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Logics for Knowability

Abstract. In this paper, we propose three knowability logics **LK**, **LK**$^-$, and **LK**$^=$. In the single-agent case, **LK** is equally expressive as arbitrary public announcement logic **APAL** and public announcement logic **PAL**, whereas in the multi-agent case, **LK** is more expressive than **PAL**. In contrast, both **LK**$^-$ and **LK**$^=$ are equally expressive as classical propositional logic **PL**. We present the axiomatizations of the three knowability logics and show their soundness and completeness. We show that all three knowability logics possess the properties of Church-Rosser and McKinsey. Although **LK** is undecidable when at least three agents are involved, **LK**$^-$ and **LK**$^=$ are both decidable.

Keywords: knowability; public announcement logic; expressivity; arbitrary public announcement logic; axiomatizations; decidability

1. Introduction

Intuitively, a proposition is known to you, if you know it; in contrast, a proposition is knowable for you, if you can get to know it. The knowability paradox is that if all truths are knowable, then all truths are actually known. The standard references for the knowability paradox are [8, 13]. However, following Salerno's archival efforts the obligatory precursor to that Church's 'anonymous' referee report of what (much) later became [13]:

> [. . .] there is always a true proposition which it is empirically impossible for a to know at time t. For let k be a true proposition which is unknown to a at time t, and let k' be the proposition that k is true but unknown to a at time t. Then k' is true. But it would seem that if a knows k'

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> at time t, then a must know k at time t, and must also know that he does not know k at time t. [9], reprinted in [20]

Fitch finally writes:

> If there is some true proposition which nobody knows (or has known, or will know) to be true, then there is some true proposition that nobody can know to be true. [13, p. 139]

Formally, 'proposition φ is knowable' later became $\Diamond K\varphi$ [8], where $\Diamond$ is some modal diamond, representing a process, or time, or some alethic modality of truth. This modal diamond does not yet occur in [13]. Let us sketch the paradox. The existence of unknown truths is semi-formalized as "there is a proposition φ such that $\varphi \wedge \neg K\varphi$". That all truths are knowable is semi-formalized as "for all propositions ψ, $\psi \rightarrow \Diamond K\psi$". Fitch's paradox is that the existence of unknown truths is inconsistent with the requirement that all truths are knowable. This can now be easily shown: let ψ be $\varphi \wedge \neg K\varphi$, then we get $(\varphi \wedge \neg K\varphi) \rightarrow \Diamond K(\varphi \wedge \neg K\varphi)$. On the assumption of $\varphi \wedge \neg K\varphi$, we therefore obtain $\Diamond K(\varphi \wedge \neg K\varphi)$. Whatever the interpretation of $\Diamond$, this will result in having to evaluate $K(\varphi \wedge \neg K\varphi)$. But this is inconsistent for knowledge, as can be shown by very simple means: since knowing a conjunction entails knowing each of the conjuncts, we obtain $K\varphi$ and $K\neg K\varphi$ from this, and from the latter and that knowledge entails truth, $\neg K\varphi$, and $K\varphi \wedge \neg K\varphi$ is inconsistent.[1] This is of course Church's argument cited above. It is also inconsistent for belief, as was already observed by Hintikka [16].

Knowability is a subjective concept; it is possible that a proposition is knowable for an agent but not for another. Take the proposition "it is raining but Alice does not know it" as an example. This proposition is not knowable for Alice, as above. But the proposition is knowable for another agent Bob, who may be aware of Alice's ignorance. We are moving from $\Diamond K\varphi$ to $\Diamond K_a\varphi$ and $\Diamond K_b\varphi$.

Since Fitch's 1963 publication, the topic of *knowability* has done the rounds of philosophical communities [see, e.g., 11, 20, 21]. The knowability paradox is relevant in verificationism and in anti-realism. The verification principle requires a non-analytic, meaningful true sentence to be empirically verifiable [4]. Replace 'empirically verifiable' for 'knowable' (or recall 'empirically impossible for a to know', cited above) and

[1] Instead of using the two properties of knowledge in question, one can show in the monotone logic of unknown truths [12] that the unknown truths $\varphi \wedge \neg K\varphi$ is not known.

we are there. Anti-realism or non-realism is the philosophy that denies the existence of an objective reality of entities. In other words, there are no true unknowable propositions: a true proposition about the objective reality that has no counterpart in a knowing subject would be such an unknowable proposition [10].

A dynamic view for knowability was subsequently proposed by van Benthem [22]. According to this dynamic view, knowable means 'known after some announcement', where 'announcement' is the truthfully public announcement of what is indeed known as *public announcement logic* [19] (PAL). A logic extending public announcement logic with this notion of knowability was proposed in the logic APAL (for 'Arbitrary Public Announcement Logic') [5].

Unlike PAL, APAL is undecidable, has an infinitary axiomatization, and even model checking is already highly complex (PSPACE complete [1]). In [27] it was subsequently shown that after all *everything is knowable* in the sense that in this logic, $\Diamond K\varphi \vee \Diamond K\neg\varphi$ is valid; in other words, everything is knowable to be true or false. But some kind of cheating is involved: for example, $p \wedge \neg K_a p$ is 'knowable' in this sense, because after Bob announcing this to Alice it has become false: Alice now knows p, $K_a p$, which entails $\neg(p \wedge \neg K_a p)$.

In this investigation we will consider the combination $\Diamond K$ as a primitive modality in the logical language, and investigate the properties of various logics with this modality. Instead of $\Diamond K$, or rather $\Diamond K_i$ for an agent i, we will then write $\Diamond_i^K$, but this is mere syntactic sugar: the point is that we are not allowed to use the $\Diamond$ modality independently, but only followed by K_i. This technique of packing or bundling a knowledge modality with another modality (or a quantifier) was pioneered in works by Wang and collaborators [17, 18, 29]. As one may see, this packing can help us see the logical properties of knowability, such as McKinsey and Church-Rosser, more clearly. As can be expected, this may affect the properties of the logic, for example its expressivity, or complexity, or even the existence of an axiomatization. Such logics with a primitive 'knowability' modality $\Diamond_i^K$ will be called *logics for knowability.*[2] We will focus on matters involving expressivity, axiomatization and decidability of such knowability logics. In particular, we show that the logic that is

[2] Although the method to pack two modalities into one is different from the usual modelling of the knowability paradox, the formalization of the paradox still requires two modalities, namely the novel knowability modality as well as the knowledge modality (see Corollary 3.6 below).

like **APAL** but instead of the $\Diamond$ modality has the packed $\Diamond_i^K$ modality also has a complete axiomatization, and we demonstrate various logical properties of $\Diamond_i^K$. Moreover, as we will show, although the full knowability logic is undecidable for at least three agents, two of its fragments *are* decidable, since both of them are equally expressive as the classical propositional logic.

The remainder is organized as follows. After introducing the syntax and semantics of knowability logics and other related logics (Section 2), we investigate the logical properties of knowability and also a fragment of positive formulas in Section 3. Section 4 introduces the bisimulation for a knowability logic **LK** and compares the relative expressivity of **LK** and some related logics. Section 5 proposes an axiomatization of **LK** and shows its soundness. Section 6 shows its completeness of $\mathbb{LK}$, and explore the decidability of **LK**, which turns out to be undecidable when there are at least three agents. We then propose two decidable knowability logics, which are both equally expressive as the classical propositional logic **PL**, and axiomatize them in Section 7. Finally we conclude with some future work in Section 8.

2. Syntax and Semantics

In what follows, we let **P** denote a denumerable set of propositional variables, and **Ag** a finite set of agents.

Definition 2.1 (Languages). We consider various fragments of the following recursively defined language $\mathcal{L}$:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \Diamond_i^K\varphi \mid \Diamond\varphi$$

where $p \in \mathbf{P}$ and $i \in \mathbf{Ag}$.

Without the construct $\Diamond\varphi$, we obtain the language **LK** of *knowability logic*; without the construct $K_i\varphi$ as well, we obtain the language $\mathbf{LK}^-$; without the construct $\langle\varphi\rangle\varphi$ further, we obtain the language $\mathbf{LK}^=$. Without the construct $\Diamond_i^K\varphi$, we obtain the language **APAL** of arbitrary public announcement logic; without additionally the construct $\Diamond\varphi$, we obtain the language **PAL** of public announcement logic; without additionally the construct $\langle\varphi\rangle\varphi$, we obtain the language **EL**; without even the construct $K_i\varphi$ additionally, we obtain the language **PL** of classical propositional logic.

Although we have different primitives $\Diamond_i^K$ and $\Diamond$, we could alternatively have defined $\Diamond_i^K$ by abbreviation as the 'packing' or 'bundling' of K_i and $\Diamond$, namely as $\Diamond_i^K\varphi := \Diamond K_i\varphi$, such that the inductive definition of **LK** could have been given as $\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid \Diamond K_i\varphi$. Instead, we will now after the presentation of the semantics have this as a property of the complete language $\mathcal{L}$. The main focus of our investigations is the logic **LK**.

Intuitively, $K_i\varphi$, $\langle\psi\rangle\varphi$, $\Diamond_i^K\varphi$, and $\Diamond\varphi$ are read, respectively, "agent i knows that φ", "after some truthful announcement of ψ, it holds that φ", "φ is knowable for agent i", "after some truthful announcement, it holds that φ". Other connectives are defined as usual. In particular, we abbreviate $\hat{K}_i\varphi$, $[\psi]\varphi$, $\Box_i^K\varphi$, and $\Box\varphi$ as, respectively, $\neg K_i\neg\varphi$, $\neg\langle\psi\rangle\neg\varphi$, $\neg\Diamond_i^K\neg\varphi$, and $\neg\Diamond\neg\varphi$. Moreover, $var(\varphi)$ is the set of propositional variables occurring in φ.

DEFINITION 2.2 (Models and Frames). A *model* is a tuple $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{Ag}\}, V\rangle$, where S is a nonempty set of states, for each $i \in \mathbf{Ag}$, R_i is an equivalence relation over S, that is, R_i is reflexive, transitive, and symmetric, and V is a valuation function. Given any $s \in S$, $R_i(s)$ is the set of all successors of s with respect to R_i; in symbol, $R_i(s) = \{t \in S \mid sR_it\}$. A *frame* is a model without a valuation.

DEFINITION 2.3 (Semantics). Given a model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{Ag}\}, V\rangle$ and a state $s \in S$, the formulas of $\mathcal{L}$ are interpreted recursively as follows:

$$\begin{array}{lcl}
\mathcal{M}, s \vDash p & \iff & s \in V(p) \\
\mathcal{M}, s \vDash \neg\varphi & \iff & \mathcal{M}, s \nvDash \varphi \\
\mathcal{M}, s \vDash \varphi \wedge \psi & \iff & \mathcal{M}, s \vDash \varphi \text{ and } \mathcal{M}, s \vDash \psi \\
\mathcal{M}, s \vDash K_i\varphi & \iff & \mathcal{M}, t \vDash \varphi \text{ for all } t \in R_i(s) \\
\mathcal{M}, s \vDash \langle\psi\rangle\varphi & \iff & \mathcal{M}, s \vDash \psi \text{ and } \mathcal{M}|_\psi, s \vDash \varphi \\
\mathcal{M}, s \vDash \Diamond_i^K\varphi & \iff & \text{for some formula } \psi \in \mathbf{EL} : \mathcal{M}, s \vDash \langle\psi\rangle K_i\varphi \\
\mathcal{M}, s \vDash \Diamond\varphi & \iff & \text{for some formula } \psi \in \mathbf{EL} : \mathcal{M}, s \vDash \langle\psi\rangle\varphi
\end{array}$$

where $\mathcal{M}|_\psi = \langle S', \{R'_i \mid i \in \mathbf{Ag}\}, V'\rangle$ is such that $S' = [\![\varphi]\!]_\mathcal{M} = \{s \in S \mid \mathcal{M}, s \vDash \varphi\}$, $R'_i = R_i \cap ([\![\varphi]\!]_\mathcal{M} \times [\![\varphi]\!]_\mathcal{M})$, and $V'(p) = V(p) \cap [\![\varphi]\!]_\mathcal{M}$.

A formula φ is *valid*, notation: $\vDash \varphi$, if for all models $\mathcal{M}$ and all states s in $\mathcal{M}$, we have $\mathcal{M}, s \vDash \varphi$. Given any two states s, t in $\mathcal{M}$ and any formula φ, we say that s and t *agree on* φ, if $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}, t \vDash \varphi$.

Note that in the semantic definition of $\Diamond_i^K\varphi$, the quantification is restricted to **EL**-formulas. This is to avoid circularity of the definition.

As **EL** is expressively equivalent to **PAL**, we can also define the semantics of $\Diamond_i^K$ as follows:

$$\mathcal{M}, s \vDash \Diamond_i^K \varphi \iff \text{for some formula } \psi \in \mathbf{PAL} : \mathcal{M}, s \vDash \langle \psi \rangle K_i \varphi.$$

For convenience, we also give the semantics of $\Box_i^K$ as follows.

$$\mathcal{M}, s \vDash \Box_i^K \varphi \iff \text{for all formulas } \psi \in \mathbf{EL} : \mathcal{M}, s \vDash [\psi] \hat{K}_i \varphi.$$

From Definition 2.3 it follows that $\vDash \langle \psi \rangle K_i \varphi \to \Diamond_i^K \varphi$, where $\psi \in \mathbf{EL}$. We can also use its equivalent version $\Box_i \varphi \to [\psi] \hat{K}_i \varphi$ (where $\psi \in \mathbf{EL}$), which means intuitively that if $\neg\varphi$ is unknowable ($\neg \Diamond_i^K \neg \varphi$), then after any announcement $\neg\varphi$ is unknown ($[\psi] \neg K_i \neg \varphi$).

By definition of the semantics we obtain:

Proposition 2.4. *For all* $\varphi \in \mathcal{L}$, $\vDash \Diamond_i^K \varphi \leftrightarrow \Diamond K_i \varphi$.

Due to the presence of the knowability operators, in the completeness proof, we need to use a method of induction with, on one hand, the size of formulas (as usual), and on the other hand, the depth of knowability operators. These two notions are combined into the notion of complexity. This notion and the next proposition will be also used in proving the proof theoretical results in Proposition 3.19 and Sec. 5.2.

Definition 2.5 (Complexity). The *complexity of a formula* consists of two aspects: size and $\Diamond^K$-depth, which are defined as follows.

