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On the Axiom of Canonicity

Abstract. The axiom of canonicity was introduced by the famous Polish logician Roman Suszko in 1951 as an explication of Skolem's Paradox (without reference to the Löwenheim-Skolem theorem) and a precise representation of the axiom of restriction in set theory proposed much earlier by Abraham Fraenkel. I discuss the main features of Suszko's contribution and hint at its possible further applications. The objective of the paper is rather modest: I only try to recall Suszko's ideas, and I add certain commentaries, hoping that they might inspire younger scientists to develop them further.

Keywords: extremal axiom; axiom of restriction; constructible set; Skolem's paradox; countability; uncountability; model of set theory

1. Introduction

1.1. Motivation

Roman Suszko was one of the most prominent Polish logicians. He is known primarily as the founder of non-Fregean logic as well as one of the most influential persons in the domain of algebraic logic, developed initially mainly in Poland. But Suszko also contributed to model theory, formal analysis of natural language, formal epistemology, and the theory of definitions, among others. He wrote only one work on set theory [Suszko, 1951], and it is precisely that publication that interests us here. I dare to claim that this contribution by Suszko still deserves attention and should be analyzed anew, in particular with respect to modern views on the foundations of set theory. I also believe that the main constructs in Suszko's paper could be applied in other mathematical domains.

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As Suszko himself writes at the beginning of the paper, his main goal was to give a precise formulation to certain considerations related to Skolem's paradox. He emphasizes the fact that his analysis makes no use of the Löwenheim-Skolem theorem, and cites a passage from Carnap's *Logical syntax of language* [1937] where Carnap recalls Fraenkel's axiom of restriction and its connection to Skolem's paradox. Later in the paper, Suszko claims that his considerations make it possible to express Fraenkel's axiom of restriction with precision. I provide a few words of explanation on Skolem's paradox and the axiom of restriction below.

It seems that Suszko achieved much more in the paper in question. First of all, he explicitly showed how to effectively and consistently apply Tarski's metalogical ideas to investigations into set theory. He also proposed an original notion of constructibility, along the lines used only a few years earlier by Gödel, and successfully applied his theory of definitions presented three years earlier in his doctoral dissertation. Finally, he formulated a couple of important open problems concerning canonic models that deserve further attention.

1.2. Skolem's paradox

Already a hundred years ago Thoralf Skolem observed that the theorem now called the Löwenheim-Skolem theorem bore a touch of paradox, which was then named Skolem's paradox. However, it is not a paradox in the proper sense of the term, and should rather be called Skolem's effect, as it is merely a feature of theories formulated in the language of first-order logic. The situation is laid out in brief below.

We accept in set theory (say, first-order Zermelo-Fraenkel set theory) the axiom of infinity, requiring the existence of at least one infinite set. The celebrated Cantor's theorem says that no set is equinumerous with its own powerset. As a consequence, the powerset of a countably infinite set must be uncountable and hence one can prove in set theory that uncountable sets exist. The Löwenheim-Skolem theorem, in turn, says (in its modern formulation) that if a theory in a first-order language has an infinite model, then it also has a countable model. Now, if set theory is consistent, then it has a model and because of the axiom of infinity such a model must be infinite. According to the Löwenheim-Skolem theorem, set theory must also have a countable model. Of course, this countable model satisfies the sentence expressing the existence of uncountable sets. This is an allegedly paradoxical situation. However, the explication is very simple, and Skolem himself was already aware of it. A set X is uncountable in a model $\mathfrak{M}$ if X is infinite in $\mathfrak{M}$ and there exists no bijection in $\mathfrak{M}$ between X and the set of natural numbers in $\mathfrak{M}$. If there are not enough bijections in a model $\mathfrak{M}$, then $\mathfrak{M}$, though countable (from the external point of view), can contain sets that are uncountable (from the point of view of the model $\mathfrak{M}$ itself). This fact is widely recognized in literature. In modern terms, one says that the notion of countability (and uncountability as well) is not an *absolute* notion. Another example of a notion that is not absolute is that of the full powerset operation. Examples of absolute notions, in turn, include the empty set and finite ordinal number.

But Skolem's paradox also has a different aspect, to which Suszko refers in his paper, citing a corresponding passage from [Carnap, 1937], where Carnap recalls Fraenkel's axiom of restriction (Carnap uses a synonymous term, "axiom of limitation"). Roughly summarizing Carnap's argumentation, this axiom says that only those sets exist in the universe of all sets whose existence is required by the remaining axioms. This is the case of the empty set and at least one countably infinite set. As we have only a finite number of constructional steps of further sets (forming pairs, sums, powersets, subsets on the basis of the comprehension axiom, sets obtained by a replacement schema, and sets obtained by the axiom of choice), only countably many sets can be formed in this way, claims Carnap. In conclusion he says that there can exist only countably many sets in the set-theoretical universe. Suszko observes that this type of argumentation has not yet been formally analyzed, and promises to do so later in the paper.

One of the referees of this paper demanded that the Carnap's argumentation mentioned above should be cited here, so I include the corresponding fragment:

> Let us take as object-language S the system of axioms used in Fraenkel's Theory of Aggregates supplemented by a sentential and functional calculus (in the word-language). The theorem that more than one transfinite cardinal exists depends upon the theorem that the aggregate $U(M)$ of the sub-aggregates of an aggregate M has a higher cardinal number than has M; this theorem is based upon what is known as Cantor's theorem, which maintains that M and $U(M)$ cannot have the same cardinal number. Fraenkel has given a proof of this theorem which remains valid for his system S even though it contains the so-called Axiom of Limitation. On the other hand, however, we arrive at the con-

> trary result as a consequence of the following argument. The Axiom of Limitation means that in the aggregate-domain which is treated in S — let us call it B — only those aggregates occur of which the existence is required by the other axioms. Therefore, only the following aggregates are existent in B: in the first place, two initial aggregates, namely, the null-aggregate and the denumerably infinite aggregate, Z, required by Axiom VII; and secondly, those aggregates which can be constructed on the basis of these initial aggregates by applying an arbitrary but finite number of times certain constructional procedures. There are only six kinds of these constructional steps (namely, the formation of the pair-aggregate, of the sum-aggregate, of the aggregate of sub-aggregates, of the aggregate of Aussonderung, of the aggregate of selection, and of the aggregate of replacement). Since only a denumerable multiplicity of aggregates can be constructed in this way, there is in B, according to the Axiom of Limitation, only a denumerable multiplicity of aggregates, and consequently, at the most, only a denumerable multiplicity of sub-aggregates of Z. Therefore $U(Z)$ cannot have a higher cardinal number than Z. [Carnap, 1937, 267–268]; citing [Suszko, 1951, 302–303]
> .

1.3. Fraenkel's axiom of restriction

Before presenting Suszko's solution, let me say a few more words about Fraenkel's axiom of restriction, its motivation, and its place among other extremal axioms. As far as I know, Suszko's contribution has not been yet analyzed as related to extremal axioms.

The first formulation of Fraenkel's axiom can be found in the article "Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre" ("On the foundations of Cantor-Zermelo set theory" [Fraenkel, 1922]). In it, Frankel points out that the Zermelo system of set theory from 1908 does not exclude certain types of set, which — in his words — are irrelevant for mathematical purposes: sets consisting of physical elements and non-well-founded sets. Fraenkel also claims that this is responsible for the non-categorical character of the set theory in question. This was what led him to propose his axiom of restriction, expressing the idea that there are no more sets than those whose existence follows from the axioms of set theory.

