarily closed and is free for x in A;
(∃2) $\forall x(A \to B) \to (\exists x A \to B)$, where x does not occur free in B.

The inference rules of HRP∀ are:

$$\frac{A; \quad A \to B}{B}\ (\mathrm{mp}), \qquad \frac{A}{\forall x A}\ (\mathrm{gen}).$$

HŁ∀ is obtained from this formulation of HRP∀ by requiring A, B, and C to be Ł∀-formulas and removing the axiom schemes (tc1) and (tc2).

THEOREM 6.1 ([19]). HRP∀ *is a conservative extension of* HŁ∀; *i.e., for any* Ł∀*-formula* A, $\vdash_{\mathrm{HRP}\forall} A$ *iff* $\vdash_{\mathrm{HŁ}\forall} A$.

As a hypersequent counterpart of the rule (mp) of HRP∀, we consider the following cut rule (cf., e.g., [23, Section 4.2]):

$$\frac{\mathcal{G} \,|\, \Gamma_1 \Rightarrow C, \Delta_1; \quad \mathcal{G} \,|\, \Gamma_2, C \Rightarrow \Delta_2}{\mathcal{G} \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{cut}).$$

Let $(\mathrm{cut})^{Ł}$ be the version of the rule (cut) whose premises and conclusion are restricted to Ł∀-hypersequents.

THEOREM 6.2 ([2]).
(a) *The rule* $(\mathrm{cut})^{Ł}$ *is not admissible for* GŁ∀.
(b) *For any* Ł∀*-sentence* A, $\vdash_{\mathrm{GŁ}\forall+(\mathrm{cut})^{Ł}} A$ *iff* $\vdash_{\mathrm{HŁ}\forall} A$.

PROPOSITION 6.3. *Let* $\mathfrak{C}$ *be any of the calculi* GRP∀ *and* G^iRP∀, *where* $i \in \{0, 1, \hat{1}, 3\}$. *Then the rule* $(\mathrm{cut})^{\text{Ł}}$ (*and hence* (cut)) *is not admissible for* $\mathfrak{C}$.

PROOF. In [21, p. 268], for the Ł∀-sentence $A = \exists x \forall y(P(x) \to P(y))$ with P being a unary predicate symbol, it is shown that $\nvdash_{\mathrm{GŁ}\forall} A$, and a proof in GŁ∀+$(\mathrm{cut})^{\text{Ł}}$ is constructed of the form

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \mathcal{H}_1; & \mathcal{H}_2 \end{array}}{\Rightarrow A}\ (\mathrm{cut})^{\text{Ł}},$$

where D_1 and D_2 are GŁ∀-proofs for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. By Corollary 5.12, we have $\vdash_{\mathfrak{C}} \mathcal{H}_1$ and $\vdash_{\mathfrak{C}} \mathcal{H}_2$, whence $\vdash_{\mathfrak{C}+(\mathrm{cut})^{\text{Ł}}} A$. But by the same corollary, $\nvdash_{\mathrm{GŁ}\forall} A$ implies $\nvdash_{\mathfrak{C}} A$. ⊣

The rest of this section is devoted to a proof of the next theorem.

THEOREM 6.4. *For any* RPL∀*-sentence* A*, the following are equivalent*: $\vdash_{\mathrm{HRP}\forall} A$; $\vdash_{\mathrm{G^3RP}\forall+(\mathrm{cut})} A$; $\vdash_{\mathrm{G^0RP}\forall+(\mathrm{cut})} A$; $\vdash_{\mathrm{GRP}\forall+(\mathrm{cut})} A$.

In proving this theorem, we will employ the calculus $\widehat{\mathrm{H}}$RP∀ obtained from HRP∀ thus: t in the axiom schemes (∀1) and (∃1) is taken to be a closed term, and the inference rule (gen) is replaced by the rule

$$\frac{(A)^x_a}{\forall x A}\ (\widehat{\mathrm{gen}}),$$

where a is a parameter not occurring in A.

We will also use the cumulative cancellation rule

$$\frac{\mathcal{G} \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma, C \Rightarrow C, \Delta}{\mathcal{G} \,|\, \Gamma \Rightarrow \Delta}\ (\mathrm{ccan});$$

cf. [10, Section 4.1] and [23, Section 4.3.5]. We remark that the (noncumulative) cancellation rule was introduced in [10] as a variant of the cut rule to establish cut elimination for the propositional fragment of the calculus GŁ∀ via elimination of the cancellation rule.

PROOF of THEOREM 6.4. Given an RPL∀-sentence A, we demonstrate the following chain of implications:

$$\vdash_{\mathrm{HRP}\forall} A \overset{(6.5)}{\Longrightarrow} \vdash_{\widehat{\mathrm{H}}\mathrm{RP}\forall} A \overset{(6.6)}{\Longrightarrow} \vdash_{\mathrm{G^3RP}\forall+(\mathrm{cut})} A \overset{(6.8)}{\Longrightarrow} \vdash_{\mathrm{G^0RP}\forall+(\mathrm{ccan})} A$$
$$\overset{(6.9)}{\Longrightarrow} \vdash_{\mathrm{G^0RP}\forall+(\mathrm{cut})} A \overset{(6.10)}{\Longrightarrow} \vdash_{\mathrm{GRP}\forall+(\mathrm{cut})} A \overset{(6.11)}{\Longrightarrow} \vdash_{\mathrm{HRP}\forall} A.$$

Implications (6.5), (6.6), (6.8)–(6.10) are established, respectively, in Lemmas 6.5, 6.6, 6.8–6.10 of Subsection 6.1; and implication (6.11), in Lemma 6.11 of Subsection 6.2. ⊣

Before going into the details of this proof, it should be observed that, in general, adding the same inference rule to equivalent calculi (i.e., those that prove exactly the same objects) may produce nonequivalent calculi.

For example, let $\mathfrak{C}_1$ be the calculus with the only axiom a and the only inference rule a/b, and let $\mathfrak{C}_2$ be $\mathfrak{C}_1 + c/d$ (where c/d is another rule). Then the calculi $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equivalent as each of them proves exactly a and b. However, the calculi $\mathfrak{C}_1 + b/c$ and $\mathfrak{C}_2 + b/c$ are nonequivalent as the latter proves d, which is unprovable in the former.

So in proving, e.g., that $\vdash_{\mathrm{G^3RP\forall+(cut)}} A$ implies $\vdash_{\mathrm{G^0RP\forall+(cut)}} A$, we have to rely on the particular features of the calculi involved.

6.1. Comparing G^{3}RP∀ with HRP∀: the syntactic part

In this subsection we establish implications (6.5), (6.6), (6.8)–(6.10), given in the above plan of the proof of Theorem 6.4, by demonstrating the respective lemmas. Here one or another lemma may assert not only that the respective implication holds but also that its converse holds if the latter is not hard to prove syntactically.

LEMMA 6.5. *For any* RPL∀*-sentence* A, $\vdash_{\mathrm{HRP\forall}} A$ *iff* $\vdash_{\widehat{\mathrm{H}}\mathrm{RP\forall}} A$.

We omit the proof of Lemma 6.5 because the proof is not complicated and does not differ from that of the similar assertion for appropriate variants of classical first-order Hilbert-type calculi.

LEMMA 6.6. *For any* RPL∀*-formula* A, *if* $\vdash_{\widehat{\mathrm{H}}\mathrm{RP\forall}} A$, *then* $\vdash_{\mathrm{G^3RP\forall+(cut)}} A$.

PROOF. The rule $(\widehat{\mathrm{gen}})$ of $\widehat{\mathrm{H}}$RP∀ is derivable in G^{3}RP∀ since G^{3}RP∀ contains the rule $(\Rightarrow\forall)^3$.

On the left in Figure 5, we obtain the conclusion of the rule (mp) from its premises and the hypersequent $\mathcal{H} = (A, A \to B \Rightarrow B)$ using the rule (cut); and on the right, we give a GRP∀-proof of $\mathcal{H}$. But $\vdash_{\mathrm{G^3RP\forall}} \mathcal{H}$ by Corollary 5.12. So (mp) is derivable in G^{3}RP∀+(cut).

To finish the proof, it is enough to show that all the axioms of $\widehat{\mathrm{H}}$RP∀ are G^{3}RP∀-provable.