The *size* of a formula φ, notation: $Size(\varphi)$, is a positive natural number, defined recursively as follows:

$$\begin{array}{lcl} Size(p) & = & 1 \\ Size(\neg\varphi) & = & 1 + Size(\varphi) \\ Size(\varphi \wedge \psi) & = & 1 + \max\{Size(\varphi), Size(\psi)\} \\ Size(K_i\varphi) & = & 3 + Size(\varphi) \\ Size(\langle\psi\rangle\varphi) & = & Size(\psi) + 3 \cdot Size(\varphi) \\ Size(\Diamond_i^K\varphi) & = & 1 + Size(\varphi) \end{array}$$

The $\Diamond^K$-*depth* of a formula φ, notation $d_\Diamond^K(\varphi)$, is a natural number, defined recursively as follows:

$$\begin{array}{lcl} d_\Diamond^K(p) & = & 0 \\ d_\Diamond^K(\neg\varphi) & = & d_\Diamond^K(\varphi) \\ d_\Diamond^K(\varphi \wedge \psi) & = & \max\{d_\Diamond^K(\varphi), d_\Diamond^K(\psi)\} \\ d_\Diamond^K(K_i\varphi) & = & d_\Diamond^K(\varphi) \\ d_\Diamond^K(\langle\psi\rangle\varphi) & = & d_\Diamond^K(\psi) + d_\Diamond^K(\varphi) \\ d_\Diamond^K(\Diamond_i^K\varphi) & = & 1 + d_\Diamond^K(\varphi) \end{array}$$

With the definitions of size and $\Diamond^K$-depth in hand, we define $<^S_\Diamond$ as a binary relation between formulas such that

$$\varphi <^S_\Diamond \psi \iff \text{either } d^K_\Diamond(\varphi) < d^K_\Diamond(\psi), \text{ or } d^K_\Diamond(\varphi) = d^K_\Diamond(\psi) \text{ and } Size(\varphi) < Size(\psi).$$

If $\varphi <^S_\Diamond \psi$, then we say that φ *is less complex than* ψ.

One may easily show by induction that $d^K_\Diamond(\varphi) = 0$ for all $\varphi \in \mathbf{EL}$. And also, it is easily computed that $Size([\psi]\varphi) = 4 + Size(\psi) + 3 \cdot Size(\varphi)$.

PROPOSITION 2.6. *In 5 and 17, $\psi \in \mathbf{EL}$.*

1. $\varphi <^S_\Diamond \neg\varphi$
2. $\varphi <^S_\Diamond \varphi \wedge \psi$
3. $\psi <^S_\Diamond \varphi \wedge \psi$
4. $\varphi <^S_\Diamond K_i\varphi$
5. $\langle\psi\rangle K_i\varphi <^S_\Diamond \Diamond^K_i\varphi$
6. $\psi <^S_\Diamond \langle\psi\rangle\varphi$
7. $\psi <^S_\Diamond \langle\psi\rangle p$
8. $p <^S_\Diamond \langle\psi\rangle p$
9. $\psi <^S_\Diamond \langle\psi\rangle\neg\varphi$
10. $\langle\psi\rangle\varphi <^S_\Diamond \langle\psi\rangle\neg\varphi$
11. $\langle\psi\rangle\varphi <^S_\Diamond \langle\psi\rangle(\varphi \wedge \chi)$
12. $\varphi <^S_\Diamond \langle\psi\rangle\varphi$
13. $\langle\psi\rangle\chi <^S_\Diamond \langle\psi\rangle(\varphi \wedge \chi)$
14. $\psi <^S_\Diamond \langle\psi\rangle K_i\varphi$
15. $K_i[\psi]\varphi <^S_\Diamond \langle\psi\rangle K_i\varphi$
16. $\langle\langle\psi\rangle\chi\rangle\varphi <^S_\Diamond \langle\psi\rangle\langle\chi\rangle\varphi$
17. $\langle\chi\rangle\langle\psi\rangle K_i\varphi <^S_\Diamond \langle\chi\rangle\Diamond^K_i\varphi$

PROOF. We take some of them as examples.

5: It is because $d^K_\Diamond(\langle\psi\rangle K_i\varphi) = d^K_\Diamond(\varphi) < 1 + d^K_\Diamond(\varphi) = d^K_\Diamond(\Diamond^K_i\varphi)$.

8: This is because $d^K_\Diamond(p) \leq d^K_\Diamond(\psi) + d^K_\Diamond(p) = d^K_\Diamond(\langle\psi\rangle p)$ and $Size(p) = 1 < Size(\psi) + 3 \cdot Size(p) = Size(\langle\psi\rangle p)$.

15: This is because $d^K_\Diamond(K_i[\psi]\varphi) = d^K_\Diamond(\psi) + d^K_\Diamond(\varphi) = d^K_\Diamond(\langle\psi\rangle K_i\varphi)$, and $Size(K_i[\psi]\varphi) = 3 + 4 + Size(\psi) + 3 \cdot Size(\varphi) = 7 + Size(\psi) + 3 \cdot Size(\varphi) < 9 + Size(\psi) + 3 \cdot Size(\varphi) = Size(\langle\psi\rangle K_i\varphi)$.

16: It is since $d^K_\Diamond(\langle\langle\psi\rangle\chi\rangle\varphi) = d^K_\Diamond(\psi) + d^K_\Diamond(\chi) + d^K_\Diamond(\varphi) = d^K_\Diamond(\langle\psi\rangle\langle\chi\rangle\varphi)$, and $\mathit{Size}(\langle\langle\psi\rangle\chi\rangle\varphi) = \mathit{Size}(\psi) + 3\cdot\mathit{Size}(\chi) + 3\cdot\mathit{Size}(\varphi) < \mathit{Size}(\psi) + 3\cdot\mathit{Size}(\chi) + 9\cdot\mathit{Size}(\varphi) = \mathit{Size}(\langle\psi\rangle\langle\chi\rangle\varphi)$. ⊣

Note that in the definition of $\mathit{Size}(K_i\varphi)$, the number 3 is the least natural number to provide $K_i[\psi]\varphi <^S_\Diamond \langle\psi\rangle K_i\varphi$. In contrast, in [6], $\mathit{Size}(K_i\varphi)$ is defined to be $1 + \mathit{Size}(\varphi)$, in other words, plus 1 rather than plus 3.

3. Logical properties of knowability

This section explores the logical properties of the knowability operator in the logic **LK**.

It has been shown in [27] that *everything is knowable*, in the sense that $\Diamond K_i\varphi \vee \Diamond K_i\neg\varphi$ is valid. In **LK** this becomes $\Diamond^K_i\varphi \vee \Diamond^K_i\neg\varphi$ and indeed this is also valid, by a very similar proof (only the case quantifier is occasionally different). For clarity we give the entire proof.

Given a model $\mathcal{M}$, the valuation of propositional variable p is *constant* on its domain S if $V(p) = S$ or $V(p) = \emptyset$, i.e., if any two states in S agree on the value of p.

Proposition 3.1 (5, Lemma 3.2). *Let $\varphi \in$ **LK**, and let $\mathcal{M}$ be a model with constant values for all variables occurring in φ. Then $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$.*

Proof. Suppose that each propositional variable occurring in φ has constant value on $\mathcal{M}$. If $V(p) = S$, that is, $\mathcal{M} \vDash p \leftrightarrow \top$, then $\mathcal{M} \vDash \varphi \leftrightarrow \varphi(\top/p)$; if $V(p) = \emptyset$, that is, $\mathcal{M} \vDash p \leftrightarrow \bot$, then $\mathcal{M} \vDash \varphi \leftrightarrow \varphi(\bot/p)$. We denote the result obtained by substituting $\top$ or $\bot$ for all propositional variables in φ in that way as $\varphi^\emptyset$. Obviously, $\mathcal{M} \vDash \varphi \leftrightarrow \varphi^\emptyset$. Note that $\varphi^\emptyset$ contains no propositional variables.

We now show by induction on the structure of φ that $\vDash \varphi^\emptyset \leftrightarrow \top$ or $\vDash \varphi^\emptyset \leftrightarrow \bot$. Cases atom, conjunction and negation are trivial. Further:

- $\vDash K_i\top \leftrightarrow \top$ and $\vDash K_i\bot \leftrightarrow \bot$;
- $\vDash \langle\top\rangle\top \leftrightarrow \top$, $\vDash \langle\top\rangle\bot \leftrightarrow \bot$, $\vDash \langle\bot\rangle\top \leftrightarrow \bot$, and $\vDash \langle\bot\rangle\bot \leftrightarrow \bot$;
- $\vDash \Diamond^K_i\top \leftrightarrow \top$ and $\vDash \Diamond^K_i\bot \leftrightarrow \bot$ (in particular, $\vDash \top \to \Diamond^K_i\top$ follows from the correctness of knowledge after the trivial announcement of $\top$).

Therefore $\vDash \varphi^\emptyset \leftrightarrow \top$ or $\vDash \varphi^\emptyset \leftrightarrow \bot$. Combining this with $\mathcal{M} \vDash \varphi \leftrightarrow \varphi^\emptyset$, we derive that $\mathcal{M} \vDash \varphi \leftrightarrow \top$ or $\mathcal{M} \vDash \varphi \leftrightarrow \bot$, that is, $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \neg\varphi$, respectively. ⊣

Theorem 3.2 (27, Thm. 1). *For all* $\varphi \in \mathbf{LK}$, *we have*

$$\models \Diamond_i^K \varphi \vee \Diamond_i^K \neg\varphi.$$

Proof. Given any model $\mathcal{M}$ and s in $\mathcal{M}$, define δ_s^φ as the characteristic formula of the restriction of the valuation in s to $var(\varphi)$:

$$\delta_s^\varphi = \bigwedge\{p \mid p \in var(\varphi) \text{ and } \mathcal{M}, s \models p\} \wedge \bigwedge\{\neg p \mid p \in var(\varphi) \text{ and } \mathcal{M}, s \not\models p\}.$$

For all $p \in var(\varphi)$, we obviously have

$$\mathcal{M}, s \models p \text{ or } \mathcal{M}, s \models \neg p,$$

and therefore

$$\mathcal{M}|_{\delta_s^\varphi}, s \models p \text{ or } \mathcal{M}|_{\delta_s^\varphi}, s \models \neg p$$

and even

$$\mathcal{M}|_{\delta_s^\varphi} \models p \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models \neg p.$$

Then by Proposition 3.1, we have

$$\mathcal{M}|_{\delta_s^\varphi} \models \varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models \neg\varphi.$$

Thus

$$\mathcal{M}|_{\delta_s^\varphi} \models K_i\varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi} \models K_i\neg\varphi.$$

Since $s \in \mathcal{M}|_{\delta_s^\varphi}$, we have

$$\mathcal{M}|_{\delta_s^\varphi}, s \models K_i\varphi \text{ or } \mathcal{M}|_{\delta_s^\varphi}, s \models K_i\neg\varphi.$$

Therefore,

$$\mathcal{M}, s \models \langle\delta_s^\varphi\rangle K_i\varphi \text{ or } \mathcal{M}, s \models \langle\delta_s^\varphi\rangle K_i\neg\varphi,$$

that is,

$$\mathcal{M}, s \models \Diamond_i^K \varphi \vee \Diamond_i^K \neg\varphi.$$

As $\mathcal{M}$ and s are arbitrary, we now conclude that

$$\models \Diamond_i^K \varphi \vee \Diamond_i^K \neg\varphi. \qquad \dashv$$

Since $\Diamond_i^K \varphi \vee \Diamond_i^K \neg\varphi$ is equivalent to $\neg\Diamond_i^K \neg\varphi \rightarrow \Diamond_i^K \varphi$, and since $\Box_i^K$ is the dual of $\Diamond_i^K$, we immediately have

Corollary 3.3. *For all* $\varphi \in \mathbf{LK}$, $\models \Box_i^K \varphi \rightarrow \Diamond_i^K \varphi$.

However, we recall that although every formula is knowable in the sense of Thm. 3.2, this does not mean that every true formula is knowable (to be true), as the announcement may 'flip' the value of the formula in question. Fitch [13] showed that there is an unknowable truth, for example $\not\models (p \wedge \neg K_i p) \to \Diamond_i^K (p \wedge \neg K_i p)$. In fact, we have a stronger result: every unknown truth is unknowable; in Salerno's term in [20, p. 32], this says that "Fitch-conjunctions are unknowable."

Proposition 3.4. $\models \neg \Diamond_i^K (\varphi \wedge \neg K_i \varphi)$.

Proof. Suppose not, that is, there is a pointed model $(\mathcal{M}, s)$ such that $\mathcal{M}, s \not\models \neg \Diamond_i^K (\varphi \wedge \neg K_i \varphi)$, then $\mathcal{M}, s \models \Diamond_i^K (\varphi \wedge \neg K_i \varphi)$. This means that for some formula $\psi \in \mathbf{EL}$ such that $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models K_i(\varphi \wedge \neg K_i \varphi)$. The latter entails that $\mathcal{M}|_\psi, s \models K_i \varphi \wedge K_i \neg K_i \varphi$. Since $\models K_i \varphi \to \varphi$, we have $\mathcal{M}|_\psi, s \models K_i \varphi \wedge \neg K_i \varphi$: a contradiction. ⊣

Consequently, we have $\models (\varphi \wedge \neg K_i \varphi) \to \neg \Diamond_i^K (\varphi \wedge \neg K_i \varphi) \wedge (\varphi \wedge \neg K_i \varphi)$, which says that if it is an unknown truth that φ, it is an unknowable truth that it is an unknown truth that φ; in short, every unknown truth is itself unknowable, see [13, Thm. 2] and [30, p. 154].

Corollary 3.5. *$\Diamond_i^K (\varphi \wedge \neg K_i \varphi)$ is unsatisfiable. That is, there is no pointed model satisfying $\Diamond_i^K (\varphi \wedge \neg K_i \varphi)$.*

In comparison, $\Diamond_j^K (p \wedge \neg K_i p)$ is satisfiable, as one may easily check. This tells us that the notion of knowability is a subjective concept: the proposition $p \wedge \neg K_i p$ is unknowable for the agent i but knowable for another agent j, as mentioned in the introduction.

Also, as we mentioned in the introduction, the knowability paradox says that if all truths are knowable, then all truths are actually known. This can be shown semantically as follows.

Corollary 3.6. *If $\models \varphi \to \Diamond_i^K \varphi$ for all φ, then $\models \varphi \to K_i \varphi$ for all φ.*

Proof. Suppose that $\models \varphi \to \Diamond_i^K \varphi$ for all φ. Then of course, $\models \varphi \wedge \neg K_i \varphi \to \Diamond_i^K (\varphi \wedge \neg K_i \varphi)$ for all φ. By Proposition 3.4, we have $\models \neg(\varphi \wedge \neg K_i \varphi)$ for all φ, and therefore $\models \varphi \to K_i \varphi$ for all φ. ⊣

Proposition 3.7. $\models K_i \varphi \to \Diamond_i^K \varphi$

Proof. This is because $\models K_i \varphi \to \langle \top \rangle K_i \varphi$ and $\models \langle \top \rangle K_i \varphi \to \Diamond_i^K \varphi$. ⊣

We continue our survey of the properties of the knowability operator with a number of validities only involving that operator.