Obviously, such a statement does not belong to the object language of set theory and Fraenkel was very well aware of that. He returned to the idea of restriction of the set-theoretical universe several times in his later works [see, e..g., Fraenkel, 1928; Fraenkel and Bar-Hillel, 1958]. Carnap also tried to formalize Fraenkel's idea, first in the paper [Carnap and Bachmann, 1936] (where the term "extremal axiom" was introduced) and then in the monograph [Carnap, 1954, p. 154]. A very detailed analysis of Fraenkel's and Carnap's ideas is presented in the recent papers [Schiemer, 2010a,b].

Fraenkel's informal axiom expresses the idea that the set-theoretical universe should be as narrow as possible. Such minimality conditions were considered earlier by Dedekind [1888] and Peano [1889] with respect to the universe of natural numbers. In this case it was the principle of mathematical induction which was supposed to express the minimality of the universe in question.

Contrastingly, Hilbert in *Grundlagen der Geometrie* [1899] proposed an axiom of maximality of the geometric universe, namely the famous axiom of completeness, which stated that: *To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms.* Again, this is not a statement in the object language of the system, as it concerns rather models of the system of Hilbert's geometry. The axiom of completeness was later replaced by the continuity axiom, known from the theory of real numbers. Let me add in the margins that a generalization of the continuity axiom in Dedekind's formulation presented in [Ehrlich, 2012] is another example of a maximality axiom.

Certain minimal and maximal axioms mentioned above were investigated in full generality in [Carnap and Bachmann, 1936], where the authors tried to formalize the idea of an extremal axiom using the apparatus of type theory. The paper remained almost unnoticed for a long time, but has recently been receiving more and more attention [see, e.g., Awodey and Reck, 2002; Hintikka, 1986; Pogonowski, 2019; Schiemer, 2012]. Suszko does not cite that paper, but cites [Fraenkel, 1928] as well as [Gödel, 1940]; note that Gödel's axiom of constructibility is another example of an axiom of restriction in set theory. However, all axioms of restriction were sharply criticized in [Fraenkel, Bar-Hillel and Levy, 1973]. Gödel had already suggested that set theory should look rather for new maximality axioms (in analogy with Hilbert's completeness axiom in geometry). There has recently been investigation into several axioms of the existence of large cardinal numbers that are often thought to maximize the universe of set theory and, in a sense, demand that this universe should be as rich as possible [see, e.g., Kanamori, 1994].

Weakly inaccessible cardinals were introduced already by Felix Hausdorff in [1908], strongly inaccessible cardinals were described by Wacław Sierpiński and Alfred Tarski in [1930]. Ernst Zermelo argued that the spirit of set theory requires consideration of a transfinite hierarchy of strongly inaccessible cardinals [Zermelo, 1930]. Recently inaccessible cardinals are only a beginning of a really huge transfinite scale of large cardinals.

Extremal axioms (both minimal and maximal) are also related to the problem of a unique characterization of intended models of mathematical theories. We know now what the possibilities and limitations of such characterization are in particular systems of logic. But in the nineteen-twenties, when Fraenkel formulated his axiom, the situation was quite different. It was a time when certain metalogical concepts began to acquire a more or less precise formulation. The concept of categoricity of a theory (that is, the existence of only one model of the theory, up to isomorphism) was introduced by Huntington and Veblen, but in a sense had already been present in Dedekind's work on natural numbers. The concept of semantic completeness, in turn, was still in statu nascendi, so to speak. Semantical completeness was initially understood as exhaustiveness of description of mathematical domains, and sometimes it was identified with categoricity. In modern terms, a theory is semantically complete if all its models are elementarily equivalent, that is they are indistinguishable semantically. The paper [Awodey and Reck, 2002] constitutes a detailed exposition of these issues, in a historical perspective. Fraenkel was investigating the notions of categoricity and semantic completeness for set theory in [Fraenkel, 1928], and he expressed there the view that his restriction axiom could be a proper step towards achieving these properties in set theory [see also Fraenkel, 1927, p. 102]. Of course, the famous limitative theorems obtained by Gödel and others in the nineteen-thirties excluded such a possibility.

2. Suszko's preliminary remarks on constructibility

Suszko's doctoral dissertation [1949] consisted of two separate parts: the first concerned logic without axioms (with the rules of inference only), while the second developed a general theory of definitions with special emphasis on inductive definitions. Certain notions and constructions in the dissertation find essential use in Suszko's paper on canonic axiomatic systems and we shall present a brief review of them here.

2.1. Categorematic names and categorematic designation

Suszko considers mathematical theories at a certain definite stage of their development. The languages of such theories are supposed to have a precisely described syntax and semantics, including the standard notion of designation. The terminology used by Suszko in [1949; 1951, in particular footnote 8 on p. 304], though precise and consequent, may confuse these readers who are accustomed with modern terminology. Therefore I am going to simplify the matter and use modern terminology, hopefully without distortion of the original Suszko's ideas. What Suszko calls a constant name is to be understood as a closed term, that is a term without free variables. These include individual constants (called by Suszko atomic names), complex terms built from function symbols and individual constants and terms obtained by the use of a description operator, that is an expression of the form *the only* x *such that* $\varphi(x)$ with the standard semantic interpretation. Constant names that do not contain a description operator are called by Suszko *categorematic names* (in brief: k-names). Further, designation by a k-name is called *categorematic designation* (in brief: k-designation). Note that k-names are those and only those constant names (closed terms) that are built from atomic names (individual constants) and from those name-creating functors that have nominal arguments.

2.2. Constructible objects

Suszko characterizes constructible objects of a mathematical system at first informally [see 1951, pp. 304–305]. Objects belonging to the universe of a system are divided into two kinds: named and unnamed in the system. An object is named in the system if it belongs to the universe of this system and is designated by a constant name.

By a *constructible object* in a given mathematical system Suszko understands any object from the universe of this system that is k-designated by a k-name. A system whose universe consists of all objects constructible in the system and only such objects is called a *canonic system*. It follows from the properties of k-names and k-designation that in the universe of a canonic system there is at most a denumerable multiplicity of elements. This is an obvious consequence of the fact that expressions are finite sequences of symbols from a finite (or at most countable) alphabet.

Suszko concludes this informal discussion on constructible objects with the following remark:

> The notions introduced above suggest the question, what mathematical theories can be formulated in canonic systems. The aim of this paper is to show that, in principle, every mathematical theory at any stage of its development can be given the form of an axiomatic canonic system. Once we have shown this, we shall arrive indirectly at the Löwenheim-Skolem paradox. For this paradox reduces in our formulation to stating that there exists a canonic system of set theory in which it is possible to prove the existence of nondenumerably many objects.
>
> [Suszko, 1951, p. 305]

The term "denumerable" occurring here (as well as later in the text) is synonymous with the recently more popular term "countable".

2.3. Inductive definitions

I need one more technical notion from [Suszko, 1949] before I present the main part of Suszko's paper on canonic systems. It is related to the theory of definitions proposed in [Suszko, 1949].

Suszko observes that in order to obtain uniqueness of notions introduced by certain inductive definitions we must enrich the system in question with new rules of inference. Let me recall one of his examples illustrating this situation. Suppose that we expand our system by a new axiom U_H, in which H is the only new descriptive term not occurring in the system. The axiom U_H alone does not suffice to characterize the term H in a unique way, because if we introduce another term K with the corresponding axiom U_K (in the same way as in the case of H), then we are still unable to prove in the extended system that: $H(\vec{x}) \equiv K(\vec{x})$, where $\vec{x}$ is an appropriate sequence of arguments. But the equivalence in question will be provable in the system if we add a new rule of inference in the following form:

$$\frac{U_G}{H(\vec{x}) \to G(\vec{x})}.$$

Suszko calls such rules the *rules of complete induction* and shows that their acceptance results in certain decomposition theorems.