Take an instance L of any of the axiom schemes (Ł1)–(Ł4), (tc1), (tc2). Since L is valid even if in L the RPL∀-formulas A, B, C from the

$$\dfrac{\Rightarrow A; \qquad \dfrac{\Rightarrow A \to B; \quad A, A \to B \Rightarrow B}{A \Rightarrow B}\,(\mathrm{cut})}{\Rightarrow B}\,(\mathrm{cut}) \qquad\qquad \dfrac{\dfrac{A \Rightarrow A; \quad B \Rightarrow B}{A, B \Rightarrow A, B}\,(\mathrm{mix})^{\mathrm{P}}}{A, A \to B \Rightarrow B}\,(\to\Rightarrow)^{\mathrm{P}}$$

Figure 5. Proofs for showing the derivability of (mp) in G^{3}RP∀+(cut).

$$\dfrac{\Rightarrow; \qquad \dfrac{(\mathrm{wl})^{\mathrm{P}}\,\dfrac{\Rightarrow}{\forall x(A \to B) \Rightarrow;} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(A)^x_a \Rightarrow (A)^x_a; \quad B \Rightarrow B}{(A)^x_a, B \Rightarrow (A)^x_a, B}\,(\mathrm{mix})^{\mathrm{P}}}{(A)^x_a \to B, (A)^x_a \Rightarrow B}\,(\to\Rightarrow)^{\mathrm{P}}}{\forall x(A \to B), (A)^x_a \Rightarrow B}\,(\forall\Rightarrow)^{\mathrm{P}}}{\forall x(A \to B), \exists x A \Rightarrow B}\,(\exists\Rightarrow)^{\mathrm{P}}}{\forall x(A \to B) \Rightarrow (\exists x A \to B)}\,(\Rightarrow\to)^{\mathrm{P}}}{\Rightarrow \forall x(A \to B) \to (\exists x A \to B)}\,(\Rightarrow\to)^{\mathrm{P}}$$

Figure 6. A GRP∀-proof of an instance of the axiom scheme (∃2).

formulation of the schemes are treated as distinct propositional variables, it follows by Lemma 3.6 that $\vdash_{\mathrm{GRP}\forall} L$. Now by Corollary 5.12, we get $\vdash_{\mathrm{G^3RP}\forall} L$.

Finally, let Q be an instance of the axiom scheme (∀1), (∀2), (∃1), or (∃2). Then we can construct a GRP∀-proof of Q. Indeed, in the cases of (∀1) and (∃1), this is trivial; in the case of (∃2), such a GRP∀-proof is shown in Figure 6 (where a does not occur in A, B); and in the case of (∀2), a GRP∀-proof of Q is constructed similarly. By Corollary 5.12, $\vdash_{\mathrm{GRP}\forall} Q$ implies $\vdash_{\mathrm{G^3RP}\forall} Q$ as required. ⊣

LEMMA 6.7. *Lemma 4.2 holds for* G^{0}RP∀+(ccan) *in place of* G^{0}RP∀.

PROOF. For each rule mentioned in Lemma 4.2, except for the rule (split)*, its admissibility or hp-admissibility for G^{0}RP∀+(ccan) is established just as in the proof of Lemma 4.2. So, in particular, the rule (ec)* is hp-admissible for G^{0}RP∀+(ccan).

Given the hp-admissibility of (ec)* for G^{0}RP∀+(ccan), the proof of the hp-admissibility of (split)* for G^{0}RP∀+(ccan) is similar to item 4 of the proof of Lemma 4.2 (and thus to the proof of Lemma 7 in [16]); we only need to consider one more case. As in the proof of Lemma 7 in [16], by induction on the height of a proof D_1 of $\mathcal{G} \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ (in G^{0}RP∀+(ccan) now), we show that D_1 can be transformed into a proof of $\mathcal{G} \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2$ whose height is no greater than the height of D_1. We add the case where the proof D_1 has the form:

$$\dfrac{\overset{D_0}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid \Gamma_1, \Gamma_2, A \Rightarrow A, \Delta_1, \Delta_2}}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\text{ccan}).$$

In this case, using the induction hypothesis twice, we split the two sequent occurrences distinguished in the bottom hypersequent of the proof D_0 to obtain a proof of

$$\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2, A \Rightarrow A, \Delta_2;$$

whence by the hp-admissible rule (ec)*, we construct a proof of

$$\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_2, A \Rightarrow A, \Delta_2;$$

and by (ccan), we get the desired proof of $\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$. ⊣

LEMMA 6.8. *If* $\vdash_{\mathrm{G}^3\mathrm{RP}\forall+(\mathrm{cut})} \mathcal{H}_0$, *then* $\vdash_{\mathrm{G}^0\mathrm{RP}\forall+(\mathrm{ccan})} \mathcal{H}_0$.

PROOF. It suffices to demonstrate that all the rules of $\mathrm{G}^3\mathrm{RP}\forall$+(cut) are admissible for $\mathrm{G}^0\mathrm{RP}\forall$+(ccan).

1. Let us first establish the admissibility for $\mathrm{G}^0\mathrm{RP}\forall$+(ccan) of the rules ($\mathrm{den}_1$) and ($\mathrm{den}_0$), which are formulated at the beginning of Section 5.

By Lemma 6.7, the rules (ew)*, (ec)*, (split)*, and (mix)* are admissible for $\mathrm{G}^0\mathrm{RP}\forall$+(ccan). Then we proceed as in Lemmas 5.3 and 5.8, adding to item 2.2 of the proof of Lemma 5.3 one more case 2.2.6 where $\mathcal{R}$ is (ccan) and the proof D (in $\mathrm{G}^0\mathrm{RP}\forall$+(ccan) now) looks like:

$$\dfrac{\overset{D_1}{\mathcal{H} \mid \Gamma_1, A, \mathfrak{p}_1 \Rightarrow A, \Delta_1}}{\mathcal{H}}\,(\text{ccan}).$$

In this case, using the induction hypothesis, we transform D_1 into a proof of

$$\mathcal{H}' \mid [\Gamma_1, A, \Pi_j \Rightarrow A, \Delta_1, \Sigma_j]_{j \in 1..n},$$

whence we get the desired proof of $\mathcal{H}'$ by n applications of (ccan).

2. Now the admissibility for $\mathrm{G}^0\mathrm{RP}\forall$+(ccan) of each rule of $\mathrm{G}^3\mathrm{RP}\forall$ can be shown just as in the proof of Lemma 5.1. Finally, (cut) is admissible for $\mathrm{G}^0\mathrm{RP}\forall$+(ccan). Indeed, the conclusion of (cut) is obtained from its premises thus:

$$\dfrac{\dfrac{\mathcal{G}\,|\,\Gamma_1 \Rightarrow C, \Delta_1; \quad \mathcal{G}\,|\,\Gamma_2, C \Rightarrow \Delta_2}{\mathcal{G}\,|\,\Gamma_1, \Gamma_2, C \Rightarrow C, \Delta_1, \Delta_2}\,(\text{mix})^*}{\dfrac{\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2\,|\,\Gamma_1, \Gamma_2, C \Rightarrow C, \Delta_1, \Delta_2}{\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\text{ccan})}\,(\text{ew})^*,$$

$(\text{mix})^*$ and $(\text{ew})^*$ being admissible for $\mathrm{G^0RP\forall{+}(ccan)}$ by Lemma 6.7. ⊣

LEMMA 6.9. *$\vdash_{\mathrm{G^0RP\forall+(ccan)}} \mathcal{H}$ if and only if $\vdash_{\mathrm{G^0RP\forall+(cut)}} \mathcal{H}$.*

PROOF. ONLY IF. It is enough to demonstrate that (ccan) is admissible for $\mathrm{G^0RP\forall{+}(cut)}$. The conclusion of (ccan) is obtained from its premise and the hypersequents $\Rightarrow$ and $C \Rightarrow C$ by rules that are admissible for $\mathrm{G^0RP\forall{+}(cut)}$ as follows (cf. [10, Section 4.1]):

$$(\text{ew})^*, (\Rightarrow\to)^0\,\dfrac{\dfrac{\Rightarrow; \quad C \Rightarrow C}{\Rightarrow C \to C;} \qquad \dfrac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta\,|\,\Gamma, C \Rightarrow C, \Delta}{\mathcal{G}\,|\,\Gamma, C \to C \Rightarrow \Delta}\,(\text{ew})^*, (\to\Rightarrow)^0}{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}\,(\text{ew})^*, (\text{cut}).$$

The hypersequents $\Rightarrow$ and $C \Rightarrow C$ are $\mathrm{GRP\forall}$-axioms, hence are $\mathrm{G^0RP\forall}$-provable by Lemma 4.3; and we are finished with the left-to-right direction.