THEOREM 3.8. $\models \Diamond_i^K \Diamond_i^K \varphi \to \Diamond_i^K \varphi$

PROOF. Let $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{Ag}\}, V\rangle$ and $s \in S$. First, suppose that $\mathcal{M}, s \models \Diamond_i^K \Diamond_i^K \varphi$, then for some $\psi \in \mathbf{EL}$: $\mathcal{M}, s \models \langle\psi\rangle K_i \Diamond_i^K \varphi$. This means that $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models K_i \Diamond_i^K \varphi$. Since R_i is an equivalence relation and equivalence relations are closed under public announcements, $R_i|_\psi$ is an equivalence relation as well. Thus $\mathcal{M}|_\psi, s \models \Diamond_i^K \varphi$, which entails that for some $\chi \in \mathbf{EL}$: $\mathcal{M}|_\psi, s \models \langle\chi\rangle K_i\varphi$, which amounts to saying that $\mathcal{M}|_\psi, s \models \chi$ and $(\mathcal{M}|_\psi)|_\chi, s \models K_i\varphi$.

Summarizing the above results, we have that for some $\psi, \chi \in \mathbf{EL}$: $\mathcal{M}, s \models \psi$ and $\mathcal{M}|_\psi, s \models \chi$ and $(\mathcal{M}|_\psi)|_\chi, s \models K_i\varphi$. As a sequence of two announcements is an announcement [28, Proposition 4.17], it directly follows that $\mathcal{M}|_{\langle\psi\rangle\chi}, s \models K_i\varphi$. From $\mathcal{M}, s \models \langle\psi\rangle\chi$ and $\mathcal{M}|_{\langle\psi\rangle\chi}, s \models K_i\varphi$ it now follows that $\mathcal{M}, s \models \Diamond_i^K \varphi$. ⊣

The following result indicates that $\Diamond_i^K$ (and thus $\Box_i^K$) are monotone. Straightforward from the semantics we obtain:

PROPOSITION 3.9. *If* $\models \varphi \to \psi$, *then* $\models \Diamond_i^K \varphi \to \Diamond_i^K \psi$ *and* $\models \Box_i^K \varphi \to \Box_i^K \psi$.

Note that $\Diamond_i^K$ is not *regular*. In other words, $\not\models \Diamond_i^K \varphi \wedge \Diamond_i^K \psi \to \Diamond_i^K (\varphi \wedge \psi)$: one may easily construct a pointed model $(\mathcal{M}, s)$ such that $\mathcal{M}, s \models \Diamond_i^K p$ and $\mathcal{M}, s \models \Diamond_i^K \neg p$ but $\mathcal{M}, s \not\models \Diamond_i^K (p \wedge \neg p)$.

The next result states that unknowable truths are themselves unknowable.

COROLLARY 3.10. $\models \neg\Diamond_i^K(\varphi \wedge \neg\Diamond_i^K \varphi)$.

PROOF. By Proposition 3.7, $\models K_i\varphi \to \Diamond_i^K \varphi$, thus $\models \varphi \wedge \neg\Diamond_i^K\varphi \to \varphi \wedge \neg K_i\varphi$. Then from Proposition 3.9, it follows that $\models \Diamond_i^K(\varphi \wedge \neg\Diamond_i^K\varphi) \to \Diamond_i^K(\varphi \wedge \neg K_i\varphi)$. Finally, using Proposition 3.4, we conclude that $\models \neg\Diamond_i^K(\varphi \wedge \neg\Diamond_i^K\varphi)$. ⊣

PROPOSITION 3.11. $\models \Diamond_i^K \varphi \to \Diamond_i^K K_i\varphi$.

PROOF. Let $\mathcal{M} = \langle S, R, V\rangle$ and $s \in S$. Suppose that $\mathcal{M}, s \models \Diamond_i^K \varphi$, then for some $\psi \in \mathbf{EL}$, $\mathcal{M}, s \models \langle\psi\rangle K_i\varphi$. Since R_i is an equivalence relation, $\models K_i\varphi \to K_iK_i\varphi$, and thus $\mathcal{M}, s \models \langle\psi\rangle K_iK_i\varphi$. Therefore $\mathcal{M}, s \models \Diamond_i^K K_i\varphi$. ⊣

THEOREM 3.12. $\models \Diamond_i^K \varphi \to \Diamond_i^K \Diamond_i^K \varphi$.

PROOF. By Proposition 3.7, $\models K_i\varphi \to \Diamond_i^K\varphi$. Then by Proposition 3.9, $\models \Diamond_i^K K_i\varphi \to \Diamond_i^K\Diamond_i^K\varphi$. Now due to Proposition 3.11, $\models \Diamond_i^K\varphi \to \Diamond_i^K\Diamond_i^K\varphi$. ⊣

Corollary 3.13. *$\vDash \Diamond_i^K \varphi \leftrightarrow \Diamond_i^K \Diamond_i^K \varphi$, and thus $\vDash \Diamond_i^K \Diamond_i^K \varphi \leftrightarrow \Diamond_i^K K_i \varphi$, $\vDash \Diamond_i^K K_i \varphi \leftrightarrow \Diamond_i^K \varphi$, and $\vDash K_i \Diamond_i^K \varphi \rightarrow \Diamond_i^K K_i \varphi$.*

Corollary 3.14. *$\vDash \Box_i^K \varphi \rightarrow \Diamond_i^K \Box_i^K \varphi$ and $\vDash \Box_i^K \varphi \rightarrow \Box_i^K \Diamond_i^K \varphi$. As a consequence, $\vDash \Box_i^K \Diamond_i^K \varphi \rightarrow \Diamond_i^K \varphi$ and $\vDash \Diamond_i^K \Box_i^K \varphi \rightarrow \Diamond_i^K \varphi$.*

Proof. By Corollary 3.13, we have $\vDash \Box_i^K \varphi \leftrightarrow \Box_i^K \Box_i^K \varphi$. By Coro. 3.3, we infer that $\vDash \Box_i^K \Box_i^K \varphi \rightarrow \Diamond_i^K \Box_i^K \varphi$, and therefore $\vDash \Box_i^K \varphi \rightarrow \Diamond_i^K \Box_i^K \varphi$; by Corollary 3.3 and Proposition 3.9, $\vDash \Box_i^K \Box_i^K \varphi \rightarrow \Box_i^K \Diamond_i^K \varphi$, and therefore $\vDash \Box_i^K \varphi \rightarrow \Box_i^K \Diamond_i^K \varphi$. ⊣

We have shown that $\vDash \Box_i^K \Diamond_i^K \varphi \rightarrow \Diamond_i^K \varphi$. However, $\Box_i^K \varphi \rightarrow \varphi$ is not valid, since its equivalent $\varphi \rightarrow \Diamond_i^K \varphi$ is not valid. Proposition 3.4 demonstrated that some true propositions are not knowable, for example $\varphi = p \wedge \neg K_i p$. This also shows that $\vDash \Diamond_i^K \varphi \leftrightarrow \varphi$ does *not* hold for all $\varphi \in \mathcal{L}_{\mathbf{LK}}$, though it does hold for all $\varphi \in \mathcal{L}_{\mathbf{PL}}$ [5, Proposition 3.11.2].

Lemma 3.15. *Let $\varphi \in$ **LK**, and let $\mathcal{M}$ be a model where all states agree on each propositional variable occurring in φ. Then $\mathcal{M} \vDash \varphi \rightarrow \Box_i^K \varphi$.*

Proof. Let s be any state in $\mathcal{M}$, and $\mathcal{M}, s \vDash \varphi$. Now consider any **EL**-formula ψ such that $\mathcal{M}, s \vDash \psi$. Let $\mathcal{M}'$ be the disjoint union of $\mathcal{M}$ and $\mathcal{M}|_\psi$. The valuation of atoms in $var(\varphi)$ is also constant on $\mathcal{M}'$. By Proposition 3.1, it follows that $\mathcal{M}' \vDash \varphi$ or $\mathcal{M}' \vDash \neg\varphi$. If $\mathcal{M}' \vDash \neg\varphi$, then it contradicts $\mathcal{M}, s \vDash \varphi$. Thus $\mathcal{M}' \vDash \varphi$, and therefore $\mathcal{M}|_\psi \vDash \varphi$. That is to say, for any state t such that sR_it in $\mathcal{M}|_\psi$, we have $\mathcal{M}|_\psi, t \vDash \varphi$. By semantics, it follows that $\mathcal{M}|_\psi, s \vDash K_i\varphi$, and thus $\mathcal{M}|_\psi, s \vDash \hat{K}_i\varphi$. As ψ is arbitrary, by semantics we know that $\mathcal{M}, s \vDash \Box_i^K \varphi$. So far we have shown that $\mathcal{M}, s \vDash \varphi \rightarrow \Box_i^K \varphi$. As s is arbitrary in $\mathcal{M}$, $\mathcal{M} \vDash \varphi \rightarrow \Box_i^K \varphi$. ⊣

In what follows, we show that the McKinsey property (MK) and the Church–Rosser property (CR) hold for **LK**.

Theorem 3.16 (MK). *$\vDash \Box_i^K \Diamond_i^K \varphi \rightarrow \Diamond_i^K \Box_i^K \varphi$*

Proof. Let a model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{Ag}\}, V\rangle$ and a state $s \in S$ be given. Suppose that $\mathcal{M}, s \vDash \Box_i^K \Diamond_i^K \varphi$. Then by the semantics, for all $\psi \in$ **EL**, we have $\mathcal{M}, s \vDash [\psi]\hat{K}_i \Diamond_i^K \varphi$. Consider δ_s^φ in the proof of Thm. 3.2. It is obvious that $\mathcal{M}, s \vDash \delta_s^\varphi$ and $\delta_s^\varphi \in$ **EL**, thus $\mathcal{M}|_{\delta_s^\varphi}, s \vDash \hat{K}_i \Diamond_i^K \varphi$. Since all states in $\mathcal{M}|_{\delta_s^\varphi}$ have constant values for variables in φ, by Lemma 3.15 we have $\mathcal{M}|_{\delta_s^\varphi} \vDash \varphi \rightarrow \Box_i^K \varphi$ and its dual $\mathcal{M}|_{\delta_s^\varphi} \vDash \Diamond_i^K \varphi \rightarrow \varphi$, therefore $\mathcal{M}|_{\delta_s^\varphi} \vDash \Diamond_i^K \varphi \rightarrow \Box_i^K \varphi$. Note that all states in $\mathcal{M}|_{\delta_s^\varphi}$ also have constant values for variables in $\Diamond_i^K \varphi$. Then by Prop. 3.1 we have $\mathcal{M}|_{\delta_s^\varphi} \vDash \Diamond_i^K \varphi$

or $\mathcal{M}|_{\delta_s^\varphi} \vDash \neg\Diamond_i^K\varphi$. As $\mathcal{M}|_{\delta_s^\varphi}, s \vDash \hat{K}_i\Diamond_i^K\varphi$, there is a state t such that $\mathcal{M}|_{\delta_s^\varphi}, t \vDash \Diamond_i^K\varphi$, contradicting $\mathcal{M}|_{\delta_s^\varphi} \vDash \neg\Diamond_i^K\varphi$. Thus $\mathcal{M}|_{\delta_s^\varphi} \vDash \Diamond_i^K\varphi$. From that and $\mathcal{M}|_{\delta_s^\varphi} \vDash \Diamond_i^K\varphi \to \Box_i^K\varphi$ already obtained above, it follows that $\mathcal{M}|_{\delta_s^\varphi} \vDash \Box_i^K\varphi$. Therefore, for any state s' such that sR_is' in $\mathcal{M}|_{\delta_s^\varphi}$ we have $\mathcal{M}|_{\delta_s^\varphi}, s' \vDash \Box_i^K\varphi$. By semantics, $\mathcal{M}|_{\delta_s^\varphi}, s \vDash K_i\Box_i^K\varphi$, and therefore $\mathcal{M}, s \vDash \Diamond_i^K\Box_i^K\varphi$. ⊣

THEOREM 3.17 (CR). $\vDash \Diamond_i^K\Box_i^K\varphi \to \Box_i^K\Diamond_i^K\varphi$

PROOF. Let a model $\mathcal{M} = \langle S, \{R_i \mid i \in \mathbf{Ag}\}, V\rangle$ and a state $s \in S$ be given. Suppose that $\mathcal{M}, s \vDash \Diamond_i^K\Box_i^K\varphi$. By semantics, for some $\psi \in \mathbf{EL}$: $\mathcal{M}, s \vDash \langle\psi\rangle K_i\Box_i^K\varphi$. Then $\mathcal{M}, s \vDash \psi$ and for any t in $\mathcal{M}|_\psi$ such that sR_it, $\mathcal{M}|_\psi, t \vDash \Box_i^K\varphi$. Consider δ_s^φ in the proof of Theorem 3.2, it is an $\mathbf{EL}$-formula and thus $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t \vDash \hat{K}_i\varphi$.

Let $\eta \in \mathbf{EL}$ be arbitrary such that $\mathcal{M}, s \vDash \eta$. The valuation of atoms in $var(\varphi)$ is constant on $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. By Prop. 3.1, we have $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \vDash \varphi$ or $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \vDash \neg\varphi$. Since $\psi \in \mathbf{EL}$ and $\mathcal{M}, s \vDash \psi$, we have also $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \vDash \varphi$ or $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \vDash \neg\varphi$. As $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t \vDash \hat{K}_i\varphi$, there must be a t' such that $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}, t' \vDash \varphi$ which contradicts $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \vDash \neg\varphi$. Thus we obtain that $(\mathcal{M}|_\psi)|_{\delta_s^\varphi} \vDash \varphi$. Consider the disjoint union $\mathcal{M}'$ of $(\mathcal{M}|_\psi)|_{\delta_s^\varphi}$ and $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. Since $\mathcal{M}'$ has constant values for variables in φ as well, we conclude that $\mathcal{M}' \vDash \varphi$, and therefore $(\mathcal{M}|_\eta)|_{\delta_s^\varphi} \vDash \varphi$. Let s' be any state such that sR_is' in $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}$. Now we know that $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}, s' \vDash \varphi$. Then $(\mathcal{M}|_\eta)|_{\delta_s^\varphi}, s \vDash K_i\varphi$, and thus $\mathcal{M}|_\eta, s \vDash \Diamond_i^K\varphi$. This follows that $\mathcal{M}|_\eta, s \vDash \hat{K}_i\Diamond_i^K\varphi$. As $\eta \in \mathbf{EL}$ is arbitrary, we conclude that $\mathcal{M}, s \vDash \Box_i^K\Diamond_i^K\varphi$. ⊣

As we have seen above, not every true formula is knowable. In contrast, every valid formula is knowable, in symbol: $\vDash \varphi$ implies $\vDash \Diamond_i^K\varphi$, as easily shown. This then follows that $\vDash \Diamond_i^K\top$. Besides, it may be worth noting that the knowability operators are *not* normal.