For example, if 0 (zero) and $s(x)$ (successor of x) are characterized by the axioms $\neg(s(x) = 0)$ and $s(x) = s(y) \to x = y$ and we expand the system by introducing a new term N (we read $N(x)$: x is a natural number) and axioms characterizing it, that is $N(0)$ and $N(x) \to N(s(x))$, then in the expanded system with a new inference rule

$$\frac{G(0) \quad G(x) \to G(s(x))}{N(y) \to G(y)}$$

we can prove the theorem:

$$N(x) \equiv [x = 0 \vee \exists z\ (N(z) \wedge x = s(z))].$$

Inference rules of the kind described above, that is rules of complete induction, play an essential role in his canonic system of set theory. As explained above, they are necessary for the understanding of newly introduced terms in a unique way.

3. Suszko's system of set theory

The starting point for the construction of Suszko's system M of set theory was a system proposed earlier by Paul Bernays in a series of articles [1937; 1941; 1942a; 1942b; 1943; 1948], together with the system proposed in [Gödel, 1940], with a slight modification inspired by the papers [Quine, 1941, 1946]. I am not going to present Suszko's system in full, limiting myself to the notions necessary for an understanding of the main ideas in his paper. I shall also replace Suszko's original notation with that recently widely adopted. I believe that these simplifications do not distort the content of the discussed issues.

The vocabulary of the system M includes: the predicates $\in$ (binary) and El (unary); atomic names ι_0 (naming identity) and ε_0 (naming elementhood); four name-creating functors of one nominal argument, namely τ (needed in the axiom of foundation), dom (domain), cnv (converse), and cpr (a certain operation on ordered triples, called coupling to the right: $a \in \mathrm{cpr}(b)$ if and only if $El(a)$ and there exist u, v, w such that $El(u)$, $El(v)$, $El(w)$ and $a = (u, (v, w))$ while $b = ((u, v), w)$); three name-creating functors of two arguments, namely $\cup$ (set union), $-$ (set difference), and $\times$ (Cartesian product); and several terms defined in the standard manner (among others: union of a family of sets, powerset of a given set, binary relation, function, and so on). The expression $a \in b$ reads: a is an element of b. In turn, $El(a)$ reads: a is an element (in general). The only rules of inference of the system M are logical rules. Suszko also uses quantifiers relativized to the class of all elements. In modern usage, sets that are not elements are called classes, while those that are elements are called sets.

3.1. Axioms and main definitions

Axioms of the system M are divided into three groups and are presented together with the definitions of some terms occurring in the axioms. The first group consists of the axioms of extensionality, the axiom of foundation and a condition saying that predecessors of $\in$ are elements. The second group features certain set-theoretical constructs: union, difference, Cartesian product, domain (of a binary relation), converse (of a binary relation), and the operation of coupling to the right. Conditions concerning ι_0 (the class of all ordered pairs such that the first element is identical with the second one) and ε_0 (the class of all ordered pairs such that the first element is an element of the second one) also belong to this group. The third group comprises the axioms of pair, sum, powerset, replacement, and the axiom $El(\omega_0)$, where ω_0 is defined as the set of all finite ordinal numbers, understood in the sense of the von Neumann definition. At this moment, Suszko does not yet include the axiom of choice in the body of axioms, but discusses it later in his work.

Definitions of certain concepts will be used later, so let me present them here in brief. V denotes the class of all sets and Λ the empty class. In Suszko's notation, the domain of a relation consists of all successors of ordered pairs belonging to the relation, while counterdomain consists of all predecessors of these pairs (conversely to the widely accepted recent convention). The symbol $b > c$ refers to all a such that $El(a)$, $El(c)$ and the ordered pair $\langle a, c\rangle$ belongs to b. The symbol $\varphi(b, c)$ refers to all a such that $El(a)$, $a \in b$ and a does not belong to $c > a$. These notions will be used in the proof of a generalization of Cantor's theorem. Intuitively speaking, $b > c$ is the b-image of c, while $\varphi(b, c)$ consists of all a from b which do not belong to the c-image of a.

3.2. A generalization of Cantor's theorem

Suszko constructs formulas expressing the notions of denumerability and non-denumerability in the system M. He introduces the expression $\Phi_x(a, b, G(x))$ understood as the conjunction of the following expressions:

A a formula expressing the fact that a is a relation with the domain contained in ω_0 and counterdomain equal b;

B a formula expressing the fact that for each element y in the domain of a the set $a < y$ has the property G, i.e. $G(a < y)$;

C a formula expressing the fact that a-images of distinct elements from the domain of a are distinct;

D a formula expressing the fact that every non-empty subset of b which has the property G is an a-image of some element x from the domain of a.

As Suszko writes on page 308, an expression of the form $\Phi_x(a, b, G(x))$ can thus be read: the relation a establishes a biunique correspondence between the set of all those non-empty subsets x of the set b that fulfill the condition $G(x)$ and a subset of the set of finite ordinal numbers. This could also be paraphrased in short as: the relation a shows the (at most) denumerability of the set of all those non-empty subsets x of the set b that fulfill the condition $G(x)$.

Therefore an expression of the form $\neg\exists y \Phi_x(y, b, G(x))$ can be read as: there does not exist a relation showing the (at most) denumerability of the set of all those non-empty subsets x of the set b that fulfill the condition $G(x)$, or in an equivalent shorter form: the set of those non-empty subsets x of the set b that fulfill the condition $G(x)$ is nondenumerable.

It is worth noting that the above formulas are very general, partly because the arbitrary formula G from the underlying language is involved in their construction. Two special cases are the following:

1. $\neg\exists y \Phi_t(y, \omega_0, t = t)$. This formula expresses the fact that the class of all non-empty sets of finite ordinal numbers is nondenumerable.
2. $\neg\exists y \Phi_t(y, V, t = t)$. This formula expresses the fact that the class of all non-empty sets is nondenumerable.

Let me present a proof of the first of these formulas, expanding the very sketchy proof originally given by Suszko [see Pogonowski, 2019, pp. 207–209]. The proof proceeds by contradiction. Suppose that $\exists y\ \Phi_t(y, \omega_0, t = t)$ is a theorem of M. Let us consider a relation a such that $\Phi_t(a, \omega_0, t = t)$. I shall show that the first and fourth conditions in the definition of the formula $\Phi_t(a, \omega_0, t = t)$ already lead to a contradiction. The two conditions written in symbolic form look as follows:

A $\mathrm{dom}(a) \subset \omega_0$

D $\forall z\ (z \subset \omega_0 \wedge z \neq \emptyset) \rightarrow \exists w\ (w \in \mathrm{dom}(a) \wedge a > w = z)$,

where the existential quantifier is relativized to sets (in Suszko's terminology: to elements). The relation a satisfies the conditions in the definition of the sentence $\Phi_t[a, \omega_0, t = t]$ which means that a is a one-one relation with the domain contained in ω_0 and the counterdomain equal to the set of all sets of finite ordinal numbers. This, together with the condition **D**, implies that the set $\mathrm{dom}(a)$ contains at least two distinct elements, say x_1 and x_2.

Let us now consider three sets: $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2\}$. Because a is a one-one relation and its range includes the set of all non-empty subsets of the set ω_0, there exist three different elements $y_1, y_2, y_3 \in \omega_0$ such that (remember that the domain of binary relation is related to the successors of ordered pairs):

1. $\langle\{x_1\}, y_1\rangle \in a$
2. $\langle\{x_2\}, y_2\rangle \in a$
3. $\langle\{x_1, x_2\}, y_3\rangle \in a$.