IF. It suffices to show that (cut) is admissible for $\mathrm{G^0RP\forall{+}(ccan)}$. But this is done in item 2 of the proof of Lemma 6.8. ⊣

LEMMA 6.10. *For any* RPL∀*-hypersequent $\mathcal{H}$, $\vdash_{\mathrm{G^0RP\forall+(cut)}} \mathcal{H}$ if and only if $\vdash_{\mathrm{GRP\forall+(cut)}} \mathcal{H}$.*

PROOF. ONLY IF. By Lemma 4.1, every RPL∀-hypersequent that is an axiom of $\mathrm{G^0RP\forall}$ is provable in GRP∀, and each rule of $\mathrm{G^0RP\forall}$ is derivable in GRP∀ if its premises and conclusion are restricted to RPL∀-hypersequents. Hence the required result follows.

IF. By Lemma 4.3, all the axioms of GRP∀ are $\mathrm{G^0RP\forall}$-provable. By Lemma 6.7, all the rules of GRP∀ are admissible for $\mathrm{G^0RP\forall{+}(ccan)}$, and hence by Lemma 6.9, for $\mathrm{G^0RP\forall{+}(cut)}$. ⊣

6.2. Comparing $\mathrm{G^3RP\forall}$ (and GRP∀) with HRP∀: the semantic part

To finish the proof of Theorem 6.4, in this subsection we establish

LEMMA 6.11. *For any* RPL∀*-formula A, if $\vdash_{\mathrm{GRP\forall+(cut)}} A$, then $\vdash_{\mathrm{HRP\forall}} A$.*

We are going to employ the completeness of HRP∀ with respect to the algebraic semantics over so-called MV-chains containing the rational unit interval. Let us describe the semantics, following [12], [18], and [2].

An *MV-algebra* is an algebra $\mathbf{L} = \langle L, \oplus, \neg, 0\rangle$ such that the reduct $\langle L, \oplus, 0\rangle$ is an Abelian (or commutative) monoid, and the three identities hold:

- $\neg\neg y = y$,
- $y \oplus \neg 0 = \neg 0$,
- $\neg(\neg y \oplus z) \oplus z = \neg(\neg z \oplus y) \oplus y$.

An MV-algebra is *nontrivial* if its universe contains more than one element.

Given an MV-algebra $\mathbf{L} = \langle L, \oplus, \neg, 0\rangle$, by definition, put:

- $1 = \neg 0$,
- $(y \to z) = (\neg y \oplus z)$,
- $(y \leqslant z)$ iff $(y \to z) = 1$.

As shown, e.g., in [12, Section 1.1], the relation $\leqslant$ is a partial order on L, called the *natural order* of $\mathbf{L}$.

An *MV-chain* is an MV-algebra whose natural order is linear. Consider the following two examples of MV-chains.

First, $[0,1]_{Ł} = \langle [0,1], \oplus, \neg, 0\rangle$, where $[0,1]$ is the real unit interval, and the operations are defined thus: $y \oplus z = \min(1,\ y+z)$, $\neg y = 1-y$; and so $y \to z = \min(1,\ 1-y+z)$. Note that the standard semantics of the logic RPL∀ (see Section 2) is defined over this MV-chain.

Second, $\mathbb{Q} \cap [0,1]_{Ł} = \langle \mathbb{Q} \cap [0,1], \oplus, \neg, 0\rangle$, where $\mathbb{Q}$ is the set of all rational numbers, and the operations are defined analogously.

An MV-chain $\mathbf{L}$ is said to *contain the rational unit interval* if the MV-chain $\mathbb{Q} \cap [0,1]_{Ł}$ is a subalgebra of $\mathbf{L}$.

Remark 6.1. For any MV-chain $\langle L, \oplus, \neg, 0\rangle$ containing the rational unit interval, the elements 0 and $\neg 0 = 1$ of L are the integers 0 and 1, respectively.

Let $\mathbf{L} = \langle L, \oplus, \neg, 0\rangle$ be an MV-chain containing the rational unit interval. We take $\mathbf{L}$ as the set of truth values.

An $\mathbf{L}$-*interpretation* M is defined just as an interpretation (see Section 2), except that now predicates assume values from L.

The *truth value* $|A|^{\mathbf{L}}_{M,\nu}$ *of an* RPL∀*-formula* A under an $\mathbf{L}$-interpretation $M = \langle \mathcal{D}, \mu\rangle$ and an M-valuation ν is defined thus:

- $|\bar{r}|^{\mathbf{L}}_{M,\nu} = r$;

- $|P(t_1,\dots,t_n)|^{\mathbf{L}}_{M,\nu} = \mu(P)(|t_1|_{M,\nu},\dots,|t_n|_{M,\nu})$ for an n-ary predicate symbol P and terms $t_1,\dots,t_n$ not necessarily closed;
- $|B \to C|^{\mathbf{L}}_{M,\nu} = |B|^{\mathbf{L}}_{M,\nu} \to |C|^{\mathbf{L}}_{M,\nu}$ if $|B|^{\mathbf{L}}_{M,\nu}$ and $|C|^{\mathbf{L}}_{M,\nu}$ are defined, otherwise $|B \to C|^{\mathbf{L}}_{M,\nu}$ is undefined;
- $|\forall x B|^{\mathbf{L}}_{M,\nu} = \inf_{d\in\mathcal{D}} |B|^{\mathbf{L}}_{M,\nu[x\mapsto d]}$ if $|B|^{\mathbf{L}}_{M,\nu[x\mapsto d]}$ is defined for all $d \in \mathcal{D}$ and the infimum exists, otherwise $|\forall x B|^{\mathbf{L}}_{M,\nu}$ is undefined;
- $|\exists x B|^{\mathbf{L}}_{M,\nu} = \sup_{d\in\mathcal{D}} |B|^{\mathbf{L}}_{M,\nu[x\mapsto d]}$ if $|B|^{\mathbf{L}}_{M,\nu[x\mapsto d]}$ is defined for all $d \in \mathcal{D}$ and the supremum exists, otherwise $|\exists x B|^{\mathbf{L}}_{M,\nu}$ is undefined.

The truth value of an RPL∀-formula (under an **L**-interpretation M and an M-valuation) may be undefined because some infima or suprema involved in the above definition may not exist. To avoid this, we restrict ourselves to so-called safe **L**-interpretations. An **L**-interpretation M is called *safe* if $|A|^{\mathbf{L}}_{M,\nu}$ is defined for all RPL∀-formulas A (over the signature being used) and all M-valuations ν.

An RPL∀-formula A is **L**-*valid*, or in symbols $\models_{\mathbf{L}} A$, if $|A|^{\mathbf{L}}_{M,\nu} = 1$ for all safe **L**-interpretations M and all M-valuations ν.

The following Theorem 6.12 is a special case of Theorem 5.2.9 in [18] and is given below in the formulation used in [19].

THEOREM 6.12 (completeness of HRP∀ [18, Theorem 5.2.9]). *For any* RPL∀-*formula* A, $\vdash_{\mathrm{HRP}\forall} A$ *iff* $\models_{\mathbf{L}} A$ *for all MV-chains* **L** *containing the rational unit interval.*

With Theorem 6.12, in order to establish Lemma 6.11, it remains to prove

LEMMA 6.13 (soundness of GRP∀+(cut)). *For any* RPL∀-*formula* A, *if* $\vdash_{\mathrm{GRP}\forall+(\mathrm{cut})} A$, *then* $\models_{\mathbf{L}} A$ *for all MV-chains* **L** *containing the rational unit interval.*

To prove Lemma 6.13, we need the notion of an o-group and the next Theorem 6.14, which connects MV-chains with o-groups.