PROPOSITION 3.18. $\nvDash \Diamond_i^K(\varphi \to \psi) \to (\Diamond_i^K\varphi \to \Diamond_i^K\psi)$

PROOF. Consider the following model $\mathcal{M}$:

$$\underline{s}: p \;\text{———}\, i \,\text{———}\; t: \neg p$$

- $\mathcal{M}, s \vDash \Diamond_i^K(p \to p \wedge \neg K_ip)$: firstly, note that $\mathcal{M}, s \vDash p \to p \wedge \neg K_ip$ and $\mathcal{M}, t \vDash p \to p \wedge \neg K_ip$, thus $\mathcal{M}, s \vDash K_i(p \to p \wedge \neg K_ip)$. By Prop. 3.7, $\mathcal{M}, s \vDash \Diamond_i^K(p \to p \wedge \neg K_ip)$.
- $\mathcal{M}, s \vDash \Diamond_i^Kp$: clearly, $\mathcal{M}, s \vDash \langle p\rangle K_ip$, thus $\mathcal{M}, s \vDash \Diamond_i^Kp$.
- $\mathcal{M}, s \nvDash \Diamond_i^K(p \wedge \neg K_ip)$: this follows directly from Prop. 3.4. ⊣

This refutes the claim that "knowable-in-principle, knowability, is closed under consequence" in [3].

We conclude this section with the fragment of the *positive formulas* in **LK**. The fragment, denoted $\mathbf{LK}^+$, is inductively defined as follows:

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_i\varphi \mid [\neg\varphi]\varphi \mid \Box_i^K \varphi$$

In modal logic, the fragment of the language where negations do not bind (box-type) epistemic modalities is known as the *positive* fragment [5, 23, 26]. It corresponds to the universal fragment in first-order logic. It has the property that it preserves truth under submodels. Intuitively, this is because a box modality says that something is true in all accessible worlds, so if you go to a submodel it is still true in all remaining accessible worlds, whatever remains. The result we present here is a generalization of a similar result in [5]. We should point out the surprising negation in the inductive clause $[\neg\varphi]\varphi$. This has to do with the semantics of public announcement. Note that we have that $\mathcal{M}, s \vDash [\neg\varphi]\psi$, iff (by the semantics of public announcement) $\mathcal{M}, s \vDash \neg\varphi$ implies $\mathcal{M}|_{\neg\varphi}, s \vDash \psi$, iff (propositionally) $\mathcal{M}, s \vDash \varphi$ or $\mathcal{M}|_{\neg\varphi}, s \vDash \psi$. In the last formulation the negation has disappeared! This aspect will also play a role in the proof of the subsequent proposition.

We say that φ is *successful*, if after being announced, φ still holds; in symbol, $\vDash [\varphi]\varphi$. The following result states that positive formulas are successful.

Proposition 3.19. *For all $\varphi \in \mathbf{LK}^+$, we have $\vDash [\varphi]\varphi$.*

Proof. We show the following claim: For any $\mathcal{M}'$ and $\mathcal{M}''$ with $\mathcal{M}'' \subseteq \mathcal{M}'$, $s \in S^{\mathcal{M}''}$ and $\varphi \in \mathbf{LK}^+$: If $\mathcal{M}', s \vDash \varphi$, then $\mathcal{M}'', s \vDash \varphi$.

The proof is by induction on the complexity of φ. Recall that the notion of complexity is given in Def. 2.5.

- φ is atomic: Since the valuation of atoms is local, it is trivial.
- Boolean cases: It is straightforward by induction hypothesis.
- φ is $K_i\psi$: Suppose $\mathcal{M}', s \vDash K_i\psi$, by semantics $\mathcal{M}', s' \vDash \psi$ for any s' such that $sR_i^{\mathcal{M}'}s'$. Consider any t such that $sR_i^{\mathcal{M}''}t$. Since $\mathcal{M}'' \subseteq \mathcal{M}'$, we have $sR_i^{\mathcal{M}'}t$. Thus $\mathcal{M}', t \vDash \psi$, and then by inductive hypothesis $\mathcal{M}'', t \vDash \psi$. By semantics again, it follows that $\mathcal{M}'', s \vDash K_i\psi$.
- φ is $[\neg\psi_1]\psi_2$. Suppose $\mathcal{M}', s \vDash [\neg\psi_1]\psi_2$ and $\mathcal{M}'', s \vDash \neg\psi_1$. By induction hypothesis, $\mathcal{M}', s \vDash \neg\psi_1$. By semantics, $\mathcal{M}'|_{\neg\psi_1}, s \vDash \psi_2$. Note that $\mathcal{M}''|_{\neg\psi_1} \subseteq \mathcal{M}'|_{\neg\psi_1}$, then by induction hypothesis $\mathcal{M}''|_{\neg\psi_1}, s \vDash \psi_2$. By semantics $\mathcal{M}'', s \vDash [\neg\psi_1]\psi_2$.

• φ is $\Box_i^K\psi$. Suppose $\mathcal{M}', s \vDash \Box_i^K\psi$. Assume, for reductio, that $\mathcal{M}'', s \nvDash \Box_i^K\psi$. By semantics, there is a $\chi \in$ **EL** such that $\mathcal{M}'', s \vDash \chi$ and $\mathcal{M}''|_\chi, s \nvDash \hat{K}_i\psi$. As $\mathcal{M}''|_\chi \subseteq \mathcal{M}'' \subseteq \mathcal{M}'$, by induction hypothesis, we infer that $\mathcal{M}', s \nvDash \hat{K}_i\psi$, that is, $\mathcal{M}', s \nvDash [\top]\hat{K}_i\psi$. Then $\mathcal{M}', s \nvDash \Box_i^K\psi$, contrary to the supposition.

Given any $\varphi \in \mathbf{LK}^+$, for any model $\mathcal{M}$ and $s \in S^{\mathcal{M}}$: If $\mathcal{M}, s \vDash \varphi$, then no matter what submodel of $\mathcal{M}$ that φ defines, it follow that $\mathcal{M}|_\varphi, s \vDash \varphi$ by the above claim. By semantics it means $\mathcal{M}, s \vDash [\varphi]\varphi$. Since $(\mathcal{M}, s)$ is arbitrary, we conclude $\vDash [\varphi]\varphi$. ⊣

4. Bisimulation and Expressivity

4.1. Bisimulation

In this part, we show that the notion of bisimilarity is tailored for the logic of knowability **LK**. That is, **LK** is invariant under bisimulation, and the Hennessy-Milner Theorem (H-M for short) holds for **LK**. First, we introduce the notion of bisimulation.

DEFINITION 4.1 (Bisimulation). Let $\mathcal{M} = \langle S^{\mathcal{M}}, \{R_i^{\mathcal{M}} \mid i \in \mathbf{Ag}\}, V^{\mathcal{M}}\rangle$ and $\mathcal{N} = \langle S^{\mathcal{N}}, \{R_i^{\mathcal{N}} \mid i \in \mathbf{Ag}\}, V^{\mathcal{N}}\rangle$ be models. A non-empty relation $Z \subseteq S^{\mathcal{M}} \times S^{\mathcal{N}}$ is a *bisimulation* between $\mathcal{M}$ and $\mathcal{N}$ if for all Zst, $p \in \mathbf{P}$ and $i \in \mathbf{Ag}$:

- *atoms*: $s \in V^{\mathcal{M}}(p)$ iff $t \in V^{\mathcal{N}}(p)$.
- *forth*: if $sR_i^{\mathcal{M}}s'$, then there is a $t' \in S^{\mathcal{N}}$ such that $tR_i^{\mathcal{N}}t'$ and $Zs't'$.
- *back*: if $tR_i^{\mathcal{N}}t'$, then there is a $s' \in S^{\mathcal{M}}$ such that $sR_i^{\mathcal{M}}s'$ and $Zs't'$.

If there exists a bisimulation Z between $\mathcal{M}$ and $\mathcal{N}$ we write $\mathcal{M} \leftrightarroweq \mathcal{N}$ (or $Z\colon \mathcal{M} \leftrightarroweq \mathcal{N}$, to indicate the relation), and if it contains the pair (s, t), we write $(\mathcal{M}, s) \leftrightarroweq (\mathcal{N}, t)$.

Given pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ and a language L, $(\mathcal{M}, s) \equiv_L (\mathcal{N}, t)$ denotes: for all $\varphi \in \mathcal{L}_L$, $\mathcal{M}, s \models \varphi$ iff $\mathcal{N}, t \models \varphi$.

PROPOSITION 4.2. *For all pointed models* $(\mathcal{M}, s)$ *and* $(\mathcal{M}', s')$, *if* $(\mathcal{M}, s) \leftrightarroweq (\mathcal{M}', s')$, *then* $(\mathcal{M}, s) \equiv_{\mathbf{LK}} (\mathcal{M}', s')$.

PROOF. Suppose that $(\mathcal{M}, s) \leftrightarroweq (\mathcal{M}', s')$, we show for all $\varphi \in$ **LK**: $\mathcal{M}, s \vDash \varphi$ if and only if $\mathcal{M}', s' \vDash \varphi$. The proof proceeds with induction on the structure of φ. As it is known that **PAL** is invariant for bisimulation, we need only present the case $\Diamond_i^K\psi$.

Assume that $\mathcal{M}, s \vDash \Diamond_i^K \psi$. Then there is an **EL**-formula χ such that $\mathcal{M}, s \vDash \chi$ and $\mathcal{M}|_\chi, s \vDash K_i\psi$. As $(\mathcal{M}, s) \leftrightarroweq (\mathcal{M}', s')$ and χ is an **EL**-formula, $\mathcal{M}', s' \vDash \chi$. Consider a relation $Z|\chi$ as the bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ restricted to $\mathcal{M}|_\chi$ and $\mathcal{M}'|_\chi$. We can check $Z|\chi$ is also a bisimulation and $(s, s') \in Z|\chi$. Therefore, for any $t' \in \mathcal{M}'|_\chi$ such that $s'R_i't'$, there is a $t \in \mathcal{M}|_\chi$ such that sR_it and $(\mathcal{M}|_\chi, t) \leftrightarroweq (\mathcal{M}'|_\chi, t')$, which by induction hypothesis implies that $\mathcal{M}|_\chi, t \vDash \psi$ if and only if $\mathcal{M}'|_\chi, t' \vDash \psi$. Since $\mathcal{M}|_\chi, s \vDash K_i\psi$, for any $t \in \mathcal{M}|_\chi$ such that sR_it: $\mathcal{M}|_\chi, t \vDash \psi$. Then by induction hypothesis, $\mathcal{M}'|_\chi, t' \vDash \psi$, and hence $\mathcal{M}'|_\chi, s' \vDash K_i\psi$. We have now shown $\mathcal{M}', s' \vDash \chi$ and $\mathcal{M}'|_\chi, s' \vDash K_i\psi$. It then follows that $\mathcal{M}', s' \vDash \Diamond_i^K\psi$. The other direction is similar. ⊣

Proposition 4.3. *For all image-finite models $\mathcal{M}$ and $\mathcal{N}$, for all s in $\mathcal{M}$ and t in $\mathcal{N}$, if $(\mathcal{M}, s) \equiv_{\mathbf{LK}} (\mathcal{N}, t)$, then $(\mathcal{M}, s) \leftrightarroweq (\mathcal{N}, t)$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be image-finite. Suppose that $(\mathcal{M}, s) \equiv_{\mathbf{LK}} (\mathcal{N}, t)$. Since **LK** is an extension of **EL**, it follows that $(\mathcal{M}, s) \equiv_{\mathbf{EL}} (\mathcal{N}, t)$. By the Hennessy-Milner theorem of **EL** [see, e.g., 7], we have $(\mathcal{M}, s) \leftrightarroweq (\mathcal{N}, t)$, as desired. ⊣

4.2. Expressivity

In this part, we shall compare the expressive powers of our logic **LK**, **PAL**, and **APAL**. It turns out that in the case of single-agent, the three logics are equally expressive; however, in the case of multi-agent, **LK** is more expressive than **PAL**. First, we introduce the definition of related concepts.

Definition 4.4 (Expressivity). Let **L** and **L**$'$ be two logics are interpreted over models.

- **L** is *at least as expressive as* **L**$'$, notation: $\mathbf{L} \preceq \mathbf{L}'$, if for $\varphi \in \mathbf{L}$ there is a $\varphi' \in \mathbf{L}'$ such that φ' is equivalent to φ over the class of $\mathcal{S}5$-models.
- **L** and **L**$'$ are *equally expressive*, notation: $\mathbf{L} \equiv \mathbf{L}'$, if $\mathbf{L} \preceq \mathbf{L}'$ and $\mathbf{L}' \preceq \mathbf{L}$.
- **L** is *less expressive than* **L**$'$, or **L**$'$ is *more expressive than* **L**, notation: $\mathbf{L} \prec \mathbf{L}'$, if $\mathbf{L} \preceq \mathbf{L}'$ but $\mathbf{L}' \not\preceq \mathbf{L}$.
- **L** and **L**$'$ are *incomparable (in expressivity)*, notation: $\mathbf{L} \asymp \mathbf{L}'$, if $\mathbf{L} \not\preceq \mathbf{L}'$ and $\mathbf{L}' \not\preceq \mathbf{L}$.

Proposition 4.5. *In the single-agent case,* **LK** *and* **APAL** *are equally expressive. As a corollary,* **LK** *and* **PAL** *are equally expressive on the single-agent case.*

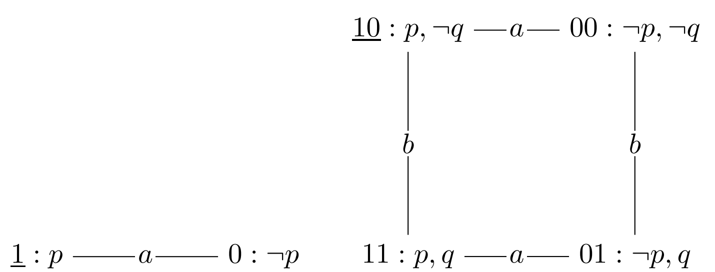
PROOF. Recall that in the single-agent case, **APAL** is equally expressive as **EL** (thus **PAL**) [5, Proposition 3.12]. Moreover, **LK** is an extension of **EL**. This entails that **LK** is at least as expressive as **APAL** in single-agent case. Besides, as **LK** is a fragment of **APAL** due to the definability of $\Diamond^K$ in terms of $\Diamond$ and K, **APAL** is at least as expressive as **LK**. Therefore, in the single-agent case, **LK** and **APAL** are equally expressive. ⊣

The following result is shown as in the proof of [5, Proposition 3.13] via slight revisions. To make the exposition self-contained, we prove it in the following.

PROPOSITION 4.6. **LK** *is more expressive than* **PAL**.

PROOF. First, as **LK** is an extension of **PAL** with the knowability operators, **PAL** $\preceq$ **LK**. It suffices to show that **LK** $\not\preceq$ **PAL**. We show that $\Diamond_a^K(p \wedge \neg K_b K_a p)$ is *not* equivalent to any **PAL**-formula.