It is easy to see that the following cases are mutually exclusive:

1. $\{x_1, x_2\} \cap \{y_1, y_2, y_3\} = \Lambda$
2. $\{x_1, x_2\} \cap \{y_1, y_2, y_3\} \neq \Lambda$.

Therefore there exists at least one element $t \in \mathrm{dom}(a)$ such that $t \notin a > t$. This means that t does not belong to the a-image of t. In the first of the above two cases we take for t any element of the set $\{y_1, y_2, y_3\}$. In the second case we take for t any element belonging to $\{y_1, y_2, y_3\} - \{x_1, x_2\}$. This choice is essential for the non-emptiness of the diagonal set to be defined in a moment.

Due to the condition **A** we have $t \in \omega_0$. We now construct the diagonal set:

$$\varphi(\omega_0, a) = \{x : \ El(x) \wedge x \in \omega_0 \wedge x \notin a > x\}.$$

This set is non-empty and due to its definition it is a subset of ω_0. One can apply the condition **D** to it: there exists an element w such that $w \in \mathrm{dom}(a)$ and

$$\varphi(\omega_0, a) = a > w = \{z : \ El(z) \wedge \langle z, w\rangle \in a\}. \qquad (*)$$

But due to the definition of the diagonal set we have for each z:

$$z \in \varphi(\omega_0, a) \text{ if and only if } z \notin a > z. \qquad (**)$$

We have already shown that the diagonal set is non-empty. Let us ask whether $w \in \varphi(\omega_0, a)$:

1. Due to $(*)$, $w \in \varphi(\omega_0, a)$ if and only if $\langle w, w\rangle \in a$.
2. Due to $(**)$, $w \in \varphi(\omega_0, a)$, if and only if $\langle w, w\rangle \notin a$.

We thus get a contradiction and hence the supposition of the existence of a relation a with the properties given above should be rejected. This means, in turn, that $\neg\exists x\ \Phi_t[x, \omega_0, t = t]$ is a theorem in the system.

4. Metatheoretical constructions

4.1. The system μM

Suszko begins his metatheoretical considerations with a synopsis of the procedure of including morphology of the object language in the corresponding metalanguage. The procedure itself is known from [Tarski, 1935], and Suszko also refers to [Tarski, 1933] and [Quine, 1946]. Several modern logical textbooks contain information about the procedure [see, e.g., Batóg, 1994, pp. 233–237]. Following the advice of the referees of this paper, let me in brief explain its main ideas for these readers who are not familiar with the procedure in question.

Tarski proposed an axiomatic approach to metatheoretical considerations. Given a formal theory in an object language, one can built its metatheory in the corresponding metalanguage which itself is a formal theory. The metalanguage should include logical constants and variables representing objects about which we talk in the metatheory. Actually, one needs several sorts of variables: one referring to the expressions of the object language, another one referring to the objects being discussed in the initial formal theory, and possibly also some auxiliary objects (for instance, set-theoretical constructs). The set of all names (structural descriptions) of expressions of the object language is then characterized axiomatically, with the use of the concatenation operation and a suitable closure condition, which has the form of a rule of complete induction. Metatheory contains also definitions of such syntactic notions as being a thesis of the underlying theory and semantic notions such as designation, among others. For more details the reader may consult the original Tarski's works cited above.

The metasystem μM for M proposed by Suszko has two sorts of expressions, referring to the objects of M and to the expressions of M, respectively. The metasystem μM also contains certain syntactic and semantic notions related to the object system M. Suszko defines the notions of a k-name in M and k-designation in M. Both definitions are ancestral, and are supplied with the corresponding rules of complete induction (see section 2.3 above). The expression $k\text{-}Nom^M(a)$ reads: a is a k-name in M, and the expression $k\text{-}Des^M(a,b)$ reads: a k-designates b in M. On the basis of these definitions and rules of inference, one can prove theorems in M expressing the adequacy of the introduced notions, in particular:

$$\forall x\ (k\text{-}Nom^M(x) \equiv \exists y\ k\text{-}Des^M(x,y))$$
$$\forall x \forall u \forall v\ (k\text{-}Des^M(x,u) \wedge k\text{-}Des^M(x,v) \to u = v).$$

At this stage Suszko introduces the notions of an object in M and a constructible object in M. If a is any name in M, then Suszko proposes reading the sentence $a = a$ as asserting that a is an object from the universe of the system M. The sentence $\exists x\ k\text{-}Des^M(x,a)$, in turn, is to be read: a is a constructible object in M. If it is the case that the sentence $\forall t \exists x\ k\text{-}Des^M(x,t)$ holds in μM, then Suszko proposes calling the system M a canonic system.

Suszko stresses the universality of the construction of metasystems of this sort. In a footnote on page 315 he mentions the fact that morphology of the object system built in the corresponding metasystem can be shown to contain the syntax of the object system, as explained in the works [Quine, 1941, 1946].

4.2. An extension of the system M

Suszko considers further systems related to the system M, beginning with M^* which is obtained from M by adding to M a new unary predicate M^*n, an axiom characterizing it (in the form of an ancestral definition), and a new rule of inference being the corresponding rule of complete induction necessary for uniqueness of the introduced predicate. An expression of the form $M^*n(a)$ is to be read: a is a k-set; and an expression of the form $El(a) \wedge M^*n(a)$ is to be read: a is a k-element.

In order to show that the notion of a k-set is an “objective” counterpart of the semantic notion of an object constructible in M, Suszko builds a common metasystem for the systems M and M^* in the same way as the system μM was built for M. In this metasystem one can prove that the notion of a k-set is equivalent to the notion of an object constructible in M (and in M^* as well).

All theorems of M can be obtained in M^*. Furthermore, in M^* one can prove the following theorems:

$\neg\exists x(M^*n(x) \wedge \Phi_t(x, \omega_0, M^*n(t)))$. It says that: there does not exist a relation being a k-set that establishes (at most) denumerability of the totality of all non-empty subsets of ω_0 that are k-sets.

$\neg\exists x(M^*n(x) \wedge \Phi_t(x, V, M^*n(t)))$. It says that: there does not exist a relation being a k-set that establishes (at most) denumerability of the totality of all non-empty k-sets.

$\neg\exists x(M^*n(x) \wedge \Phi_t(x, V, M^*n(x) \wedge \forall z(z \in t \to M^*n(z))))$. It says that: there does not exist a relation being a k-set that establishes (at most) denumerability of the totality of all non-empty k-sets whose elements are k-elements only.

Suszko then goes on to prove that the system of k-sets is a model for M^*, using the technique of relativization of quantifiers to k-sets. This implies that every theorem of M^* relativized to k-sets is again a theorem of M^*. The general procedure used in this part of the paper (that is, the method of syntactic interpretation) is described in detail in several modern logical textbooks [see, e.g. Batóg, 1994, pp. 193–215]. Suszko shows, step by step, that relativizations to k-sets of all axioms of M^* are theorems of M^*. In the case of the axiom of extensionality this fact has a consequence that two different sets such that every k-set which is an element of one of them is also an element of the other one are not both k-sets.