A *linearly ordered Abelian group* (an *o-group* for short) is a structure $\mathbf{G} = \langle G, +, -, 0, \leqslant\rangle$ such that

- $\langle G, +, -, 0\rangle$ is an Abelian group,
- $\langle G, \leqslant\rangle$ is a chain, and
- for all $v, y, z \in G$, if $y \leqslant z$, then $y + v \leqslant z + v$;

see, e.g., [18, Section 1.6]. For elements y and z of such a group, we write $y - z$ for $y + (-z)$, and call the element y *positive* if $0 \leqslant y$ and $y \neq 0$.

For an o-group $\mathbf{G} = \langle G, +, -, 0, \leqslant\rangle$ and a positive element $e \in G$, let

- $[0, e]_G$ denote the set $\{g \in G \mid 0 \leqslant g \leqslant e\}$, and
- $MV(\mathbf{G}, e)$ denote the algebra $\langle [0, e]_G, \oplus, \neg, 0\rangle$, where $y \oplus z = \min(e,\ y + z)$ and $\neg y = e - y$.

THEOREM 6.14 ([9], cf. [18, Section 3.2]). *For any nontrivial MV-chain* $\mathbf{L}$ *there exist an o-group* $\mathbf{G} = \langle G, +, -, 0, \leqslant\rangle$ *and a positive element* $e \in G$ *such that*

- $MV(\mathbf{G}, e)$ *is an MV-chain,*
- *the natural order of* $MV(\mathbf{G}, e)$ *coincides with the restriction of the order* $\leqslant$ *of* $\mathbf{G}$ *to* $[0, e]_G$*, and*
- *the MV-chain* $\mathbf{L}$ *is isomorphic to the MV-chain* $MV(\mathbf{G}, e)$*.*

The above Theorem 6.14 is due to Chang [9], but is given in the formulation close to that in [18, Section 3.2].

PROOF of LEMMA 6.13. Suppose that A is an RPL∀-formula provable in GRP∀+(cut), and $\mathbf{L} = \langle L, \oplus, \neg, 0\rangle$ is an MV-chain containing the rational unit interval. We are to show that $\vDash_{\mathbf{L}} A$.

Applying Theorem 6.14 yields an o-group $\mathbf{G} = \langle G, +', -', 0', \leqslant'\rangle$ and a positive element $e \in G$ such that the MV-chain $\mathbf{L}$ is isomorphic to the MV-chain $MV(\mathbf{G}, e) = \langle [0', e]_G, \oplus', \neg', 0'\rangle$. Without loss of generality we can assume that $\mathbf{L}$ is exactly $MV(\mathbf{G}, e)$. Then we can use the order $\leqslant'$ of $\mathbf{G}$ as the natural order of $\mathbf{L}$. We also have $0' = 0$, $e = \neg 0 = 1$, and for all $y, z \in [0, 1]_G$,

$$y \oplus z = \min(1,\ y +' z), \quad \neg y = 1 -' y, \quad y \to z = \min(1,\ 1 -' y +' z).$$

Now we extend the notion of the validity of an RPL∀-hypersequent to $\mathbf{L}$. For a finite multiset Γ of RPL∀-formulas, a safe $\mathbf{L}$-interpretation M, and an M-valuation ν, we put

$$\|\Gamma\|^{\mathbf{L}}_{M,\nu} = {\sum_{B \in \Gamma}}' (|B|^{\mathbf{L}}_{M,\nu} -' 1),$$

where the summation $\sum'$ is carried out in the o-group $\mathbf{G}$, taking multiplicities of multiset elements into account, and $\sum'_{B \in \varnothing}(\ldots) = 0$. We say that an RPL∀-hypersequent $\mathcal{H}$ is $\mathbf{L}$-*valid* if, for every safe $\mathbf{L}$-interpretation M and every M-valuation ν, there is a sequent $\Gamma \Rightarrow \Delta$ in $\mathcal{H}$ such that

$$\|\Gamma\|^{\mathbf{L}}_{M,\nu} \leqslant' \|\Delta\|^{\mathbf{L}}_{M,\nu}.$$

Observe that $\vDash_{\mathbf{L}} A$ iff the hypersequent $\Rightarrow A$ is $\mathbf{L}$-valid.

To prove that $\vDash_{\mathbf{L}} A$, it is sufficient to demonstrate that the calculus GRP∀+(cut) is sound with respect to this semantics, i.e., proves only **L**-valid RPL∀-hypersequents. Verifying the soundness of all the rules of GRP∀ with respect to **L**-validity is performed essentially as for the standard semantics (see Proposition 3.2). The rule (cut) with its premises and conclusion restricted to RPL∀-hypersequents is easily seen to be sound with respect to **L**-validity. Further, a GRP∀-axiom of the form $B \Rightarrow B$ is clearly **L**-valid.

It remains to establish the **L**-validity of a GRP∀-axiom of the form $\bar{r}_1, \dots, \bar{r}_l \Rightarrow \bar{s}_1, \dots, \bar{s}_m, A_1, \dots, A_n,$ where

$$\sum_{i=1}^{l}(r_i - 1) \leqslant \sum_{j=1}^{m}(s_j - 1) - n. \tag{I_0}$$

Here and below $\sum$, $+$, $-$, $\leqslant$, and $<$ (used also in the form $>$) are, respectively, the usual summation, addition, subtraction, (nonstrict) order, and strict order on the set of all rational numbers. By $k{*}1$ we denote the sum $(1 +' 1 +' \dots +' 1)$ of k items, the sum being equal to 0 if $k = 0$. To finish the proof, it is enough to show that inequality (I_0) implies the inequality

$$\sum_{i=1}^{l}{}'(r_i -' 1) \leqslant' \sum_{j=1}^{m}{}'(s_j -' 1) -' n{*}1.$$

For this, in turn, it suffices to prove that, for any nonnegative integers l, m, n_1, n_2 and any rational numbers $r_1, \dots, r_l, s_1, \dots, s_m \in [0, 1]$,

$$(\mathrm{I}) \quad n_1 + \sum_{i=1}^{l} r_i \leqslant n_2 + \sum_{j=1}^{m} s_j$$

implies

$$(\mathrm{I}') \quad n_1{*}1 +' \sum_{i=1}^{l}{}' r_i \leqslant' n_2{*}1 +' \sum_{j=1}^{m}{}' s_j.$$

Let us note that this implication is not obvious, as we do not know how $+'$ and $\leqslant'$ behave outside of the rational unit interval. However, for any rational numbers $r, s \in [0, 1]$, we have:

- $r \leqslant' s$ iff $r \leqslant s$; and
- if $r + s \in [0, 1]$, then $r +' s = r + s$.

We proceed by induction on $(l + m)$, considering the following cases.

Case 1: $l \leqslant 1$ and $m \leqslant 1$. For convenience, we can harmlessly assume that $l = m = 1$, because otherwise we can add $r_1 = 0$ (resp. $s_1 = 0$) to the left-hand (resp. right-hand) sides of both inequalities (I) and (I′).

Subcase 1.1: $n_1 = n_2$. Then inequality (I) implies $r_1 \leqslant s_1$, whence $r_1 \leqslant' s_1$, and so $n_1{*}1 +' r_1 \leqslant' n_2{*}1 +' s_1$ as required.

Subcase 1.2: $n_1 < n_2$. Then $n_1 + 1 \leqslant n_2$, and hence $n_1{*}1 +' r_1 \leqslant' (n_1+1){*}1 \leqslant' n_2{*}1 \leqslant' n_2{*}1 +' s_1$.

Subcase 1.3: $n_1 > n_2$. Then from (I) it follows that $n_1 = n_2 + 1$, $r_1 = 0$, and $s_1 = 1$. Hence $n_1{*}1 +' r_1 = n_1{*}1 = (n_2+1){*}1 = n_2{*}1 +' s_1$.

Case 2: $l > 1$ or $m > 1$. We assume for definiteness that $l > 1$, and put $R = r_1 + r_2$.

Subcase 2.1: $R \leqslant 1$. Then $r_1 +' r_2 = R$. So inequalities (I) and (I′) are equivalent, respectively, to

$$n_1 + \left(R + \sum_{i=3}^{l} r_i\right) \leqslant n_2 + \sum_{j=1}^{m} s_j \qquad \text{and}$$

$$n_1{*}1 +' \left(R +' {\sum_{i=3}^{l}}' r_i\right) \leqslant' n_2{*}1 +' {\sum_{j=1}^{m}}' s_j.$$

The required result follows by the induction hypothesis applied to the last two inequalities.