Suppose not, then as **PAL** is equally expressive as **EL**, the given knowability formula is equivalent to an **EL**-formula, say ψ. Because ψ is finite, it contains only finite many propositional variables. Let q be a propositional variable not occurring in ψ. Consider the following models, where the left-hand side is $\mathcal{M}$ and the right-hand side is $\mathcal{M}'$:



Since $(\mathcal{M}, 1)$ and $(\mathcal{M}', 10)$ are bisimilar for atoms other than q, we have that $\mathcal{M}, 1 \vDash \psi$ iff $\mathcal{M}', 10 \vDash \psi$. However, $\mathcal{M}, 1 \nvDash \Diamond_a^K(p \wedge \neg K_b K_a p)$ but $\mathcal{M}', 10 \vDash \Diamond_a^K(p \wedge \neg K_b K_a p)$. The argument for the former is as follows: every announcement that makes a know that p at 1 (that is, $\mathcal{M}, 1 \vDash K_a p$) must delete the state 0, and therefore $K_a \neg K_b K_a p$ is false at 1. To see the latter, just notice that $\mathcal{M}', 10 \vDash \langle p \vee q \rangle (K_a p \wedge K_a \neg K_b K_a p)$, which is equivalent to $\mathcal{M}', 10 \vDash \langle p \vee q \rangle K_a(p \wedge \neg K_b K_a p)$, and therefore $\mathcal{M}', 10 \vDash \Diamond_a^K(p \wedge \neg K_b K_a p)$. ⊣

We conjecture that **LK** is less expressive than **APAL**. In the concluding Section 8 we will explain in some detail why this is a difficult problem.

5. Axiomatization

To present the proof system, we need a notion of 'admissible forms' originally from [15, pp. 55–56], also known as 'necessity forms' in [5, 6].

Definition 5.1 (Admissible Forms). Where $\varphi \in \mathbf{LK}$ and $i \in \mathbf{Ag}$, the set of admissible forms $\eta(\sharp)$ is defined recursively as follows:

$$\eta(\sharp) ::= \sharp \mid \varphi \to \eta(\sharp) \mid K_i\eta(\sharp) \mid [\varphi]\eta(\sharp)$$

It is worth noting that $\sharp$ is not a formula, but a placeholder. The result from replacing $\sharp$ in an admissible form $\eta(\sharp)$ by a formula ψ, denoted $\eta(\psi)$, is a formula. It is defined as follows:

$$\begin{array}{lcl}\sharp(\psi) & = & \psi \\ (\varphi \to \eta(\sharp))(\psi) & = & \varphi \to \eta(\psi) \\ (K_i\eta(\sharp))(\psi) & = & K_i\eta(\psi) \\ ([\varphi]\eta(\sharp))(\psi) & = & [\varphi]\eta(\psi)\end{array}$$

Now we are close to the proof system, denoted by $\mathbb{LK}$.

5.1. Proof system and soundness

Definition 5.2. The system $\mathbb{LK}$ consists of the following axioms and is closed under the following rules.

`TAUT`	all instances of propositional tautologies
`K`	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
`T`	$K_i\varphi \to \varphi$
`4`	$K_i\varphi \to K_iK_i\varphi$
`5`	$\neg K_i\varphi \to K_i\neg K_i\varphi$
`!ATOM`	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$
`!NEG`	$\langle\psi\rangle\neg\varphi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\varphi)$
`!CON`	$\langle\psi\rangle(\varphi \wedge \chi) \leftrightarrow (\langle\psi\rangle\varphi \wedge \langle\psi\rangle\chi)$
`!K`	$\langle\psi\rangle K_i\varphi \leftrightarrow (\psi \wedge K_i[\psi]\varphi)$
`!!`	$\langle\psi\rangle\langle\chi\rangle\varphi \leftrightarrow \langle\langle\psi\rangle\chi\rangle\varphi$
`Dual`	$\Diamond_i^K\varphi \leftrightarrow \neg\Box_i^K\neg\varphi$

AKK	$\Box_i^K \varphi \to [\psi]\hat{K}_i\varphi$, where $\psi \in \mathbf{EL}$
MP	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{K_i\varphi}$
RM$\langle\cdot\rangle$	$\dfrac{\varphi \to \psi}{\langle\chi\rangle\varphi \to \langle\chi\rangle\psi}$
RKb	$\dfrac{\eta([\psi]\hat{K}_i\varphi) \text{ for all } \psi \in \mathbf{EL}}{\eta(\Box_i^K\varphi)}$

A formula φ is a *theorem* of $\mathbb{LK}$, or φ is *provable* in $\mathbb{LK}$, notation $\vdash \varphi$, if φ is either an instantiation of an axiom, or obtained by applying inferences to axioms. We use **Thm** for the set of all theorems of $\mathbb{LK}$.

Note that although our reduction axioms are different from the more familiar ones from, e.g., [6, 28], we will show that they are provable from ours (see Proposition 5.7).

Also note that we include Dual as an axiom. This is because we are now using $\Diamond_i^K$ rather than $\Box_i^K$ as modal primitives. This is similar to some case in the minimal normal modal logic, e.g. [7, Sec. 1.6], where the possibility operator $\Diamond$ instead of the necessity operator $\Box$ is used as a modal primitive and $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ is used as an axiom. The axiom Dual will be used later, namely in the proofs of RE (Proposition 5.6), Proposition 5.12 and Proposition 5.13.

To see the intuition of AKK, we can use its dual form:

$$\langle\psi\rangle K_i\varphi \to \Diamond_i^K\varphi, \text{where } \psi \in \mathbf{EL},$$

also denoted AKK. Intuitively, this formula says that if φ is known after some announcement, then φ is knowable.

PROPOSITION 5.3. $\mathbb{LK}$ *is sound with respect to the class of all frames.*

PROOF. By the soundness of public announcement logic, it remains only to show the soundness of Dual, AKK and RKb. The soundness of Dual is obtained from the semantics of $\Diamond_i^K$ and $\Box_i^K$. The soundness of AKK is straightforward by semantics of $\Box_i^K$. To show the soundness of RKb, we show a stronger result:

$$(*) \quad \text{for all } (\mathcal{M}, s), \text{ if } \mathcal{M}, s \vDash \eta([\psi]\hat{K}_i\varphi) \text{for all } \psi \in \mathbf{EL}, \text{ then } \mathcal{M}, s \vDash \eta(\Box_i^K\varphi).$$

The proof proceeds by induction on the structure of admissible forms.

Base case $\sharp$. Since $\sharp([\psi]\hat{K}_i\varphi) = [\psi]\hat{K}_i\varphi$ and $\sharp(\Box_i^K\varphi) = \Box_i^K\varphi$, ($*$) follows directly from the semantics of $\Box_i^K\varphi$.

Inductive cases. We assume by induction hypothesis (IH) that ($*$) holds for $\eta(\sharp)$, we show that ($*$) also holds for the cases $\chi \to \eta(\sharp)$, $K_i\eta(\sharp)$ and $[\chi]\eta(\sharp)$, as follows.

- Case $\chi \to \eta(\sharp)$. Note that $(\chi \to \eta(\sharp))([\psi]\hat{K}_i\varphi) = \chi \to \eta([\psi]\hat{K}_i\varphi)$ and $(\chi \to \eta(\sharp))(\Box_i^K\varphi) = \chi \to \eta(\Box_i^K\varphi)$. Our goal is to show that for all $(\mathcal{M}, s)$, if $\mathcal{M}, s \vDash \chi \to \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$, then $\mathcal{M}, s \vDash \chi \to \eta(\Box_i^K\varphi)$. For this, suppose that $\mathcal{M}, s \vDash \chi \to \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$ and $\mathcal{M}, s \vDash \chi$, then $\mathcal{M}, s \vDash \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$. By (IH), we infer that $\mathcal{M}, s \vDash \eta(\Box_i^K\varphi)$, as desired.
- Case $K_i\eta(\sharp)$. Note that $(K_i\eta(\sharp))([\psi]\hat{K}_i\varphi) = K_i\eta([\psi]\hat{K}_i\varphi)$ and $(K_i\eta(\sharp))(\Box_i^K\varphi) = K_i\eta(\Box_i^K\varphi)$. Our goal is to show that for all $(\mathcal{M}, s)$, if $\mathcal{M}, s \vDash K_i\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$, then $\mathcal{M}, s \vDash K_i\eta(\Box_i^K\varphi)$. For this, suppose that $\mathcal{M}, s \vDash K_i\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$, and for any t in $\mathcal{M}$ such that sR_it, then $\mathcal{M}, t \vDash \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$. By (IH), we derive that $\mathcal{M}, t \vDash \eta(\Box_i^K\varphi)$. Therefore, $\mathcal{M}, s \vDash K_i\eta(\Box_i^K\varphi)$, as desired.
- Case $[\chi]\eta(\sharp)$. Note that $([\chi]\eta(\sharp))([\psi]\hat{K}_i\varphi) = [\chi]\eta([\psi]\hat{K}_i\varphi)$ and $([\chi]\eta(\sharp))(\Box_i^K\varphi) = [\chi]\eta(\Box_i^K\varphi)$. Our goal is to show that for all $(\mathcal{M}, s)$, if $\mathcal{M}, s \vDash [\chi]\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$, then $\mathcal{M}, s \vDash [\chi]\eta(\Box_i^K\varphi)$. For this, suppose that $\mathcal{M}, s \vDash [\chi]\eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$ and $\mathcal{M}, s \vDash \chi$, then $\mathcal{M}|_\chi, s \vDash \eta([\psi]\hat{K}_i\varphi)$ for all $\psi \in \mathbf{EL}$. By (IH), we obtain that $\mathcal{M}|_\chi, s \vDash \eta(\Box_i^K\varphi)$. Therefore, $\mathcal{M}, s \vDash [\chi]\eta(\Box_i^K\varphi)$, as desired. ⊣

5.2. Proof theoretical results

In this subsection we present some proof theoretical results for **LK**. Almost all proofs are in Appendix, as they are rather lengthy.

In the first place, some common alternative derivation rules are derivable in the system $\mathbb{LK}$ (where Lemma 5.5 is essential in showing Proposition 5.6).

Proposition 5.4. *The following rule is derivable in* $\mathbb{LK}$*:*

$$\texttt{RM}[\cdot] \qquad \frac{\varphi \to \psi}{[\chi]\varphi \to [\chi]\psi}$$

Proof. We have the following derivation in $\mathbb{LK}$:

(i)	$\varphi \to \psi$	assumption
(ii)	$\neg\psi \to \neg\varphi$	(i)
(iii)	$\langle\chi\rangle\neg\psi \to \langle\chi\rangle\neg\varphi$	(ii), $\texttt{RM}\langle\cdot\rangle$

(iv)	$\neg\langle\chi\rangle\neg\varphi \to \neg\langle\chi\rangle\neg\psi$	(iii)
(v)	$[\chi]\varphi \to [\chi]\psi$	(iv), Def.$[\cdot]$ ⊣

LEMMA 5.5. *For all φ, ψ and χ, if $\vdash \psi \leftrightarrow \chi$, then $\vdash \langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$.*

PROPOSITION 5.6. *The following rule called* `RE` (*for 'replacement of equivalents'*) *is derivable in* $\mathbb{LK}$*:*

$$\frac{\psi \leftrightarrow \chi}{\varphi(p/\psi) \leftrightarrow \varphi(p/\chi)}$$

Recall that the axiomatization of public announcement logic, denoted **PA**, is given in e.g. [28, Sec. 4.8].

PROPOSITION 5.7. $\mathbf{PA} \subseteq \mathbb{LK}$*.*

PROPOSITION 5.8. *The following axiom is provable*:

`AKK*` $\quad\langle\psi\rangle K_i\varphi \to \Diamond_i^K\varphi$*, where* $\psi \in \mathbf{PAL}$*.*

PROOF. It is known that for any **PAL**-formula ψ, there is an **EL**-fomula ψ' such that $\vDash \psi \leftrightarrow \psi'$. By the completeness of **PA**, we have $\vdash_{\mathbf{PA}} \psi \leftrightarrow \psi'$. By Proposition 5.7, $\mathbf{PA} \subseteq \mathbb{LK}$, thus $\vdash_{\mathbb{LK}} \psi \leftrightarrow \psi'$. Then by `AKK` and `RE`, `AKK*` is derivable. ⊣

PROPOSITION 5.9. *Let* $\varphi \in \mathbf{LK}$*.* $\vdash \varphi \leftrightarrow \langle\top\rangle\varphi$

COROLLARY 5.10. $\vdash [\top]\varphi \leftrightarrow \varphi$ *for all* $\varphi \in \mathbf{LK}$*.*

PROOF. By Proposition 5.9, $\vdash \langle\top\rangle\neg\varphi \leftrightarrow \neg\varphi$. Thus $\vdash \neg\langle\top\rangle\neg\varphi \leftrightarrow \neg\neg\varphi$. By Def. $[\cdot]$, we obtain $\vdash [\top]\varphi \leftrightarrow \varphi$. ⊣

PROPOSITION 5.11. *If* $\vdash \varphi \to \psi$*, then* $\vdash \Box_i^K\varphi \to \Box_i^K\psi$*.*

Recall that in Proposition 3.8 and Proposition 3.12 we show that $\vDash \Diamond_i^K\Diamond_i^K\varphi \to \Diamond_i^K\varphi$ and $\vDash \Diamond_i^K\varphi \to \Diamond_i^K\Diamond_i^K\varphi$, respectively. We can also give a syntactic proof of them.

PROPOSITION 5.12. $\vdash \Diamond_i^K\varphi \to \Diamond_i^K\Diamond_i^K\varphi$

PROPOSITION 5.13. $\vdash \Diamond_i^K\Diamond_i^K\varphi \to \Diamond_i^K\varphi$

We conclude this section with a derivable rule.

PROPOSITION 5.14. *The following rule is derivable in* $\mathbb{LK}$*:*

$$\frac{\varphi}{\Diamond_i^K\varphi}$$

Proof. We have the following derivation in $\mathbb{LK}$:

(i)	φ	assumption
(ii)	$K_i\varphi$	(i), NECK
(iii)	$\langle\top\rangle K_i\varphi$	(ii), Proposition 5.9
(iv)	$\Diamond_i^K\varphi$	(iii), AKK ⊣

For contrast, note that $\varphi \to \Diamond_i^K\varphi$ is *not* derivable (see the remarks before Proposition 3.4).

6. Completeness and Decidability

6.1. Completeness

This section deals with a demonstration of the completeness of $\mathbb{LK}$. The canonical model will be based on a notion of *maximal consistent theory*, rather than the more familiar notion of *maximal consistent set*. The reason of defining consistency for a theory rather than any set of formulas, is because we need the clousure condition under RKb, which is indispensable in the completeness proof.

Definition 6.1 (MCT). A set Γ of formulas is said to be a *theory*, if besides containing **Thm**, it is also closed under the rules MP and RKb. A theory Γ is said to be *consistent*, if $\bot \notin \Gamma$; Γ is said to be *maximal*, if for all φ, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Γ is a *maximal consistent theory* (MCT), if it is a theory which is consistent and maximal.

One may easily check that **Thm** is the smallest theory.

Define $s + \varphi$ as $\{\psi \mid \varphi \to \psi \in s\}$. We omit the proof details of the following result.

Proposition 6.2. *Let $\varphi \in$ **LK** and s be a theory. Then*

1. *$s + \varphi$ is a theory, and $s \cup \{\varphi\} \subseteq s + \varphi$.*
2. *$s + \varphi$ is consistent iff $\neg\varphi \notin s$.*

Lindenbaum's Lemma can be proven as [5, Lemma 4.12], with only corresponding changes of the rule RKb. Thus we omit the proof details.