4.3. The axiom of canonicity

The model built of k-sets constitutes an interpretation of the system M^* inside M^*. Adding the sentence $\forall t M^*n(t)$ to the system M^* as a new axiom, Suszko obtains a system $\overline{M^*}$. If the system M^* is consistent, then the system $\overline{M^*}$ is consistent as well. Suszko calls the sentence $\forall t M^*n(t)$ *the axiom of canonicity*. He then claims that the axiom of canonicity, together with the characterization of the predicate M^*n mentioned above, constitutes a precise formulation of Fraenkel's axiom of restriction. However, this claim is not supported by the author by any argumentation and therefore it is difficult to comment on it. In my opinion it is secure to say that Suszko's axiom of canonicity is an example of an extremal axiom, an axiom of minimality, in a sense similar to Gödel's axiom of constructibility.

The system $\overline{M^*}$ is a canonic system, which can be shown by building a metasystem $\mu\overline{M^*}$ in the way mentioned in section 4.1 and proving in it that every k-set is a constructible object in $\overline{M^*}$ and conversely, every constructible object in $\overline{M^*}$ is a k-set.

Suszko points to the possibility of forming a common metasystem $\mu M, M^*, \overline{M^*}$ in the way mentioned above, and concludes that the notions of a constructible object in M and M^* coincide and, moreover, are equivalent to the notion of a k-set. And because the axiom of canonicity holds in this metasystem, the systems M, M^* and $\overline{M^*}$ are all canonic systems.

On the basis of all hitherto results, Suszko is now able to conclude that he has achieved the goal declared at the beginning of the paper, that is a formal explication of Skolem's paradox. Canonic systems of set theory considered in the paper are countable (from the point of view of metalanguage), and one can prove in them the theorem stating the existence of uncountable sets.

Suszko describes next a general method of constructing canonic systems being extensions of systems satisfying certain natural conditions. The first steps in the construction consist of eliminating the description operator and transforming the axioms of a system X into prenex normal forms. The existential quantifiers are then eliminated by the well-known procedure of Skolemization, which introduces new individual constants and function symbols. The system Y obtained from X in this way is a conservative extension of X. Finally, system Z is obtained from X by introducing the definition of a k-object and addition of the appropriate form of the axiom of canonicity. The system Z is then a canonic system (which can be proved by forming the corresponding metasystem μZ) and an extension of the initial system X. Suszko summarizes this as follows:

> In the general case, in which, building the canonic system Z of which the system X is a fragment (simply), a certain artificiality cannot be avoided, the methodological situation is very simple. It is namely easy to show that by joining the axiom of canonicity we are not introducing any contradictions. For it may be proved in the system not containing this axiom that the k-objects are constituting a model of the system in which the axiom of canonicity is assumed. This is due to the fact that no bound variables are occurring in the axioms hence the relativization of variables to k-sets leads always from theorems to theorems. It is also obvious that the axiom of canonicity is satisfied by the model of k-objects. [Suszko, 1951, p. 324]

Suszko also considers an expansion of the initial system M by the addition of the axiom of choice which, in his formulation, takes the form of a set-theoretical definition characterizing a new atomic name τ_0 (here existential quantifiers are relativized to elements):

$$a \in \tau_0 \;\equiv\; El(a) \wedge \exists u \exists v\, (a = \langle u, v\rangle \wedge u = \tau(v)).$$

I skip a discussion of Suszko's further comments concerning the notion of a constructible object in this expanded system and a few other expansions presented by him.

4.4. Suszko's conclusions

In the last part of the paper Suszko formulates certain general problems related to previously obtained results. He writes that it could be interesting to investigate relationships between his notion of constructibility and the effectiveness of existence and constructibility as understood in intuitionistic logic. Furthermore, he poses the problem of developing his system in order to include the theory of ordinal numbers and comparing his notion of constructibility with that proposed by Gödel.

Suszko makes it explicit that although in theory one could conduct mathematical research on the basis of canonic systems only, such a possibility is only virtual, because the development of mathematical theories is dynamic, and the investigated universes are changing and expanding:

> The universe of a canonic system is limited and unchanging, and so is the range of every notion occurring in such a system. Should we limit ourselves to canonic systems, every change of such a system would necessitate the introducing of a new definition of a constructible object and of a new axiom of canonicity. As regards non-canonic systems, their universes and also the ranges of their notions, which strictly speaking are not notions but schemata of notions, are not limited.
>
> [Suszko, 1951, p. 328]

Finally, Suszko discusses in a draft form the possibility of obtaining essentially the same results as those in his paper by arithmetization of metatheoretical constructs. It should perhaps be added that John Myhill proposed a similar approach in the short note [Myhill, 1952], though without reference to Suszko's earlier paper (both results may thus be acknowledged as obtained independently of each other).

5. The significance of Suszko's contribution

Suszko never did return to the study of his canonic systems. One can only speculate as to why. Note that at that time models of set theory had not yet been discussed in detail. Gödel's ideas concerning the constructible universe were still fresh, while Cohen's independence results were only to appear fifteen years later [see 1966]. Classical model theory, promoted at first by Tarski, Vaught, and others, only began a few years after Suszko's article. Shepherdson demonstrated the limitations of the method of inner models in a series of papers [1951; 1952; 1953]. Carnap persevered in his attempts to formalize Fraenkel's axiom of restriction, developing his earlier proposals contained in [Carnap and Bachmann, 1936], for instance in [Carnap, 1954]. Axioms of restriction in set theory were criticized in [Fraenkel, Bar-Hillel and Levy, 1973]. A revival of interest in Carnap's ideas was observed quite recently, but this is not related to the content of Suszko's paper.

It might be the case that Suszko decided that he had reached the goal of his research concerning the explication of Skolem's paradox, and lost interest in the continuation of the research. However, I think the main reason for him not continuing the work was the fact that in the nineteen-fifties, after his move to Warsaw, Suszko became mostly interested in algebraic logic, which he focused on in his work. In his "Warsaw period", Suszko published numerous works on consequence relations, model theory, and the logical analysis of natural language, among others. Afterwards, for more than a decade, he was working intensively on non-Fregean logic, a system he created in the late nineteen-sixties and developed until his premature death in 1979. Following the advice of one of the referees of this paper let me say a few words about this matter. Non-Fregean logic is a two-valued, fully extensional system and it makes the weakest possible ontological assumptions. According to Suszko, sentences have not only logical values but also denotations: they describe situations and, in particular, facts. Suszko expanded the language of propositional calculus with the identity connective, which was supposed to be the linguistic counterpart of the identity of situations described by sentences. Suszko used his system of non-Fregean logic to formalize the ontology of Wittgenstein's *Tractatus*. Readers interested in Suszko's achievements in this domain may consult, for instance, [Bloom and Suszko, 1972; Omyła, 1986; Suszko, 1968b, 1975].

In [1967] Suszko published a review, entitled “Expedition against Skolemites” of Michael Resnik’s paper “On Skolem’s paradox” [1966]. Roughly speaking, by a Skolemite one means a person who claims that only countable infinities exist, and from an absolute point of view uncountable sets do not exist. For a detailed exposition and critique of the Skolemite’s position see, for instance, [Bays, 2014] or [Bellotti, 2008]:

> The Skolemite position is not easy even to state. Let us start with the following formulation (Resnik 1966, p. 425). According to the Skolemite, Skolem’s ‘paradox’ would show that axiomatic set theories prove the existence of sets which are *uncountable only relative to these set theories but countable from an absolute point of view*. We can distinguish between a *strong* and a *weak* Skolemite thesis. According to the first, no set theory can produce genuinely uncountable sets; according to the second, *either* set-theoretic concepts elude any axiomatic characterization, *or* uncountability can only be relative and all sets are denumerable from an absolute point of view. [Bellotti, 2008, p. 187]

Numerous papers on Skolem’s paradox and the Skolemite thesis appeared much later than the work [Suszko, 1951], so he cannot be counted as a participant in the later dispute. But still, his considerations are among the pioneering ones. The last sentence of [Suszko, 1951] is:

> In the light of this, as well as of the whole trend of our considerations, it becomes evident that the set-theoretical notion of denumerability should be reconsidered anew. [Suszko, 1951, p. 328]

It is worth adding that in 1992 a Polish version [Suszko, 1950] of Suszko’s dissertation [1951] was found and then published in 2002 in an archive-material volume of *Kwartalnik Filozoficzny*. The English text is a faithful translation of the Polish original.