Subcase 2.2: $R > 1$. Put $\widehat{R} = R - 1$, i.e., $\widehat{R} = r_1 + r_2 - 1$. Then $r_2 = (1 - r_1) + \widehat{R}$, where r_2, $(1-r_1)$, $\widehat{R} \in [0,1]$. Hence $r_2 = (1-r_1) +' \widehat{R}$, and so $r_1 +' r_2 = r_1 +' (1-r_1) +' \widehat{R}$; but $r_1 +' (1-r_1) = r_1 + (1-r_1) = 1$, and therefore $r_1 +' r_2 = 1 +' \widehat{R}$. Thus (I) and (I′) are equivalent, respectively, to the inequalities

$$(n_1 + 1) + \left(\widehat{R} + \sum_{i=3}^{l} r_i\right) \leqslant n_2 + \sum_{j=1}^{m} s_j \qquad \text{and}$$

$$(n_1+1){*}1 +' \left(\widehat{R} +' {\sum_{i=3}^{l}}' r_i\right) \leqslant' n_2{*}1 +' {\sum_{j=1}^{m}}' s_j,$$

to which the induction hypothesis applies. ⊣

Proof of Lemma 6.11. Follows from Theorem 6.12 and Lemma 6.13. ⊣

7. Conclusion

In the present article, we have compared the proof-search-oriented analytic hypersequent calculus $G^3RP\forall$ (for the logic RPL∀) with the analytic hypersequent calculus GŁ∀ (for the logic Ł∀) and with the Hilbert-type calculus HRP∀ (for RPL∀).

To facilitate our comparison, we have introduced (and included in this comparison) two analytic hypersequent calculi for RPL∀, namely $G^0RP\forall$ and GRP∀, which are unsuitable for proof search, but are useful in theoretical investigations, such as those given above. $G^0RP\forall$ is a simplified version of $G^3RP\forall$ and, in fact, is a predecessor of $G^3RP\forall$ that was our initial result of excluding all the structural rules from GŁ∀ but was not published previously. GRP∀ is a natural extension of GŁ∀ with axioms that handle truth constants of RPL∀ and are defined in nearly syntactic terms. Table 1 summarizes the analytic calculi just mentioned.

Calculus	Structural rules	For proof search	Short description
GŁ∀ [2]	yes	no	first analytic calculus for Ł∀
GRP∀ [this article]	yes	no	extension of GŁ∀ with "nearly syntactic" axioms for truth constants
$G^0RP\forall$ [this article]	no	no	initial result of excluding all the structural rules from GŁ∀
$G^3RP\forall$ [17]	no	yes	repetition-free, proof-search-oriented calculus, obtained from $G^0RP\forall$

Table 1. The main analytic calculi considered for the logics Ł∀ and RPL∀.

We have established that GRP∀ is a conservative extension of GŁ∀, and that $G^0RP\forall$ and $G^3RP\forall$ are equivalent and are conservative extensions of GRP∀ (see Theorem 5.10 and Corollary 5.12). We have also demonstrated that the calculi GRP∀, $G^0RP\forall$, and $G^3RP\forall$ each extended with the cut rule prove exactly the same RPL∀-sentences as the calculus HRP∀ (see Theorem 6.4).

The key part of our argument is the syntactic proofs of the admissibility for $G^0RP\forall$ of the nonstandard variants (den_1) and (den_0) of the density rule (see Lemmas 5.3 and 5.8). These proofs can be easily adapted to show the admissibility for $G^0RP\forall$ of the nonstandard density rule (see Remark 5.2 on p. 294). The given proof of the admissibility of (den$_1$) for G^{0}RP∀ provides an algorithm for transforming a proof of a hypersequent in G^{0}RP∀+(den$_1$) into a proof of the same hypersequent in G^{0}RP∀.

Let us adopt the following definition (cf., e.g., [10, 22, 23, 24]). Suppose that $\mathfrak{C}$ is a calculus, and a rule $\mathcal{R}$ is not an inference rule of $\mathfrak{C}$; then *elimination of* $\mathcal{R}$ is said to *hold for* $\mathfrak{C}+\mathcal{R}$ (as well as *for* $\mathfrak{C}$) if an algorithm is constructed that, given a proof of an object in $\mathfrak{C}+\mathcal{R}$, transforms it into a proof of the same object in $\mathfrak{C}$.

Thus, our proof of the admissibility of (den$_1$) for G^{0}RP∀ establishes the elimination of (den$_1$) for G^{0}RP∀+(den$_1$); similarly with (den$_0$) and the nonstandard density rule. One feature of these density elimination proofs is that they do not use the cut rule, which is not admissible for G^{0}RP∀ (see Proposition 6.3).

Before the present work, density elimination proofs were known for some (classes of) hypersequent calculi, though for logics different from Ł∀ and RPL∀; see [3], [1], [22], [11], [23], [8], [4], [5], [6], [24], [7], [28], and [29] (in chronological order). In all these works except [1], such proofs use the cut rule even if no application of it is in an initial formal proof. The density elimination proof for a single-conclusion hypersequent calculus for first-order Gödel logic in [1] is provided as an improvement of the earlier density elimination proof (introducing cuts) for the same calculus in [3], and does not introduce cuts if an initial formal proof is cut-free. Our technique for proving density elimination resembles that in [1], but has been rediscovered, made more explicit, and elaborated for the multiple-conclusion calculus G^{0}RP∀ for the logic RPL∀. Given these two applications of the technique, it would be nice to generalize the technique to as wide a class of hypersequent calculi as possible.

Further, the book [23] on p. 134 says that it is unclear whether density elimination can be obtained for calculi for the propositional fragment of the logic Ł∀. We have given a density elimination proof for the calculus G^{0}RP∀, which is a conservative extension of the calculus GŁ∀ (which in turn is complete for the propositional fragment of Ł∀). Moreover, to the best of our knowledge, the given proof is the first density elimination proof for a first-order multiple-conclusion hypersequent calculus in which neither the weakening rule nor the contraction rule is admissible.[3]

[3] The weakening and contraction rules are, respectively:

$$\frac{\mathcal{G}\,|\,\Gamma \Rightarrow \Delta}{\mathcal{G}\,|\,\Gamma, \Pi \Rightarrow \Sigma, \Delta} \quad \text{and} \quad \frac{\mathcal{G}\,|\,\Gamma, \Pi, \Pi \Rightarrow \Sigma, \Sigma, \Delta}{\mathcal{G}\,|\,\Gamma, \Pi \Rightarrow \Sigma, \Delta},$$

Finally, let us note that, in contrast to numerous works on the complexity of cut elimination, how complexity of formal proofs varies has not yet been investigated for any density elimination proof, which offers another problem for future research.

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where $\mathcal{G}$ is any hypersequent, and Γ, Δ, Π, and Σ are any finite multisets of formulas of a language under consideration (see [23, Section 4.3]).

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Appendix

This appendix contains some proofs omitted from the above article.

A. The analog of Lemma 3.5 for RPL∀ and GRP∀

Remark 3.2 of Section 3 says, in particular, that Lemma 3.5 readily carries over to RPL∀ and GRP∀ (instead of Ł∀ and GŁ∀, respectively). Let us show this.

First recall that, in Section 3 just before Lemma 3.5, by $\ell(\mathcal{G})$ we denote the number of distinct nonconstant atomic RPL∀-formulas occurring in the antecedents of the sequents in an atomic RPL∀-hypersequent

$\mathcal{G}$. Also there we say that Lemma 3.5 is in fact established in the proof of Theorem 6.24 in [23].

Here is the required analog of Lemma 3.5 for RPL∀ and GRP∀, with the proof adapted from that of Theorem 6.24 in [23] and supplemented with a few clarifications.