Lemma 6.3 (Lindenbaum's Lemma). *Every consistent theory can be extended to a MCT.*

Definition 6.4 (Canonical Model). The canonical model for $\mathbb{LK}$ is $\mathcal{M}^c = \langle S^c, \{R_i^c \mid i \in \mathbf{Ag}\}, V^c\rangle$, where

- S^c is the set of all MCTs;
- For all $i \in \mathbf{Ag}$, $sR_i^c t$ iff $\{\varphi \mid K_i\varphi \in s\} \subseteq t$;
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

Using axioms `T`, `4` and `5`, we can show that each R_i^c is an equivalence relation. Thus $\mathcal{M}^c$ is indeed a model.

The following proposition can be shown as in [6, Lemma 7]. Thus again, we omit the proof details.

PROPOSITION 6.5. *Let $s \in S^c$, $\psi \in \mathbf{LK}$, and $i \in \mathbf{Ag}$ such that $K_i\psi \notin s$. Then there exists $t \in S^c$ such that $sR_i^c t$ and $\psi \notin t$.*

LEMMA 6.6 (Truth Lemma). *For all $\varphi \in \mathbf{LK}$ and $s \in S^c$, we have*

$$\mathcal{M}^c, s \vDash \varphi \iff \varphi \in s.$$

PROOF. It is straightforward to show that $<_\Diamond^S$ is a well-founded strict partial order between formulas. Let $\varphi \in \mathbf{LK}$ and $s \in S^c$, we proceed with $<_\Diamond^S$-induction on φ, that is, with induction on the complexity of φ.

- $\varphi = p$. We have $\mathcal{M}^c, s \vDash p \iff s \in V^c(p) \overset{\text{Def. } V^c}{\iff} p \in s$.
- $\varphi = \neg\psi$. Recall that $\psi <_\Diamond^S \neg\psi$ (Proposition 2.6). We have

$$\begin{aligned} \mathcal{M}^c, s \vDash \neg\psi &\iff \mathcal{M}^c, s \nvDash \psi \\ &\overset{\text{IH}}{\iff} \psi \notin s \\ &\iff \neg\psi \in s. \end{aligned}$$

- $\varphi = \psi \wedge \chi$. Recall that $\psi <_\Diamond^S \psi \wedge \chi$ and $\chi <_\Diamond^S \psi \wedge \chi$ (Proposition 2.6). We have

$$\begin{aligned} \mathcal{M}^c, s \vDash \psi \wedge \chi &\iff \mathcal{M}^c, s \vDash \psi \text{ and } \mathcal{M}^c, s \vDash \chi \\ &\overset{\text{IH}}{\iff} \psi \in s \text{ and } \chi \in s \\ &\iff \psi \wedge \chi \in s. \end{aligned}$$

- $\varphi = K_i\psi$. Recall that $\psi <_\Diamond^S K_i\psi$ (Proposition 2.6). We have

$$\begin{aligned} \mathcal{M}^c, s \vDash K_i\psi &\iff \mathcal{M}^c, t \vDash \psi \text{ for all } t \in R_i^c(s) \\ &\overset{\text{IH}}{\iff} \psi \in t \text{ for all } t \in R_i^c(s) \\ &\overset{(*)}{\iff} K_i\psi \in s. \end{aligned}$$

The equivalence $(*)$ follows from the definition of R_i^c and Proposition 6.5.

• $\varphi = \langle\psi\rangle p$. Recall that $\psi <^S_\Diamond \langle\psi\rangle p$ and $p <^S_\Diamond \langle\psi\rangle p$ (Proposition 2.6). We have

$$\begin{aligned}\mathcal{M}^c, s \vDash \langle\psi\rangle p &\iff \mathcal{M}^c, s \vDash \psi \text{ and } \mathcal{M}^c, s \vDash p\\ &\overset{\text{IH}}{\iff} \psi \in s \text{ and } p \in s\\ &\iff \psi \wedge p \in s\\ &\overset{\text{Ax. !ATOM}}{\iff} \langle\psi\rangle p \in s.\end{aligned}$$

• $\varphi = \langle\psi\rangle\neg\chi$. Recall that $\psi <^S_\Diamond \langle\psi\rangle\neg\chi$ and $\langle\psi\rangle\chi <^S_\Diamond \langle\psi\rangle\neg\chi$ (Proposition 2.6). We have

$$\begin{aligned}\mathcal{M}^c, s \vDash \langle\psi\rangle\neg\chi &\iff \mathcal{M}^c, s \vDash \psi \text{ and } \mathcal{M}^c, s \nvDash \langle\psi\rangle\chi\\ &\overset{\text{IH}}{\iff} \psi \in s \text{ and } \langle\psi\rangle\chi \notin s\\ &\iff \psi \in s \text{ and } \neg\langle\psi\rangle\chi \in s\\ &\overset{\text{Ax. !NEG}}{\iff} \langle\psi\rangle\neg\chi \in s.\end{aligned}$$

• $\varphi = \langle\psi\rangle(\chi_1 \wedge \chi_2)$. Recall that $\langle\psi\rangle\chi_1 <^S_\Diamond \langle\psi\rangle(\chi_1 \wedge \chi_2)$ and $\langle\psi\rangle\chi_2 <^S_\Diamond \langle\psi\rangle(\chi_1 \wedge \chi_2)$ (Proposition 2.6). We have

$$\begin{aligned}\mathcal{M}^c, s \vDash \langle\psi\rangle(\chi_1 \wedge \chi_2) &\iff \mathcal{M}^c, s \vDash \langle\psi\rangle\chi_1 \text{ and } \mathcal{M}^c, s \vDash \langle\psi\rangle\chi_2\\ &\overset{\text{IH}}{\iff} \langle\psi\rangle\chi_1 \in s \text{ and } \langle\psi\rangle\chi_2 \in s\\ &\iff \langle\psi\rangle\chi_1 \wedge \langle\psi\rangle\chi_2 \in s\\ &\overset{\text{Ax. !CON}}{\iff} \langle\psi\rangle(\chi_1 \wedge \chi_2) \in s.\end{aligned}$$

• $\varphi = \langle\psi\rangle K_i\chi$. Recall that $\psi <^S_\Diamond \langle\psi\rangle K_i\chi$ and $K_i[\psi]\varphi <^S_\Diamond \langle\psi\rangle K_i\chi$ (Proposition 2.6). We have

$$\begin{aligned}\mathcal{M}^c, s \vDash \langle\psi\rangle K_i\chi &\iff \mathcal{M}^c, s \vDash \psi \text{ and } \mathcal{M}^c, s \vDash K_i[\psi]\chi\\ &\overset{\text{IH}}{\iff} \psi \in s \text{ and } K_i[\psi]\chi \in s\\ &\iff \psi \wedge K_i[\psi]\chi \in s\\ &\overset{\text{Ax. !CON}}{\iff} \langle\psi\rangle K_i\chi \in s.\end{aligned}$$

• $\varphi = \langle\psi\rangle\langle\chi\rangle\delta$. Recall that $\langle\langle\psi\rangle\chi\rangle\delta <^S_\Diamond \langle\psi\rangle\langle\chi\rangle\delta$ (Proposition 2.6). We have

$$\begin{aligned}\mathcal{M}^c, s \vDash \langle\psi\rangle\langle\chi\rangle\delta &\iff \mathcal{M}^c, s \vDash \langle\langle\psi\rangle\chi\rangle\delta\\ &\overset{\text{IH}}{\iff} \langle\langle\psi\rangle\chi\rangle\delta \in s\\ &\overset{\text{Ax. !!}}{\iff} \langle\psi\rangle\langle\chi\rangle\delta \in s.\end{aligned}$$

- $\varphi = \langle\psi\rangle\Diamond_i^K\chi$. We have

$$\begin{aligned}
\mathcal{M}^c, s \vDash \langle\psi\rangle\Diamond_i^K\chi &\iff \mathcal{M}^c, s \vDash \psi \text{ and } \mathcal{M}^c|_\psi, s \vDash \Diamond_i^K\chi \\
&\iff \mathcal{M}^c, s \vDash \psi,\ \mathcal{M}^c|_\psi, s \vDash \langle\delta\rangle K_i\chi \text{ for some } \delta \in \mathbf{EL} \\
&\iff \mathcal{M}^c, s \vDash \langle\psi\rangle\langle\delta\rangle K_i\chi \text{ for some } \delta \in \mathbf{EL} \\
&\overset{\text{IH}}{\iff} \langle\psi\rangle\langle\delta\rangle K_i\chi \in s \text{ for some } \delta \in \mathbf{EL} \\
&\overset{(1)}{\iff} [\psi][\delta]\hat{K}_i\neg\chi \notin s \text{ for some } \delta \in \mathbf{EL} \\
&\overset{(**)}{\iff} [\psi]\Box_i^K\neg\chi \notin s \\
&\overset{(2)}{\iff} \langle\psi\rangle\Diamond_i^K\chi \in s.
\end{aligned}$$

Recall that $\langle\psi\rangle\langle\delta\rangle K_i\chi <^S_\Diamond \langle\psi\rangle\Diamond_i^K\chi$ for any $\delta \in \mathbf{EL}$ (Proposition 2.6), thus we can use the induction hypothesis (IH) in the fourth step. In $(**)$, the left-to-right direction follows from Axiom AKK and rule RM$[\cdot]$, and the other direction is because s is closed under the rule RKb for the admissible form $[\psi]\sharp$. (1) and (2) hold due to the maximal consistency of s.

- $\varphi = \Diamond_i^K\psi$. We have

$$\begin{aligned}
\mathcal{M}^c, s \vDash \Diamond_i^K\psi &\iff \mathcal{M}^c, s \vDash \langle\chi\rangle K_i\psi \text{ for some } \chi \in \mathbf{EL} \\
&\overset{\text{IH}}{\iff} \langle\chi\rangle K_i\psi \in s \text{ for some } \chi \in \mathbf{EL} \\
&\overset{(a)}{\iff} [\chi]\hat{K}_i\neg\psi \notin s \text{ for some } \chi \in \mathbf{EL} \\
&\overset{(***)}{\iff} \Box_i^K\neg\psi \notin s \\
&\overset{(b)}{\iff} \Diamond_i^K\psi \in s.
\end{aligned}$$

Recall that $\langle\chi\rangle K_i\psi <^S_\Diamond \Diamond_i^K\psi$ for any $\chi \in \mathbf{EL}$ (Proposition 2.6), thus we can use the induction hypothesis (IH) in the second step. The equivalence $(***)$ is due to Axiom AKK and the fact that s is closed under the rule RKb for the possible form $\sharp$. (a) and (b) hold because of the maximal consistency of s. ⊣

With the Truth Lemma in mind, we obtain the completeness theorem as usual.

THEOREM 6.7 (Completeness Theorem). **LK** *is sound and complete with respect to the class of frames. That is, if* $\vDash \varphi$*, then* $\vdash \varphi$.

Proof. The soundness is immediate. For the completeness, suppose $\nvdash \varphi$, i.e. $\varphi \notin$ **Thm**. Since **Thm** is a theory, it is closed under MP, thus $\neg\neg\varphi \notin$ **Thm**. By Proposition 6.2, **Thm** $+ \{\neg\varphi\}$ is a consistent theory and $\neg\varphi \in$ **Thm** $+ \{\neg\varphi\}$. By Lindenbaum's Lemma (Lemma 6.3), there exists $t \in S^c$ with **Thm** $+ \{\neg\varphi\} \subseteq t$, and thus $\neg\varphi \in t$, that is, $\varphi \notin t$. Due to the Truth Lemma (Lemma 6.6), we obtain $\mathcal{M}^c, t \nvDash \varphi$. Moreover, as remarked before, $\mathcal{M}^c$ is a model. Therefore $\nvDash \varphi$. ⊣

6.2. Decidability

Recall that the satisfiability problem of **APAL** is shown to be undecidable when there are at least two agents [2, 14]. The approach is by reducing an undecidable tiling problem into **APAL** [2]. Following the same approach, we may infer that **LK** is also undecidable when there are at least three agents. We will sketch the main idea of the proof.

In [2] an **APAL**-formula φ is defined such that a certain finite set of tiles Γ tiles the infinite plain $\mathbb{N} \times \mathbb{N}$, if and only if φ is satisfiable on a certain model $\mathcal{M}$ defined for two agents a and b. We can transform φ into an **LK**-formula ψ by substituting all quantifiers $\Box$ in φ for knowability operators $\Box_i^K$, and we can change the model $\mathcal{M}$ into a model $\mathcal{M}^{\mathbf{LK}}$ that is the same as $\mathcal{M}$ except that we add another agent i that has the identity relation on the domain. Since for any state t in the model, t has itself as the only i-successor, it follows for any subformula θ of φ:

$$\mathcal{M}^{\mathbf{LK}} \vDash \theta \leftrightarrow \hat{K}_i\theta$$

For example, a constituent of the formula φ is:

$$c_{apal}(\heartsuit) := \heartsuit \to \Box(K_{\mathfrak{s}}(r \to (K_{\mathfrak{e}}(l \to (K_{\mathfrak{s}}(u \to K_{\mathfrak{e}}(d \to K_{\mathfrak{s}}(l \to K_{\mathfrak{e}}(r \to K_{\mathfrak{s}}(d \to K_{\mathfrak{e}}(u \to \hat{K}_{\mathfrak{s}}\heartsuit))))))))))$$

It is transformed into:

$$c_{lk}(\heartsuit) := \heartsuit \to \Box_i^K(K_{\mathfrak{s}}(r \to (K_{\mathfrak{e}}(l \to (K_{\mathfrak{s}}(u \to K_{\mathfrak{e}}(d \to K_{\mathfrak{s}}(l \to K_{\mathfrak{e}}(r \to K_{\mathfrak{s}}(d \to K_{\mathfrak{e}}(u \to \hat{K}_{\mathfrak{s}}\heartsuit))))))))))$$

and $\mathcal{M}^{LK}, t \vDash c_{lk}(\heartsuit)$ if and only if $\mathcal{M}, t \vDash c_{apal}(\heartsuit)$.

This may sufficiently demonstrate that a detailed proof of the undecidability of the satisfiability of **LK** would be nearly identical to the proof in [2]. Therefore, **LK** is undecidable for at least three agents. Whether **LK** is decidable for only two agents needs further investigation.

In what follows, we will give two decidable knowability logics.