5.1. Reception of the paper

Suszko’s article was reviewed by Jan Kalicki in *The Journal of Symbolic Logic* 17: 211–212. Several authors discussed his dissertation shortly after its publication [Fraenkel and Bernays, 1958, p. 23], [Fraenkel and Bar-Hillel, 1958, p. 116], [Fraenkel, Bar-Hillel and Levy, 1973, p. 116], [Mostowski, 1955, pp. 38–39], [Wang, 1955, pp. 64–65], displaying an appreciation for the originality of Suszko’s contribution.

However, Suszko’s paper is not cited in numerous later papers on Skolem’s paradox or Fraenkel’s axiom of restriction. This is quite surprising, because *Studia Philosophica* was among the most important journals dealing with mathematical logic and the foundations of mathematics, and was widely recognized in the academic community. Suffice to say that the first volume of the journal contained Tarski's celebrated work [1935]. An extensive bibliography of papers devoted to Skolem's paradox can be found for instance in [Bays, 2014], while many bibliographical items concerning Fraenkel's axiom of restriction are given, for example, in [Schiemer, 2010a].

Quite recently Krystian Jobczyk [2015] tried to apply the ideas contained in Suszko's dissertation to the defence of Hilary Putnam's model-theoretic argument presented in [Putnam, 1980]. Critical remarks concerning this proposal are contained in [Woleński, 2015].

Chapter 7 of [Pogonowski, 2019] contains a presentation of Suszko's canonic axiomatic systems, discussed in comparison with other proposals of extremal axioms in set theory, including Fraenkel's axiom of restriction, Gödel's axiom of constructibility, von Neumann's axiom of limitation of size, and axioms of the existence of large cardinal numbers.

5.2. Possible further developments of Suszko's ideas

One might ask whether the ideas and constructs related to canonic axiomatic systems could possibly still be relevant in the foundational research, taking into account the development of this research after Suszko's publication. Suszko himself was overtly cautious about such a possibility:

> The study of canonic systems may induce to postulating that solely canonic systems should be used in mathematics. From the theorem of canonicity it follows that this could be done without impoverishing the scope of mathematics. Since, however, mathematical theories are being incessantly enlarged with respect to their problems as well as the scope of investigated objects, the above postulate can not be considered as a practical demand, but solely as the expression of a theoretical possibility. [Suszko, 1951, p. 327–328]

It follows from the above passage that Suszko considers the construction of a given canonic axiomatic system a formal representation of a mathematical theory viewed synchronically, at a fixed point of its development. A few years later Suszko also proposed a diachronic perspective concerning formal logic and epistemology [see 1957a; 1957b; 1968a].

One could raise the problem of comparing Suszko's early ideas with the further development of mathematical aspects of set theory, such as findings on the constructible hierarchy, the method of forcing, reflection principles, findings concerning the existence of large cardinal numbers, and so on. One may also ask whether specific notions from classical and modern model theory are applicable to canonic axiomatic systems, and what mathematical results could emerge from these applications. Suszko himself did some research in model theory and published (in collaboration with Łoś and Słomiński) a series of papers concerning extensions of models, but there is no explicit connection between those works and the dissertation from 1951. As already mentioned, Suszko posited the problem of developing the theory of ordinal numbers in terms of canonic axiomatic systems, which could be the first step towards comparing his notion of constructibility with that proposed by Gödel. As far as I know, nobody has undertaken this task yet. Another challenge is to consider canonic axiomatic systems from the intuitionistic point of view. Suszko has shown the consistency of the axiom of canonicity with other axioms of the system M. He did not consider the problem of its independence from these axioms. Gödel's axiom of constructibility is known to be independent of the other axioms of Gödel's system [see, e.g., Shoenfield, 1959], and one might try to solve this problem in the case of Suszko's system.

Harvey Friedman sees the source of incompleteness of rich mathematical theories (including set theories) in the fact that these theories allow the consideration of completely arbitrary objects [1992]. Friedman introduces the notion of *points of view* in mathematics. He discusses Borel, constructive, and predicative points of view, and shows that certain statements undecidable in set theory become decidable, when restricted to sets of a prescribed form. One may of course raise the issue of the naturalness of such restrictions, but they can be claimed to play a regulative role in mathematical research. I think that Suszko's approach can be viewed as a special point of view in set theory, being a kind of a constructive point of view.

In a recent publication [Hamkins, Linetzky and Reitz, 2013] the authors address what they call "the math-tea argument": that there must be real numbers that we cannot describe or define, because there are only countably many definitions, but uncountably many reals. They then answer in the affirmative the following question: Is it consistent with the axioms of set theory that every real is definable in the language of set theory without parameters? We say that a first-order structure $\mathbf{M}$ is pointwise definable if every element of $\mathbf{M}$ is definable in $\mathbf{M}$ without parameters. The authors prove several theorems about the existence of pointwise definable models of set theory, including the following:

1. If ZFC is consistent, then there are continuum many nonisomorphic pointwise definable models of ZFC.
2. Every countable model of GBC has a pointwise definable extension, in which every set and class is first-order definable without parameters.

I believe these results are relevant to Suszko's ideas. The authors credit John Myhill with the observation that if ZFC is consistent, then there is a pointwise definable model of GBC + V=L [see Myhill, 1952], and cite the last sentences of his short note:

> One often hears it said that since there are indenumerably many sets and only denumerably many names, therefore there must be nameless sets. The above shows this argument to be fallacious.
>
> [Myhill, 1952, p. 979]

As already said, Myhill's result was obtained independently of Suszko's publication [1951], but Suszko announced the possibility of a solution in Myhill's style, using arithmetization of syntax and Skolem functions [see Suszko, 1951, p. 328]. This fact should really, out of fairness, be acknowledged by the above authors. Anyway, it could be interesting to look at canonic axiomatic systems from the point of view presented by the two of them.

In turn, let me suggest another possibility of further investigations along the lines proposed by Suszko. This is the possibility of constructing canonic axiomatic systems not for set theory but for other mathematical theories, for instance the arithmetic of natural numbers, systems of geometry (absolute, Euclidean, projective, and so on), or selected algebraic theories (for instance the theory of real numbers). Applying Suszko's approach to each of these domains might result in some light being shed on the problems of accessibility to the mathematical objects described by the corresponding theories. Still another possibility is to consider relationships between canonic axiomatic systems associated with the different stages of development of mathematical theories.