LEMMA A.1. *Let $\mathcal{G}$ be a valid atomic* RPL∀-*hypersequent with $\ell(\mathcal{G}) > 0$. Then $\mathcal{G}$ is* GRP∀-*provable from a valid atomic* RPL∀-*hypersequent $\mathcal{H}$ with $\ell(\mathcal{H}) < \ell(\mathcal{G})$.*

PROOF. We can harmlessly regard each nonconstant atomic RPL∀-formula A in $\mathcal{G}$ as a new propositional variable p_A, and specify only an interpretation M (omitting an M-valuation) when speaking about the truth values of propositional RPL∀-formulas.

We pick a propositional variable q occurring on the left of one of the sequents of $\mathcal{G}$. If q occurs on both sides in the same sequent, then we apply $(\text{mix})^{\text{P}}$ and $(\text{id})^{\text{P}}$ backwards to remove it, noting that the new hypersequent is also valid. Next, we use $(\text{ec})^{\text{P}}$ and $(\text{split})^{\text{P}}$ backwards to multiply sequents, giving (for some integers $k > 0$, $m \geqslant 0$, and $n \geqslant 0$) a hypersequent

$$\mathcal{G}' = \Big(\mathcal{G}_0 \mid [\Gamma_i, [q]^k \Rightarrow \Delta_i]_{i \in 1..n} \mid [\Pi_j \Rightarrow [q]^k, \Sigma_j]_{j \in 1..m} \Big)$$

where q does not occur in $\mathcal{G}_0$, Γ_i, Δ_i, Π_j, or Σ_j for $i \in 1..n$ and $j \in 1..m$, and $[q]^k$ stands for the multiset consisting of k copies of q.

Observe that $\vdash_{\text{GRP}\forall} \mathcal{G}$ if $\vdash_{\text{GRP}\forall} \mathcal{G}'$. Also $\vDash \mathcal{G}'$. Now let $\mathcal{H}$ be

$$\Big(\mathcal{G}_0 \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]^{i \in 1..n}_{j \in 1..m} \mid [\Gamma_i \Rightarrow \Delta_i]_{i \in 1..n} \mid [\Pi_j \Rightarrow [q]^k, \Sigma_j]_{j \in 1..m} \Big).$$

Clearly, $\ell(\mathcal{H}) < \ell(\mathcal{G})$. Also $\mathcal{G}'$ is GRP∀-provable from $\mathcal{H}$. Indeed, we apply $(\text{ec})^{\text{P}}$ and $(\text{split})^{\text{P}}$ backwards to $\mathcal{G}'$ to combine sequents of the form $(\Gamma_i, [q]^k \Rightarrow \Delta_i)$ and $(\Pi_j \Rightarrow [q]^k, \Sigma_j)$ into one: $(\Gamma_i, \Pi_j, [q]^k \Rightarrow [q]^k, \Delta_i, \Sigma_j)$. Then we apply $(\text{mix})^{\text{P}}$ and $(\text{id})^{\text{P}}$ backwards to remove the balanced occurrences of q, and $(\text{wl})^{\text{P}}$ backwards to $(\Gamma_i, [q]^k \Rightarrow \Delta_i)$ to get $(\Gamma_i \Rightarrow \Delta_i)$.

It remains to show that $\vDash \mathcal{H}$. Suppose, for a contradiction, otherwise; i.e., that there exists an interpretation M such that $\|\Gamma\|_M > \|\Delta\|_M$ for all $(\Gamma \Rightarrow \Delta) \in \mathcal{H}$. Let

$$\alpha = \max\big(\{\|\Delta_i\|_M - \|\Gamma_i\|_M : 1 \leqslant i \leqslant n\} \cup \{-k\}\big) \quad (\text{and so } -k \leqslant \alpha);$$
$$\beta = \min\big(\{\|\Pi_j\|_M - \|\Sigma_j\|_M : 1 \leqslant j \leqslant m\} \cup \{0\}\big) \quad (\text{and so } \beta \leqslant 0).$$

If $\alpha \geqslant \beta$, then we have at least one of the following cases:

(1) for some i and j, $\|\Delta_i\|_M - \|\Gamma_i\|_M \geqslant \|\Pi_j\|_M - \|\Sigma_j\|_M$;
(2) for some i, $\|\Delta_i\|_M - \|\Gamma_i\|_M \geqslant 0$;
(3) for some j, $-k \geqslant \|\Pi_j\|_M - \|\Sigma_j\|_M$.

Since $\|\Gamma\|_M > \|\Delta\|_M$ for all $(\Gamma \Rightarrow \Delta) \in \mathcal{H}$, neither of cases (1)–(3) is possible; and hence $\alpha < \beta$.

Define an interpretation M' to be like M, but (noting that $0 \leqslant \alpha/k+1$ and $\beta/k+1 \leqslant 1$) choose $|q|_{M'}$ so that $\alpha/k+1 < |q|_{M'} < \beta/k+1$, i.e., $\alpha < k(|q|_{M'}-1) < \beta$. Then for all $i \in 1..n$ and all $j \in 1..m$:

$$\|\Delta_i\|_{M'} - \|\Gamma_i\|_{M'} \leqslant \alpha < k(|q|_{M'}-1) < \beta \leqslant \|\Pi_j\|_{M'} - \|\Sigma_j\|_{M'};$$

therefore $\|\Gamma_i, [q]^k\|_{M'} > \|\Delta_i\|_{M'}$ and $\|\Pi_j\|_{M'} > \|[q]^k, \Sigma_j\|_{M'}$. So $\not\models \mathcal{G}'$, a contradiction. ⊣

B. The admissibility of the rules (split)* and (mix)* for G⁰RP∀

Item 4 of the proof of Lemma 4.2 says that the proof of the hp-admissibility of (split)* for $\mathrm{G^0RP\forall}$ is very similar to the proof of Lemma 7 in [16]. Besides, in the proof of Lemma 6.7, we extend the proof of the hp-admissibility of (split)* for $\mathrm{G^0RP\forall}$ with a new case to obtain the proof of the hp-admissibility of (split)* for $\mathrm{G^0RP\forall}$+(ccan).

Next, item 5 of the proof of Lemma 4.2 says that the proof of the admissibility of (mix)* for $\mathrm{G^0RP\forall}$ can be obtained from the proof of Lemma 8 in [16] by identifying the notion of a completable ancestor of a sequent occurrence with the notion of an ancestor of a sequent occurrence (the former notion is used in [16]).

Below we give the proof of the hp-admissibility of (split)* for $\mathrm{G^0RP\forall}$ and the proof of the admissibility of (mix)* for $\mathrm{G^0RP\forall}$, adapting the mentioned proofs from [16] (and correcting inaccuracies introduced in [16] by a translator of the original Russian article).

Lemma B.1. *The following rule is hp-admissible for* $\mathrm{G^0RP\forall}$*:*

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\mathrm{split})^*.$$

Proof. Let

$$\mathcal{H}_1 = (\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2), \quad \mathcal{H}_2 = (\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2).$$

Using induction on the height of a ($G^0RP\forall$-)proof D_1 of $\mathcal{H}_1$, we show that D_1 can be transformed into a proof of $\mathcal{H}_2$ whose height is no greater than the height of D_1.

1. If $\mathcal{H}_1$ is an axiom, then it is easy to see that $\mathcal{H}_2$ is an axiom too.

2. Let the bottom hypersequent $\mathcal{H}_1$ in D_1 be the conclusion of an application R of a rule $\mathcal{R}$. We consider the case where $\mathcal{R}$ is $(\to\Rightarrow)^0$; the remaining cases are similar.

2.1. Suppose that the principal sequent occurrence in the application R is in the distinguished occurrence of $\mathcal{G}$ in $\mathcal{H}_1$. Then the premise $\mathcal{H}_0$ of the application R has the form $\mathcal{G}_0 \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. By the induction hypothesis for the proof of $\mathcal{H}_0$ (which is a subtree of the proof tree D_1), we can construct a proof of $\mathcal{G}_0 \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2$. By applying the rule $\mathcal{R}$, we obtain the required proof of $\mathcal{H}_2$.