7. Decidable knowability logics

7.1. Logic $\mathbf{LK}^=$

We recall that the language of the logic $\mathbf{LK}^=$ was defined as the fragment

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond_i^K \varphi$$

In this fragment we can no longer quantify over all epistemic formulas, but, for a similar treatment of the quantifier, over all Booleans only. Its semantics are:

$$\mathcal{M}, s \vDash \Diamond_i^K \varphi \iff \text{there is a } \psi \in \mathbf{PL} \text{ such that } \mathcal{M}, s \vDash \psi \text{ and for all } t \in R_i(s), \mathcal{M}|_\psi, t \vDash \varphi$$

This quantification is therefore like the one in so-called Boolean arbitrary public announcement logic **BAPAL** [24] (where again $\Diamond K_i \varphi$ corresponds to $\Diamond_i^K \varphi$).

$$\mathcal{M}, s \vDash \Diamond\varphi \iff \text{there is } \psi \in \mathbf{PL} \text{ such that } \mathcal{M}, s \vDash \psi \text{ and } \mathcal{M}|_\psi, s \vDash \varphi$$

As the semantics of the quantifier in $\mathbf{LK}^=$ are different, the properties of the quantifier $\Diamond_i^K$ that were observed in Section 3 now have to be shown again. It is straightforward that $\Diamond_i^K \varphi$ implies $\Diamond\varphi$.

It may be interesting and surprising to see that the knowability operators are dispensable in classical propositional logic. That is to say, the addition of knowability operators does not increase the expressive power of classical propositional logic.

PROPOSITION 7.1. $\mathbf{LK}^=$ *is equally expressive as* $\mathbf{PL}$.

PROOF. As $\mathbf{LK}^=$ extends $\mathbf{PL}$, $\mathbf{LK}^=$ is at least as expressive as $\mathbf{PL}$. It suffices to prove that $\mathbf{PL}$ is at least as expressive as $\mathbf{LK}$.

For this, let φ be a formula in the language of $\mathbf{LK}^=$. We prove that φ is equivalent to a formula in $\mathbf{PL}$. The proof is by induction on the number of $\Diamond_i^K$ modalities in φ.

If φ contains *no* $\Diamond_i^K$ modality, then φ is already in $\mathbf{PL}$, and we are done. Otherwise, consider a subformula $\Diamond_i^K \psi$ of φ such that $\psi \in \mathbf{PL}$.

We first show that $\vDash \Diamond_i^K \psi \leftrightarrow \psi$.

Let $\mathcal{M} = \langle S, R, V \rangle$ and $s \in S$ be given.

Assume that $\mathcal{M}, s \vDash \Diamond_i^K \psi$. By definition, there is a $\chi \in \mathbf{PL}$ such that $\mathcal{M}, s \vDash \chi$ and for all $t \in R_i(s)$, $\mathcal{M}|_\chi, t \vDash \psi$. In particular, $\mathcal{M}|_\chi, s \vDash \psi$.

Therefore, as ψ is Boolean and as the valuation does not change after model restriction, we have $\mathcal{M}, s \vDash \psi$.

Conversely, assume that $\mathcal{M}, s \vDash \psi$. Consider the characteristic formula δ_s^ψ defined as in the proof of Thm. 3.2. Then $\mathcal{M}, s \vDash \delta_s^\psi$, and also $\mathcal{M}|_{\delta_s^\psi}, s \vDash \psi$. As the valuation of the variables in ψ is constant on $\mathcal{M}|_{\delta_s^\psi}$, it follows from Proposition 3.1 that $\mathcal{M}|_{\delta_s^\psi} \vDash \psi$, and therefore $\mathcal{M}|_{\delta_s^\psi}, t \vDash \psi$ for all $t \in R_i(s)$. From that and $\mathcal{M}, s \vDash \delta_s^\psi$ it follows by semantics that $\mathcal{M}, s \vDash \Diamond_i^K \psi$.

This proves $\vDash \Diamond_i^K \psi \leftrightarrow \psi$. Now replace $\Diamond_i^K \psi$ by ψ in φ. Let the result be φ'. Note that $\vDash \varphi \leftrightarrow \varphi'$. As φ' contains one less knowability modality than φ, by induction hypothesis we can conclude that φ' is equivalent to a Boolean formula φ''. From $\vDash \varphi \leftrightarrow \varphi'$ and $\vDash \varphi' \leftrightarrow \varphi''$ it follows that $\vDash \varphi \leftrightarrow \varphi''$. ⊣

It may be instructive to present an example.

Example 7.2. We will show that the formula $\Diamond_i^K \Diamond_j^K (\Box_k^K (p \to q) \vee \Box_k^K \neg r)$, read "it is knowable for i that it is knowable for j that *either* it is unknowable for k that p does not imply q *or* it is unknowable for k that r", is equivalent to a Boolean formula. The proof is as follows:

$$\begin{aligned}
\Diamond_i^K \Diamond_j^K (\Box_k^K (p \to q) \vee \Box_k^K \neg r) &\leftrightarrow \Diamond_i^K \Diamond_j^K (\neg \Diamond_k^K \neg (p \to q) \vee \neg \Diamond_k^K \neg\neg r) \\
&\leftrightarrow \Diamond_i^K \Diamond_j^K (\neg\neg (p \to q) \vee \neg\neg\neg r) \\
&\leftrightarrow \Diamond_i^K \Diamond_j^K ((p \to q) \vee \neg r) \\
&\leftrightarrow \Diamond_i^K ((p \to q) \vee \neg r) \\
&\leftrightarrow (p \to q) \vee \neg r
\end{aligned}$$

In what follows, we show the properties of Church-Rosser and McKinsey hold for $\mathbf{LK}^=$. For this, we define a translation from $\mathbf{LK}^=$ to $\mathbf{PL}$.

Definition 7.3. Define $t : \mathbf{LK}^= \to \mathbf{PL}$ as follows.

$$\begin{array}{lcl}
t(p) & = & p \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(\Diamond_i^K \varphi) & = & t(\varphi).
\end{array}$$

Intuitively, t removes every occurrence of $\Diamond_i^K$ in the formulas of $\mathbf{LK}^=$.

It is straightforward to compute that $t(\Box_i^K \varphi) = \neg\neg t(\varphi)$.

This translation helps us show the properties of Church-Rosser and McKinsey holds for $\mathbf{LK}^=$, namely, $\Diamond_i^K\Box_i^K\varphi \to \Box_i^K\Diamond_i^K\varphi$ and $\Box_i^K\Diamond_i^K\varphi \to \Diamond_i^K\Box_i^K\varphi$, respectively, are valid on the semantics of $\mathbf{LK}^=$. To see this, we first show the following result.

LEMMA 7.4. *For all* $\varphi \in \mathbf{LK}^=$, *we have*

$$\vDash \varphi \leftrightarrow t(\varphi).$$

PROOF. By induction on $\varphi \in \mathbf{LK}^=$.

- $\varphi = p \in \mathbf{P}$. Since $t(p) = p$, we obviously have $\vDash p \leftrightarrow t(p)$.
- $\varphi = \neg\psi$. By induction hypothesis, $\vDash \psi \leftrightarrow t(\psi)$. Then $\vDash \neg\psi \leftrightarrow t(\neg\psi)$.
- $\varphi = \psi \wedge \chi$. By induction hypothesis, $\vDash \psi \leftrightarrow t(\psi)$ and $\vDash \chi \leftrightarrow t(\chi)$. Then $\vDash (\psi \wedge \chi) \leftrightarrow t(\psi \wedge \chi)$.
- $\varphi = \Diamond_i^K\psi$. By induction hypothesis, $\vDash \psi \leftrightarrow t(\psi)$. Then $\vDash \Diamond_i^K\psi \leftrightarrow \Diamond_i^K t(\psi)$. Since $t(\psi) \in \mathbf{PL}$, by the proof of Proposition 7.1, $\vDash \Diamond_i^K t(\psi) \leftrightarrow t(\psi)$. This follows that $\vDash \Diamond_i^K\psi \leftrightarrow t(\psi)$.[3] As $t(\Diamond_i^K\psi) = t(\psi)$, we conclude that $\vDash \Diamond_i^K\psi \leftrightarrow t(\Diamond_i^K\psi)$. ⊣

THEOREM 7.5 (CR and MK). $\vDash \Diamond_i^K\Box_i^K\varphi \leftrightarrow \Box_i^K\Diamond_i^K\varphi$.

PROOF. Note that $t(\Diamond_i^K\Box_i^K\varphi) = t(\Box_i^K\varphi) = \neg\neg t(\varphi)$ and $t(\Box_i^K\Diamond_i^K\varphi) = \neg\neg t(\Diamond_i^K\varphi) = \neg\neg t(\varphi)$. Thus $t(\Diamond_i^K\Box_i^K\varphi) = t(\Box_i^K\Diamond_i^K\varphi)$. By Lemma 7.4, we have $\vDash \Diamond_i^K\Box_i^K\varphi \leftrightarrow t(\Diamond_i^K\Box_i^K\varphi)$ and $\vDash \Box_i^K\Diamond_i^K\varphi \leftrightarrow t(\Box_i^K\Diamond_i^K\varphi)$. Therefore, $\vDash \Diamond_i^K\Box_i^K\varphi \leftrightarrow \Box_i^K\Diamond_i^K\varphi$. ⊣

Now we add an axiomatization for $\mathbf{LK}^=$. In retrospect, Lemma 7.4 essentially gives us the following reduction-like axiom (denoted `Red`):

$$\Diamond_i^K\varphi \leftrightarrow \varphi.$$

Intuitively, `Red` removes all $\Diamond_i^K$ operators from formulas in $\mathbf{LK}^=$ within finitely many steps.

We use $\mathbb{LK}^=$ to denote $\mathbb{PL} + $ `Red`, in which $\mathbb{PL}$ is the classical propositional calculus. In what follows, we will show that $\mathbb{LK}^=$ is determined by the class of frames. For this, we first need an important result.

LEMMA 7.6. *For all* $\varphi \in \mathbf{LK}^=$, *we have* $\vdash \varphi \leftrightarrow t(\varphi)$.

[3] Note that Proposition 7.1 only shows that $\vDash \Diamond_i^K\chi \leftrightarrow \chi$ holds for every $\chi \in \mathbf{PL}$, but it does not show this statement holds for any $\mathbf{LK}^=$-formula. This is what we are doing here.

Proof. By induction on $\varphi \in \mathbf{LK}^{=}$.

- $\varphi = p \in \mathbf{P}$. As $t(p) = p$, we have $\vdash p \leftrightarrow t(p)$.
- $\varphi = \neg\psi$. By induction hypothesis, $\vdash \psi \leftrightarrow t(\psi)$, and thus $\vdash \neg\psi \leftrightarrow \neg t(\psi)$, that is, $\vdash \neg\psi \leftrightarrow t(\neg\psi)$.
- $\varphi = \psi \wedge \chi$. By induction hypothesis, we have $\vdash \psi \leftrightarrow t(\psi)$ and $\vdash \chi \leftrightarrow t(\chi)$. Therefore, $\vdash (\psi \wedge \chi) \leftrightarrow t(\psi \wedge \chi)$.
- $\varphi = \lozenge_i^K \psi$. By induction hypothesis, $\vdash \psi \leftrightarrow t(\psi)$. By axiom `Red`, $\vdash \lozenge_i^K \psi \leftrightarrow \psi$. Moreover, $t(\lozenge_i^K \psi) = t(\psi)$. Then we conclude that $\vdash \lozenge_i^K \psi \leftrightarrow t(\lozenge_i^K \psi)$. ⊣

Theorem 7.7. *$\mathbb{LK}^{=}$ is sound and complete with respect to the class of all frames.*

Proof. For the soundness, it remains only to show the validity of axiom `Red`. By Lemma 7.4, $\vDash \lozenge_i^K \varphi \leftrightarrow t(\lozenge_i^K \varphi)$ and $\vDash \varphi \leftrightarrow t(\varphi)$. As $t(\lozenge_i^K \varphi) = t(\varphi)$, we therefore obtain $\vDash \lozenge_i^K \varphi \leftrightarrow \varphi$.

As for the completeness, suppose $\vDash \varphi$, then by Lemma 7.4, $\vDash t(\varphi)$. Since $t(\varphi) \in \mathbf{PL}$, by the completeness of $\mathbf{PL}$, $\vdash_{\mathbb{PL}} t(\varphi)$. Since $\mathbb{PL} \subseteq \mathbb{LK}^{=}$, then $\vdash t(\varphi)$. Now using Lemma 7.6, we conclude that $\vdash \varphi$, as desired. ⊣

Remark 7.8. With axiom `Red` in hand, we can even give a *syntactic* proof of CR and MK in $\mathbb{LK}^{=}$ (without use of completeness), because we can derive that $\vdash \lozenge_i^K \varphi \leftrightarrow \varphi$ and $\vdash \Box_i^K \varphi \leftrightarrow \varphi$. Therefore, both $\lozenge_i^K \Box_i^K \varphi$ and $\Box_i^K \lozenge_i^K \varphi$ are provably equivalent to φ. Therefore, $\vdash \lozenge_i^K \Box_i^K \varphi \leftrightarrow \Box_i^K \lozenge_i^K \varphi$.

7.2. Logic $\mathbf{LK}^{-}$

One may naturally ask whether the announcement operators increase the expressivity in $\mathbf{LK}^{=}$. Again, the answer is negative. Recall that when the announcement operators are added to $\mathbf{LK}^{=}$, we obtain the language $\mathbf{LK}^{-}$. In other words, $\mathbf{LK}^{-}$ is defined recursively as follows.

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\varphi\rangle\varphi \mid \lozenge_i^K \varphi$$

Proposition 7.9. *$\mathbf{LK}^{-}$ is equally expressive as $\mathbf{PL}$.*

Proof. As $\mathbf{LK}^{-}$ extends $\mathbf{PL}$, $\mathbf{LK}^{-}$ is at least as expressive as $\mathbf{PL}$. It suffices to show that $\mathbf{PL}$ is at least as expressive as $\mathbf{LK}^{-}$.

For this, let φ be a formula in the language of $\mathbf{LK}^{-}$. We show that φ is equivalent to a formula in $\mathbf{PL}$. The proof is by induction on the number of $\langle\cdot\rangle$ modalities in φ.

If φ contains no $\langle\cdot\rangle$ modality, then φ is a formula in the language of $\mathbf{LK}^{=}$. As we shown in Proposition 7.1, φ is equivalent to a $\mathbf{PL}$-formula. Otherwise, consider a subformula $\langle\chi\rangle\psi$ of φ such that $\psi, \chi \in \mathbf{LK}^{=}$. By Proposition 7.1 again, each of ψ and χ is equivalent to some $\mathbf{PL}$-formula. Then by using the reduction axioms concerning announcements and Boolean formulas, we can infer that $\langle\chi\rangle\psi$ is equivalent to a $\mathbf{PL}$-formula, namely $\chi \wedge \psi$. Now replace $\langle\chi\rangle\psi$ by $\chi \wedge \psi$ in φ. Let the result be φ'. Note that $\vDash \varphi \leftrightarrow \varphi'$. As φ' contains one less $\langle\cdot\rangle$ modality than φ, by induction hypothesis we conclude that φ' is equivalent to a formula φ'' in $\mathbf{PL}$. From $\vDash \varphi \leftrightarrow \varphi'$ and $\vDash \varphi' \leftrightarrow \varphi''$, it follows that $\vDash \varphi \leftrightarrow \varphi''$. $\dashv$

Also, we give a concrete example to illustrate the result.