Could we say that Suszko's canonic system of set theory is (in a sense to be specified) an intended model, or a standard model of set theory? An answer to this question presupposes a characterization of what is meant both by an intended model and by a standard model. The terms "intended model" and "standard model" are sometimes used interchangeably in literature, but I prefer to distinguish them in the following manner [see Pogonowski, 2019, 2020]. An intended model is a structure investigated for its own sake, usually over a long period of time, so that we accumulated a sizeable amount of information about it. This in turn is responsible for "domestication" of the structure in question, meaning in particular that we have gathered stable intuitions about the investigated objects. Intended models precede the formal theories developed to characterize them. Examples of intended models in this sense include natural number series, integers, rational and real numbers, the geometric universe depicted in Euclid's *Elements*, and possibly also the universe of Cantorian set theory understood in the sense of "naive set theory", that is before its axiomatization proposed by Zermelo. Standard models, in turn, are models distinguished in the class of all models of a (consistent) mathematical theory partly on the basis of pragmatic criteria. Let us call a model of such a theory (which is ultimately an axiomatic theory) its standard model, if it is most closely similar to the intended structure investigated in advance. Examples of standard models in this sense are the standard model of first-order Peano arithmetic, and the completely ordered field of real numbers. The situation in set theory is a little bit more complicated. Set theoreticians call a model of set theory standard if the denotation of the membership predicate is the real membership relation. If set theory (say, the first-order Zermelo-Fraenkel set theory with the axiom of choice) is consistent, then it has many models. Which of them have a privileged status? It seems (to me at least) that "normal" mathematicians (that is these not working on the foundations of set theory) either believe in the existence of *the* universe V of all sets or are fully satisfied with a smaller universe, like the class L of all constructible sets. Compare the following opinion on this matter:

> The set theorist is looking for deep theoretic phenomena, and so $V = L$ is anathema since it restricts the set theoretic universe so drastically that all sorts of phenomena are demonstrably not present. Furthermore, for the set theorist, any advantage that $V = L$ has in terms of power can be obtained with more powerful axioms of the same rough type that accommodate measurable cardinals and the like—e.g., $V = L(\mu)$, or the universe is a canonical inner model of a large cardinal.
> However, for the normal mathematician, since set theory is merely a vehicle for interpreting mathematics as to establish rigor, and not mathematically interesting in its own right, the less set theoretic difficulties and phenomena the better.

> I.e., less is more and more is less. So if mathematicians were concerned with the set theoretic independence results — and they generally are not — then $V = L$ is by far the most attractive solution for them.
> This is because it appears to solve all set theoretic problems (except for those asserting the existence of sets of unrestricted cardinality), and is also demonstrably relatively consistent.
> Set theorists also say that $V = L$ has implausible consequences — e.g., there is a PCA well ordering of the reals, or there are non-measurable PCA sets.
> The set theorists claim to have a direct intuition which allows them to view these as so implausible that this provides "evidence" against $V = L$.
> However, mathematicians disclaim such direct intuition about complicated sets of reals. Some say they have no direct intuition about all multivariate functions from $\mathbb{N}$ into $\mathbb{N}$!
> [Feferman, Friedman, Maddy and Steel, 2000, pp. 436–437]

The above characterization of these two notions is to a great extent intuitive. Intended structures emerge in mathematical research practice, and they give rise to the formulation of formal theories. To call a model standard is based on our decision, dictated by its observed resemblance to the intended structure given a priori. It may well happen that the development of a given mathematical discipline forces us to extend the collection of standard models. If, for instance, analysis based on hyperreal numbers brings new deep results with a wide scope of applications, then hyperreal numbers may become a new standard, corresponding to the intended structure (the continuum) investigated for so many centuries and recently having an associated standard or "usual" real numbers.

I think that Suszko's canonic axiomatic systems should be seen in this perspective as standard models of set theory of a very special kind. The universe of a canonic system, as consisting of objects which are k-designated by k-names, embraces all objects that are linguistically accessible from the point of view of the underlying axiomatic system M of set theory, and only those objects. The metatheoretical constructions associated with the system M provide further control over the investigated objects and their theory. This applies not only to set theory but also to any mathematical theory represented in the form of a canonic axiomatic system.

I must sincerely admit that the present section contains only loose suggestions without any substantiation in the form of original results.

They can be treated as an invitation addressed to younger scientists to develop the ideas in question further, if these ideas would deserve their attention. My primary goal was to recall Suszko's metatheoretical approach to foundational matters. As far as I know from private communications, Suszko's approach to set theory visible in his later works was instrumental — he used to say that he simply *uses* set theory, similarly to these "normal" mathematicians who are not set-theoretical specialists. He wrote in his textbook on logic and set theory (mine translation from the Polish original text):

> Fundamental principles of logic contain certain knowledge about formal properties and structural relationships present in the world. We accept thus set theory in logic as a certain general and schematic knowledge about reality which decides — on the basis of semantic relations between expressions and their subjects — about logical properties and dependencies of the expressions themselves. Set theory, and the calculus of sets and relations in particular (Leiniz's *mathesis universalis*), plays thus a role of ontological assumptions of formal logic. [Suszko, 1965, p. 52]

Readers interested in Suszko's philosophical ideas may consult [1968a; 1975].

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References

Awodey, S., and E. H. Reck, 2002, "Completeness and categoricity. Part I: nineteenth-century axiomatics to twentieth-century metalogic", *History and Philosophy of Logic* 23: 1–30.

Batóg, T., 1994, *Podstawy logiki* [*Fundamentals of logic*], Poznań: Wydawnictwo Naukowe UAM.

Bays, T., 2014, "Skolem's paradox", *Stanford Encyclopedia of Philosophy*, accessed on June 15, 2021. https://plato.stanford.edu/entries/paradox-skolem/

Bellotti, L., 2008, "Skolem, the Skolem's 'paradox' and informal mathematics", *Theoria* 72: 177–220.

Bernays, P., 1937, "A system of axiomatic set theory. Part I", *The Journal of Symbolic Logic* 2: 65–77.

Bernays, P., 1941, "A system of axiomatic set theory. Part II", *The Journal of Symbolic Logic* 6: 1–17.

Bernays, P., 1942a, "A system of axiomatic set theory. Part III", *The Journal of Symbolic Logic* 7: 65–89.

Bernays, P., 1942b, "A system of axiomatic set theory. Part IV", *The Journal of Symbolic Logic* 7: 133–145.

Bernays, P., 1943, "A system of axiomatic set theory. Part V", *The Journal of Symbolic Logic* 8: 89–106.

Bernays, P., 1948, "A system of axiomatic set theory. Part VI", *The Journal of Symbolic Logic* 13: 65–79.

Bloom, S. L., and R. Suszko, 1972, "Investigations into sentential calculus with identity", *Notre Dame Journal of Symbolic Logic* 13: 289–308.

Carnap, R., 1937, *The Logical Syntax of Language*, London: Kegan Paul.

Carnap, R. 1954, *Einführung in die symbolische Logik mit besonderer Berücksichtigung ihrer Anwendungen*, Wien: Springer-Verlag.

Carnap, R., and F. Bachmann, 1936, "Über Extremalaxiome", *Erkenntnis* 6: 166–188.

Cohen, P. 1966, *Set theory and the continuum hypothesis*, New York: W. A. Benjamin.

Dedekind, R. 1888, *Was sind und was sollen die Zahlen?* Braunschweig: Friedrich Vieweg und Sohn.

Ehrlich, P. 2012, "The absolute arithmetic continuum and the unification of all numbers great and small", *The Bulletin of Symbolic Logic* 18 (1): 1–45.

Feferman, S., H. M. Friedman, P. Maddy and J. R. Steel, 2000, "Does mathematics need new axioms?", *The Bulletin of Symbolic Logic* 6 (4): 401–446.

Fraenkel, A. A., 1922, "Zu den Grundlagen der CANTOR-ZERMELOschen Mengenlehre", *Mathematische Annalen* 86, 230–237.

Fraenkel, A. A., 1927, *Zehn Vorlesungen über die Grundlegung der Mengenlehre*, Leipzig und Berlin: Springer.

Fraenkel, A. A., 1928, *Einleitung in die Mengenlehre*, Berlin: Verlag von Julius Springer.

Fraenkel, A. A., and Y. Bar-Hillel, 1958, *Foundations of Set Theory*, Amsterdam London: North-Holland Publishing Company.