2.2. Suppose that the principal sequent occurrence in the application R is the distinguished occurrence of $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in $\mathcal{H}_1$. For definiteness we assume that the principal occurrence of a formula $A_1 \to B_1$ in the application R is in Γ_1. Then $\Gamma_1 = (\Gamma_1', A_1 \to B_1)$ for some Γ_1', and the proof D_1 has the form

$$\dfrac{\begin{array}{c} D_0 \\ \mathcal{G} \,|\, \Gamma_1', A_1 \to B_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \,|\, \Gamma_1', \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \\ |\, \Gamma_1', B_1, \Gamma_2 \Rightarrow A_1, \Delta_1, \Delta_2 \end{array}}{\mathcal{G} \,|\, \Gamma_1', A_1 \to B_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\,(\to\Rightarrow)^0.$$

Using the induction hypothesis three times, we split all the three sequent occurrences that are distinguished in the bottom hypersequent of the proof D_0, thus obtaining a proof of the hypersequent

$$\begin{array}{c} \mathcal{G} \,|\, \Gamma_1', A_1 \to B_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \,|\, \Gamma_1' \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2 \\ |\, \Gamma_1', B_1 \Rightarrow A_1, \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2. \end{array}$$

From this hypersequent we eliminate two occurrences of $\Gamma_2 \Rightarrow \Delta_2$ with the help of the hp-admissible rule $(\mathrm{ec})^*$ (see Lemma 4.2 and item 2 of its proof), getting a proof of the hypersequent

$$\mathcal{G} \,|\, \Gamma_1', A_1 \to B_1 \Rightarrow \Delta_1 \,|\, \Gamma_1' \Rightarrow \Delta_1 \,|\, \Gamma_1', B_1 \Rightarrow A_1, \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2.$$

Finally, we apply the rule $(\to\Rightarrow)^0$ to the last hypersequent and obtain the required proof of $\mathcal{G} \,|\, \Gamma_1', A_1 \to B_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2$. ⊣

LEMMA B.2. *The following rule is admissible for* $\mathrm{G^0RP}\forall$:

$$\frac{\mathcal{G}\,|\,\Gamma_1 \Rightarrow \Delta_1; \quad \mathcal{G}\,|\,\Gamma_2 \Rightarrow \Delta_2}{\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\mathrm{mix})^*.$$

PROOF. Let

$$\mathcal{H}_1 = (\mathcal{G}\,|\,\Gamma_1 \Rightarrow \Delta_1), \quad \mathcal{H}_2 = (\mathcal{G}\,|\,\Gamma_2 \Rightarrow \Delta_2),$$
$$\mathcal{H}_3 = (\mathcal{G}\,|\,\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2).$$

We suppose that $\vdash_{\mathrm{G^0RP}\forall} \mathcal{H}_1$ and $\vdash_{\mathrm{G^0RP}\forall} \mathcal{H}_2$, and show that $\vdash_{\mathrm{G^0RP}\forall} \mathcal{H}_3$. Let D_1 be a ($\mathrm{G^0RP}\forall$-)proof of $\mathcal{H}_1$ such that no proper parameter from D_1 occurs in $\Gamma_2 \Rightarrow \Delta_2$.

We obtain a proof search tree D_3^0 for $\mathcal{H}_3$ as follows. In D_1, for each occurrence $\mathcal{S}$ of a sequent of the form $\Pi_1 \Rightarrow \Sigma_1$, if $\mathcal{S}$ is an ancestor of the distinguished occurrence of the sequent $\Gamma_1 \Rightarrow \Delta_1$ in the root of D_1, then we replace $\mathcal{S}$ by an occurrence $\mathcal{S}'$ of the sequent $\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$. We also mark $\mathcal{S}'$ if $\mathcal{S}$ is an atomic sequent occurrence in a leaf of D_1. Let $\mathcal{S}_i$, $i = 0, \dots, l-1$, be all distinct marked sequent occurrences in D_3^0.

We expand D_3^0, proceeding for each $i = 0, \dots, l-1$ as follows.

(0) Let $\mathcal{S}_i$ be an occurrence of a sequent of the form $\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$.

(1) We construct a proof D_2 of $\mathcal{H}_2$ such that no proper parameter from D_2 occurs in D_3^i.

(2) We obtain a proof search tree $\widehat{D}_2$ for $\mathcal{G}\,|\,\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2$ thus: in D_2, for each occurrence of a sequent of the form $\Pi_2 \Rightarrow \Sigma_2$, if this occurrence is an ancestor of the distinguished occurrence of the sequent $\Gamma_2 \Rightarrow \Delta_2$ in the root of D_2, then we replace this occurrence by $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$.

(3) We expand each branch of D_3^i containing the occurrence $\mathcal{S}_i$ as follows: we identify the top node of this branch, which represents an occurrence of a hypersequent of the form $\mathcal{G}\,|\,\Pi_1, \Gamma_2 \Rightarrow \Sigma_1, \Delta_2\,|\,\mathcal{H}$ for some $\mathcal{H}$, with the root of the tree obtained from $\widehat{D}_2$ by appending "$|\,\mathcal{H}$" to each node hypersequent. By D_3^{i+1} we denote the tree resulting from this expansion of D_3^i.

It is not difficult to see that the tree D_3^l is a proof search tree for $\mathcal{H}_3$. It remains to show that D_3^l is a proof.

We consider an arbitrary leaf $\mathcal{L}_3$ of D_3^l and show that $\mathcal{L}_3$ is an axiom. Given $\mathcal{L}_3$, we find a unique leaf $\mathcal{L}_1$ of D_1 that transforms into a leaf of D_3^0 that, in turn, transforms (in expanding D_3^0) into a node of D_3^l belonging to the same branch as $\mathcal{L}_3$.

Let $\Pi_{1,i} \Rightarrow \Sigma_{1,i}$, $i \in I$, be all atomic sequents whose occurrences in $\mathcal{L}_1$ are ancestors of the distinguished occurrence of $\Gamma_1 \Rightarrow \Delta_1$ in the root of D_1. By the construction of D_3^l, for each $i \in I$, there exist a proof D_2^i of $\mathcal{H}_2$ and its leaf $\mathcal{L}_2^i$ such that, for each $j \in J_i$, an atomic sequent $\Pi_{1,i}, \Pi_{2,j}^i \Rightarrow \Sigma_{1,i}, \Sigma_{2,j}^i$ occurs in $\mathcal{L}_3$, where $\Pi_{2,j}^i \Rightarrow \Sigma_{2,j}^i$, $j \in J_i$, are all atomic sequents whose occurrences in $\mathcal{L}_2^i$ are ancestors of the distinguished occurrence of $\Gamma_2 \Rightarrow \Delta_2$ in the root of D_2^i.

In addition, $\mathcal{L}_3$ contains all atomic sequents $S_{1,k}$, $k \in K$, whose occurrences in $\mathcal{L}_1$ are ancestors of sequent occurrences in the distinguished occurrence of $\mathcal{G}$ in the root of D_1.

Finally, for each $i \in I$, the leaf $\mathcal{L}_3$ contains all atomic sequents $S_{2,m}^i$, $m \in M_i$, whose occurrences in $\mathcal{L}_2^i$ are ancestors of sequent occurrences in the distinguished occurrence of $\mathcal{G}$ in the root of D_2^i.

The leaf $\mathcal{L}_1$ of the proof D_1 is an axiom and contains exactly the following atomic sequents: $\Pi_{1,i} \Rightarrow \Sigma_{1,i}$ for each $i \in I$ and $S_{1,k}$ for each $k \in K$. For each $i \in I$, the leaf $\mathcal{L}_2^i$ of the proof D_2^i is an axiom and contains exactly the following atomic sequents: $\Pi_{2,j}^i \Rightarrow \Sigma_{2,j}^i$ for each $j \in J_i$ and $S_{2,m}^i$ for each $m \in M_i$. Therefore, the leaf $\mathcal{L}_3$ of D_3^l, which contains the above-mentioned atomic sequents, is an axiom too. ⊣

C. The soundness of the nonstandard density rule

Remark 5.1 on p. 284 says that the rule

$$\frac{\mathcal{G} \,|\, \Gamma, \mathfrak{p} \Rightarrow \Delta \,|\, \Pi \Rightarrow \mathfrak{p}, \Sigma}{\mathcal{G} \,|\, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (den)}$$

(1) is unsound if $\mathfrak{p}$ is a propositional variable not occurring in the conclusion, but (2) becomes sound if we expand the notion of a hypersequent by adding new-type semipropositional variables interpreted by any real numbers, and require $\mathfrak{p}$ to be such a variable not occurring in the conclusion.

Let us prove (1). Recall that the propositional variable $\mathfrak{p}$ is interpreted by any real number in $[0,1]$. Consider the following application of (den):

$$\frac{\mathfrak{p} \Rightarrow \bar{0}, \bar{0} \,|\, \bar{0} \Rightarrow \mathfrak{p}}{\bar{0} \Rightarrow \bar{0}, \bar{0}}.$$

The premise of this application is valid (because $\bar{0} \Rightarrow \mathfrak{p}$ is), but its conclusion is not. ⊣

Now let us prove (2), i.e., that $\vDash (\mathcal{G} \mid \Gamma, \mathfrak{p} \Rightarrow \Delta \mid \Pi \Rightarrow \mathfrak{p}, \Sigma)$ implies $\vDash (\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma)$ under the specified condition on $\mathfrak{p}$. To make this proof shorter, we assume harmlessly that the hypersequent $\mathcal{G}$ is empty.

(a) $\nvDash (\Gamma, \Pi \Rightarrow \Delta, \Sigma) \iff$ for some hs-interpretation M and M-valuation ν, $\|\Delta\|_{M,\nu} - \|\Gamma\|_{M,\nu} < \|\Pi\|_{M,\nu} - \|\Sigma\|_{M,\nu} \iff$ (by the density of the set of all real numbers) for some hs-interpretation M, M-valuation ν, and real number ξ, $\|\Delta\|_{M,\nu} - \|\Gamma\|_{M,\nu} < \xi - 1 < \|\Pi\|_{M,\nu} - \|\Sigma\|_{M,\nu}$.

(b) $\nvDash (\Gamma, \mathfrak{p} \Rightarrow \Delta \mid \Pi \Rightarrow \mathfrak{p}, \Sigma) \iff$ for some hs-interpretation M' and M'-valuation ν', $\|\Delta\|_{M',\nu'} - \|\Gamma\|_{M',\nu'} < |\mathfrak{p}|_{M'} - 1 < \|\Pi\|_{M',\nu'} - \|\Sigma\|_{M',\nu'}$.

It is easy to see that (a) implies (b): define M' to be the same as M but set $|\mathfrak{p}|_{M'} = \xi$, and take $\nu' = \nu$. ⊣

D. The admissibility of the nonstandard density rule for $\mathrm{G}^0\mathrm{RP}\forall$

Remark 5.2 on p. 294 says that the proofs of Lemmas 5.3 and 5.8 can be easily combined to establish the admissibility for $\mathrm{G}^0\mathrm{RP}\forall$ of the rule

$$\frac{\mathcal{G} \mid \Gamma, \mathfrak{p} \Rightarrow \Delta \mid \Pi \Rightarrow \mathfrak{p}, \Sigma}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma} \text{ (den)},$$

provided that the notion of a hypersequent is expanded by adding new-type semipropositional variables interpreted by any real numbers, and $\mathfrak{p}$ is such a variable not occurring in the conclusion.

Let us prove the following lemma on the admissibility of a generalization of (den) for $\mathrm{G}^0\mathrm{RP}\forall$, denoting by $\mathfrak{p}$ a special variable that can assume any real values under hs-interpretations.

LEMMA D.1 (admissibility of a generalization of (den) for $\mathrm{G}^0\mathrm{RP}\forall$). *Suppose that $m \geqslant 1$, $n \geqslant 1$,*

$$\mathcal{H} = \Big(\mathcal{G} \mid [\Gamma_i, \mathfrak{p} \Rightarrow \Delta_i]_{i \in 1..m} \mid [\Pi_j \Rightarrow \mathfrak{p}, \Sigma_j]_{j \in 1..n}\Big),$$
$$\mathcal{H}' = \Big(\mathcal{G} \mid [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]^{i \in 1..m}_{j \in 1..n}\Big),$$

$\mathfrak{p}$ does not occur in $\mathcal{H}'$, and $\vdash_{\mathrm{G}^0\mathrm{RP}\forall} \mathcal{H}$. Then $\vdash_{\mathrm{G}^0\mathrm{RP}\forall} \mathcal{H}'$.

PROOF. Take a ($\mathrm{G}^0\mathrm{RP}\forall$-)proof D of $\mathcal{H}$ and proceed by induction on the height of D.

1. Suppose that $\mathcal{H}$ is an axiom (of $\mathrm{G}^0\mathrm{RP}\forall$); i.e., $\vDash \mathcal{H}_{at}$. Without loss of generality we assume that

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Gamma_i, \mathfrak{p} \Rightarrow \Delta_i]_{i \in 1..k} \,\big|\, [\Pi_j \Rightarrow \mathfrak{p}, \Sigma_j]_{j \in 1..l} \Big),$$

where $0 \leqslant k \leqslant m$ and $0 \leqslant l \leqslant n$. Let $\mathcal{H}'_{at} = (\mathcal{H}')_{at}$. Consider the following cases 1.1–1.4.

Case 1.1: $k \neq 0$ *and* $l \neq 0$. We have

$$\mathcal{H}'_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j]^{i \in 1..k}_{j \in 1..l} \Big)$$

and want to show that $\vDash \mathcal{H}'_{at}$.

Suppose otherwise; i.e., for some hs-interpretation M and some M-valuation ν, there is no true sequent in $\mathcal{G}_{at}$, and for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

By the density of the set $\mathbb{R}$ of all real numbers, there exists $\xi \in \mathbb{R}$ such that, for all $i \in 1..k$ and all $j \in 1..l$,

$$\|\Delta_i\|_{M,\nu} - \|\Gamma_i\|_{M,\nu} < \xi - 1 < \|\Pi_j\|_{M,\nu} - \|\Sigma_j\|_{M,\nu}.$$

Define an hs-interpretation M_1 to be like M but set $|\mathfrak{p}|_{M_1} = \xi$. Since $\mathfrak{p}$ does not occur in $\mathcal{G}_{at}$, Γ_i, Δ_i $(i \in 1..k)$, Π_j, Σ_j $(j \in 1..l)$, we see that no sequent in $\mathcal{H}_{at}$ is true under the hs-interpretation M_1 and M_1-valuation ν. Hence $\nvDash \mathcal{H}_{at}$, a contradiction.

Therefore $\vDash \mathcal{H}'_{at}$, and so $\mathcal{H}'$ is an axiom.

Case 1.2: $k = 0$ *and* $l \neq 0$. Then

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Pi_j \Rightarrow \mathfrak{p}, \Sigma_j]_{j \in 1..l} \Big)$$

and $\mathcal{H}'_{at} = \mathcal{G}_{at}$. Since $\mathfrak{p}$ does not occur in $\mathcal{G}_{at}$, Π_j, Σ_j $(j \in 1..l)$, and hs-interpretations can take $\mathfrak{p}$ to negative real numbers whose absolute values are arbitrarily large, we conclude that $\vDash \mathcal{H}_{at}$ implies $\vDash \mathcal{G}_{at}$. Thus $\vDash \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

Case 1.3: $k \neq 0$ *and* $l = 0$. Then

$$\mathcal{H}_{at} = \Big(\mathcal{G}_{at} \,\big|\, [\Gamma_i, \mathfrak{p} \Rightarrow \Delta_i]_{i \in 1..k} \Big)$$

and $\mathcal{H}'_{at} = \mathcal{G}_{at}$. Since $\mathfrak{p}$ does not occur in $\mathcal{G}_{at}$, Γ_i, Δ_i $(i \in 1..k)$, and $\mathfrak{p}$ can assume arbitrarily large values under hs-interpretations, we see that $\vDash \mathcal{H}_{at}$ implies $\vDash \mathcal{G}_{at}$. So $\vDash \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

Case 1.4: $k = 0$ and $l = 0$. Then $\mathcal{H}_{at} = \mathcal{G}_{at} = \mathcal{H}'_{at}$. Thus $\models \mathcal{H}_{at}$ means that $\models \mathcal{H}'_{at}$, and $\mathcal{H}'$ is an axiom.

2. It remains to consider the case where the root hypersequent $\mathcal{H}$ in D is the conclusion of a rule application. But the argument for this case can be obtained from item 2 of the proof of Lemma 5.3 by replacing $\mathfrak{p}_1$ with $\mathfrak{p}$. ⊣

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