Fraenkel, A. A., and P. Bernays, 1958, *Axiomatic Set Theory*, Amsterdam: North-Holland Publishing Company.

Fraenkel, A. A., Y. Bar-Hillel and A. Levy, 1973, *Foundations of Set Theory*, Amsterdam London: North-Holland Publishing Company.

Friedman, H., 1992, "The incompleteness phenomena", pages 49–84 in *Proceedings of the AMS Centennial Symposium, August 8–12, 1988*, American Mathematical Society.

Gödel, K., 1940, "The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory", Princeton: *Annals of Mathematics Studies* 3.

Hamkins, J. D., D. Linetzky and J. Reitz, 2013, "Pointwise definable models of set theory", *The Journal of Symbolic Logic* 78 (1): 139–156.

Hausdorff, F., 1908, "Grundzüge einer Theorie der geordneten Mengen", *Mathematische Annalen* 65 (4): 435–505.

Hilbert, D., 1899, *Grundlagen der Geometrie*, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Leipzig: Teubner.

Hintikka, J., 1986, "Extremality assumptions in the foundations of mathematics", *Philosophy of Science Association* 2: 247–252.

Jobczyk, K., 2015, "Putnam i jego argument teoriomodelowy. Polemiczne uwagi do Jana Woleńskiego krytyki antyrealizmu semantycznego" [Putnam and his model-theoretic argument. Polemic remarks concerning Jan Woleński's critique of semantic antirealism], *Filozofia Nauki* 3 (91): 133–143.

Kanamori, A., 1994, *The Higher Infinite. Large Cardinals in Set Theory from their Beginnings*, Berlin: Springer-Verlag.

Mostowski, A., 1955, "Współczesny stan badań nad podstawami matematyki" [Recent state of research in the foundations of mathematics], *Prace matematyczne* 1: 13–55.

Myhill, J. 1952, "The hypothesis that all classes are nameable", *Proceedings of the National Academy of Sciences of the United States of America* 38: 979.

Omyła, M. 1986, *Zarys logiki niefregowskiej*[*An Outline of Non-Fregean Logic*], Warszawa: Państwowe Wydawnictwo Naukowe.

Peano, G., 1889, *Arithmetices Principia, Nova Methodo Exposita*, Torino: Bocca.

Pogonowski, J., 2019, *Extremal Axioms. Logical, Mathematical and Cognitive Aspects*, Poznań: Wydawnictwo Nauk Społecznych i Humanistycznych UAM.

Pogonowski, J., 2020, "A note on intended and standard models", *Studia Humana* 9 (3–4): 131–139.

Putnam, H., 1980, "Models and reality", *The Journal of Symbolic Logic* 45: 464–482.

Quine, W. V. O., 1941, “Element and number”, *The Journal of Symbolic Logic* 6 (4): 135–149.

Quine, W. V. O., 1946, “Concatenation as a basis for arithmetic”, *The Journal of Symbolic Logic* 11 (4): 105–114.

Resnik, M. 1966, “On Skolem’s paradox”, *The Journal of Philosophy* 63 (15): 425–438.

Schiemer, G., 2010a, “Fraenkel’s axiom of restriction: axiom choice, intended models, and categoricity”, pages 307–340 in B. Löwe and T. Müller (eds.), *Philosophy of Mathematics: Sociological Aspects and Mathematical Practice*, London: College Publications.

Schiemer, G., 2010b, “Carnap’s early semantics”, PhD Dissertation, Universität Wien.

Schiemer, G., 2012, “Carnap on extremal axioms, “completeness of models”, and categoricity”, *The Review of Symbolic Logic* 5 (4): 613–641.

Shepherdson, J. C., 1951, “Inner models for set theory. Part I”, *The Journal of Symbolic Logic* 16: 161–190.

Shepherdson, J. C., 1952, “Inner models for set theory. Part II”, *The Journal of Symbolic Logic* 17: 225–237.

Shepherdson, J. C. 1953, “Inner models for set theory. Part III”, *The Journal of Symbolic Logic* 18: 145–167.

Shoenfield, J. R., 1959, “On the independence of the axiom of constructibility”, *American Journal of Mathematics* 81 (3): 537–540.

Sierpiński, W., and A. Tarski, 1930, “Sur une propriété caractéristique des nombres inaccessibles”, *Fundamenta Mathematicae* 15: 292–300.

Suszko, R., 1949, “O analitycznych aksjomatach i logicznych regułach wnioskowania. Z teorii definicji [On analytical axioms and logical rules of inference. From the theory of definitions]”, Poznań: Poznańskie Towarzystwo Przyjaciół Nauk, Prace Komisji Filozoficznej 7 (5): 1–57.

Suszko, R., 1950, “Konstruowalne przedmioty i kanoniczne systemy aksjomatyczne” [Constructible objects and canonic axiomatic systems], *Kwartalnik Filozoficzny* 19 (3/4): 331–359. Published in an archive volume by Polska Akademia Umiejętności and Uniwersytet Jagielloński in 2002.

Suszko, R., 1951, “Canonic axiomatic systems”, *Studia Philosophica* 4: 301–330.

Suszko, R., 1957a, “Logika formalna a niektóre zagadnienia teorii poznania” [Formal logic and some problems of the theory of knowledge], *Myśl Filozoficzna* 2 (28): 27–56.

Suszko, R., 1957b, "Logika formalna a niektóre zagadnienia teorii poznania. Diachroniczna logika formalna" [Formal logic and some problems of the theory of knowledge. Diachronic formal logic], *Myśl Filozoficzna* 3 (29): 34–67.

Suszko, R., 1965, *Wykłady z logiki formalnej. Część I: Wstęp do zagadnień logiki. Elementy teorii mnogości* [*Lectures on formal logic. Part I: Introduction to logic. Elements of set theory*], Warszawa: Państwowe Wydawnictwo Naukowe.

Suszko, R., 1967, "Wyprawa przeciw Skolemitom" [Expedition against Skolemites], *Studia Filozoficzne* 2 (49): 264–266). Review of Resnik [1966].

Suszko, R., 1968, "Formal logic and the development of knowledge", pages 210–222 in I. Lakatos and A. Musgrave (eds.), *Problems in the Philosophy of Science. Proceedings of the International Colloquium in the Philosophy of Science*, Amsterdam: North-Holland.

Suszko, R., 1968, "Ontology in the *Tractatus* of L. Wittgenstein", *Notre Dame Journal of Formal Logic* 9: 7–33.

Suszko, R., 1975, "Abolition of the Fregean axiom", pages 169–239 in R. Parikh (ed.), *Logic Colloquium*, *Lecture Notes in Mathematics* 453, Springer Verlag.

Tarski, A., 1933, "Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und ω-Vollständigkeit", *Monatshefte für Mathematik und Physik* 40: 97–112.

Tarski, A., 1935, "Der Wahrheitsbegriff in den formalisierten Sprachen", *Studia Philosophica* 1: 261–405.

Wang, H., 1955, "On denumerable bases of formal systems", pages 57–84 in T. Skolem, G. Hasenjaeger, G. Kreisel, A. Robinson, H. Wang, L. Henkin and J. Łoś (eds.), *Mathematical Interpretation of Formal Systems*, Amsterdam: North-Holland Publishing Company.

Woleński, J., 2015, "W odpowiedzi Krystianowi Jobczykowi" [In replay to Krystian Jobczyk], *Filozofia Nauki* 3 (91): 145–149.

Zermelo, E., 1930, "Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre", *Fundamenta Mathematicae* 16: 29–47.

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