Example 7.10. We will show that the formula $\Diamond_i^K\langle p\rangle\Diamond_j^K\langle\Diamond_i^K(q \wedge r)\rangle(p \to q)$, read "it is knowable for i that after a truthful announcement of p, it is knowable for j that after a truthful announcement of the fact that the conjunction of q and r is knowable for i, p implies q", is equivalent to a Boolean formula, as follows:

$$\begin{aligned}
\Diamond_i^K\langle p\rangle\Diamond_j^K\langle\Diamond_i^K(q \wedge r)\rangle(p \to q) &\leftrightarrow \Diamond_i^K\langle p\rangle\Diamond_j^K\langle q \wedge r\rangle(p \to q)\\
&\leftrightarrow \Diamond_i^K\langle p\rangle\Diamond_j^K((q \wedge r) \wedge (p \to q))\\
&\leftrightarrow \Diamond_i^K\langle p\rangle((q \wedge r) \wedge (p \to q))\\
&\leftrightarrow \Diamond_i^K(p \wedge (q \wedge r) \wedge (p \to q))\\
&\leftrightarrow p \wedge (q \wedge r) \wedge (p \to q)\\
&\leftrightarrow p \wedge q \wedge r
\end{aligned}$$

Also, we can axiomatize $\mathbf{LK}^{-}$ over the class of all frames. Define $\mathbb{LK}^{-}$ as the smallest extension of $\mathbb{LK}^{=}$ plus the following axiom `Red'`:

$$\langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge \psi).$$

In what follows, we show the properties of Church-Rosser and McKinsey also hold for $\mathbf{LK}^{-}$. For this, we define a translation from $\mathbf{LK}^{-}$ to $\mathbf{PL}$.

DEFINITION 7.11. Define $t' : \mathbf{LK}^{-} \to \mathbf{PL}$ as follows.

$$\begin{array}{lcl}
t'(p) & = & p\\
t'(\neg\varphi) & = & \neg t'(\varphi)\\
t'(\varphi \wedge \psi) & = & t'(\varphi) \wedge t'(\psi)\\
t'(\langle\varphi\rangle\psi) & = & t'(\varphi) \wedge t'(\psi)\\
t'(\Diamond_i^K\varphi) & = & t'(\varphi)
\end{array}$$

That is, t' extends t for the fragment $\mathbf{LK}^{=}$ in Def. 7.3 with the extra case $\langle\varphi\rangle\psi$.

LEMMA 7.12. *For all* $\varphi \in \mathbf{LK}^{-}$, *we have* $\vDash \varphi \leftrightarrow t'(\varphi)$.

PROOF. By induction on $\varphi \in \mathbf{LK}^{-}$. By Lemma 7.4, it suffices to show the case that $\varphi = \langle\psi\rangle\chi$.

By induction hypothesis, $\vDash \psi \leftrightarrow t'(\psi)$ and $\vDash \chi \leftrightarrow t'(\chi)$. Thus $\vDash \langle\psi\rangle\chi \leftrightarrow \langle t'(\psi)\rangle t'(\chi)$. Since $t'(\chi) \in \mathbf{PL}$, $\vDash \langle t'(\psi)\rangle t'(\chi) \leftrightarrow (t'(\psi) \wedge t'(\chi))$. As $t'(\langle\psi\rangle\chi) = t'(\psi) \wedge t'(\chi)$, we conclude that $\vDash \langle\psi\rangle\chi \leftrightarrow t'(\langle\psi\rangle\chi)$. ⊣

Then as in Theorem 7.5, we can show that the properties of Church-Rosser and McKinsey hold for $\mathbf{LK}^{-}$.

THEOREM 7.13 (CR and MK). $\vDash \Diamond_i^K \Box_i^K \varphi \leftrightarrow \Box_i^K \Diamond_i^K \varphi$.

In what follows, we will also show that $\mathbb{LK}^{-}$ is determined by the class of all frames. For this, we show

LEMMA 7.14. *For all* $\varphi \in \mathbf{LK}^{-}$, *we have* $\vdash \varphi \leftrightarrow t'(\varphi)$.

PROOF. By induction on $\varphi \in \mathbf{LK}^{-}$. The cases for $\varphi \in \mathbf{LK}^{=}$ formulas is similar as in Lemma 7.6. It remains only to prove the case that $\varphi = \langle\psi\rangle\chi$.

By induction hypothesis, $\vdash \psi \leftrightarrow t'(\psi)$ and $\vdash \chi \leftrightarrow t'(\chi)$. Thus $\vdash (\psi \wedge \chi) \leftrightarrow (t'(\psi) \wedge t'(\chi))$. By axiom `Red'` and definition of t', we derive that $\vdash \langle\psi\rangle\chi \leftrightarrow t'(\langle\psi\rangle\chi)$. ⊣

THEOREM 7.15. $\mathbb{LK}^{-}$ *is sound and complete with respect to the class of all frames.*

PROOF. For the soundness, by Theorem 7.7, it suffices to show the validity of axiom `Red'`. By Lemma 7.12, $\vDash \langle\varphi\rangle\psi \leftrightarrow t'(\langle\varphi\rangle\psi)$, $\vDash \varphi \leftrightarrow t'(\varphi)$, and $\vDash \psi \leftrightarrow t'(\psi)$. By definition of t', $t'(\langle\varphi\rangle\psi) = t'(\varphi) \wedge t'(\psi)$. Therefore, $\vDash \langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge \psi)$.

As for the completeness, suppose $\vDash \varphi$, then by Lemma 7.12, $\vDash t(\varphi)$. Since $t(\varphi) \in \mathbf{PL}$, by the completeness of $\mathbf{PL}$, $\vdash_{\mathbb{PL}} t'(\varphi)$. Since $\mathbb{PL} \subseteq \mathbb{LK}^{-}$, we have $\vdash t'(\varphi)$. Now using Lemma 7.14, we conclude that $\vdash \varphi$, as desired. ⊣

Similar to Remark 7.8, we can also give a *syntactic* proof of CR and MK in $\mathbf{LK}^{-}$ without use of completeness.

As both $\mathbf{LK}^{=}$ and $\mathbf{LK}^{-}$ are equally expressive as $\mathbf{PL}$, and $\mathbf{PL}$ is decidable, we have the following decidability result.

THEOREM 7.16. $\mathbf{LK}^{=}$ *and* $\mathbf{LK}^{-}$ *are both decidable.*

8. Conclusion and future work

In this paper, we proposed three knowability logics, namely **LK**, $\mathbf{LK}^-$ and $\mathbf{LK}^=$. We compared the relative expressivity of the three logics and other related logics. It turns out that in the single-agent case, **LK** is equally expressive as arbitrary public announcement logic **APAL** and public announcement logic **PAL**, whereas in the multi-agent case, **LK** is more expressive than **PAL**. In contrast, both $\mathbf{LK}^-$ and $\mathbf{LK}^=$ are equally expressive as classical propositional logic **PL**. We axiomatized the three knowability logics and showed their soundness and completeness. We showed that the properties of Church-Rosser (CR) and McKinsey (MK) holds for all three knowability logics, both syntactically and semantically. **LK** is undecidable for at least three agents; in contrast, $\mathbf{LK}^-$ and $\mathbf{LK}^=$ are both decidable for any number of agents.

We currently see three topics for future research.

Firstly, one may investigate whether **LK** is already undecidable for only two agents.

Secondly, we would wish to determine whether **LK** is less expressive than **APAL**. We have a proof that $\mathbf{LK} < \mathbf{APAL}$ on the class of reflexive models, but we have not yet managed to modify this proof to work with **S5** models. The issue with **S5** models is that they provide far less freedom to make certain states distinguishable while others are indistinguishable. For example, if s_1 and s_2 in an **S5** model are distinguishable and t_1 and t_2 are a-successors of s_1 and s_2, respectively, and only of those states, then t_1 and t_2 cannot be indistinguishable. As a consequence, potential **S5** counterexamples to **LK** being as expressive as **APAL** need to be for more complex than the counterexamples for reflexive models, and are therefore harder to find. We do still conjecture that such counterexamples exist, and therefore that $\mathbf{LK} < \mathbf{APAL}$ on **S5** models, but so far we have not managed to find them.

Finally, an remaining important open question is what the axiomatization is of the logic with the language of **LK** but without public announcements, so that the semantics of the quantifier is given directly (and equivalently). A similar open question remains for the logic **APAL** but without the public announcement in the language (see also [25] where this is discussed at some length). In such cases, we can no longer resort to the public announcement in the axiom and in the derivation rule for the quantifier, and it is very unclear how to proceed alternatively.

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Appendix

This appendix deals with the proof details in Section 5.2.

Proof of Lemma 5.5. Assume that $\vdash \psi \leftrightarrow \chi$, to show that $\vdash \langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$. The proof goes by induction on the complexity of φ (recall that the notion of the complexity of a formula is given in Definition 2.5).

Case p. We have the following derivation in $\mathbb{LK}$:

(i)	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$	!ATOM
(ii)	$\langle\chi\rangle p \leftrightarrow (\chi \wedge p)$	!ATOM
(iii)	$(\psi \wedge p) \leftrightarrow (\chi \wedge p)$	assumption
(iv)	$\langle\psi\rangle p \leftrightarrow \langle\chi\rangle p$	(i)–(iii)

Case $\neg\varphi$. Recall that φ is less complex than $\neg\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$. We have the following derivation in $\mathbb{LK}$:

(i)	$\langle\psi\rangle\neg\varphi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\varphi)$	!NEG
(ii)	$\langle\chi\rangle\neg\varphi \leftrightarrow (\chi \wedge \neg\langle\chi\rangle\varphi)$	!NEG
(iii)	$(\psi \wedge \neg\langle\psi\rangle\varphi) \leftrightarrow (\chi \wedge \neg\langle\chi\rangle\varphi)$	assumption, IH
(iv)	$\langle\psi\rangle\neg\varphi \leftrightarrow \langle\chi\rangle\neg\varphi$	(i)–(iii)

Case $\varphi_1 \wedge \varphi_2$. Recall that both φ_1 and φ_2 are less complex than $\varphi_1 \wedge \varphi_2$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\psi\rangle\varphi_1 \leftrightarrow \langle\chi\rangle\varphi_1$ and $\vdash \langle\psi\rangle\varphi_2 \leftrightarrow \langle\chi\rangle\varphi_2$. We have the following derivation in $\mathbb{LK}$:

(i)	$\langle\psi\rangle(\varphi_1 \wedge \varphi_2) \leftrightarrow (\langle\psi\rangle\varphi_1 \wedge \langle\psi\rangle\varphi_2)$	!CON
(ii)	$\langle\chi\rangle(\varphi_1 \wedge \varphi_2) \leftrightarrow (\langle\chi\rangle\varphi_1 \wedge \langle\chi\rangle\varphi_2)$	!CON

(iii)	$\langle\psi\rangle\varphi_1 \leftrightarrow \langle\chi\rangle\varphi_1$	IH
(iv)	$\langle\varphi\rangle\varphi_2 \leftrightarrow \langle\chi\rangle\varphi_2$	IH
(v)	$(\langle\psi\rangle\varphi_1 \wedge \langle\psi\rangle\varphi_2) \leftrightarrow (\langle\chi\rangle\varphi_1 \wedge \langle\chi\rangle\varphi_2)$	(iii), (iv)
(vi)	$\langle\psi\rangle(\varphi_1 \wedge \varphi_2) \leftrightarrow \langle\chi\rangle(\varphi_1 \wedge \varphi_2)$	(i), (ii), (v)

Case $K_i\varphi$. Recall that φ is less complex than $K_i\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$. We have the following derivation in $\mathbb{LK}$:

(i)	$\langle\psi\rangle K_i\varphi \leftrightarrow (\psi \wedge K_i[\psi]\varphi)$	`!K`
(ii)	$\langle\chi\rangle K_i\varphi \leftrightarrow (\chi \wedge K_i[\chi]\varphi)$	`!K`
(iii)	$\langle\psi\rangle\varphi \leftrightarrow \langle\chi\rangle\varphi$	IH
(iv)	$\langle\psi\rangle\neg\varphi \leftrightarrow \langle\chi\rangle\neg\varphi$	(iii), similar to the case $\neg\varphi$
(v)	$[\psi]\varphi \leftrightarrow [\chi]\varphi$	(iv), Def.$[\cdot]$
(vi)	$K_i[\psi]\varphi \leftrightarrow K_i[\chi]\varphi$	(v), `NECK`, `K`, `MP`
(vii)	$(\psi \wedge K_i[\psi]\varphi) \leftrightarrow (\chi \wedge K_a[\chi]\varphi)$	(vi), assumption
(viii)	$\langle\psi\rangle K_i\varphi \leftrightarrow \langle\chi\rangle K_i\varphi$	(i), (ii), (vii)

Case $\langle\varphi_1\rangle\varphi_2$. Recall that φ_1 is less complex than $\langle\varphi_1\rangle\varphi_2$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\psi\rangle\varphi_1 \leftrightarrow \langle\chi\rangle\varphi_1$. We have the following derivation in $\mathbb{LK}$:

(i)	$\langle\psi\rangle\langle\varphi_1\rangle\varphi_2 \leftrightarrow \langle\langle\psi\rangle\varphi_1\rangle\varphi_2$	`!!`
(ii)	$\langle\chi\rangle\langle\varphi_1\rangle\varphi_2 \leftrightarrow \langle\langle\chi\rangle\varphi_1\rangle\varphi_2$	`!!`
(iii)	$\langle\psi\rangle\varphi_1 \leftrightarrow \langle\chi\rangle\varphi_1$	IH
(iv)	$\langle\langle\psi\rangle\varphi_1\rangle\varphi_2 \leftrightarrow \langle\langle\chi\rangle\varphi_1\rangle\varphi_2$	IH by (iii)
(v)	$\langle\psi\rangle\langle\varphi_1\rangle\varphi_2 \leftrightarrow \langle\chi\rangle\langle\varphi_1\rangle\varphi_2$	(i), (ii), (iv)

Case $\Diamond_i^K\varphi$. Let θ be any **EL**-formula. Recall that $\langle\theta\rangle K_i\varphi$ is less complex than $\Diamond_i^K\varphi$ (Proposition 2.6), and thus $\neg[\theta]\hat{K}_i\neg\varphi$ is less expressive than $\Diamond_i^K\varphi$. By induction hypothesis (IH), $\vdash \langle\psi\rangle\neg[\theta]\hat{K}_i\neg\varphi \leftrightarrow \langle\chi\rangle\neg[\theta]\hat{K}_i\neg\varphi$. Then $\vdash \neg\langle\psi\rangle\neg[\theta]\hat{K}_i\neg\varphi \leftrightarrow \neg\langle\chi\rangle\neg[\theta]\hat{K}_i\neg\varphi$. By Def. $[\cdot]$, $\vdash [\psi][\theta]\hat{K}_i\neg\varphi \leftrightarrow [\chi][\theta]\hat{K}_i\neg\varphi$. We denote this by $(*)$. Then we have the following derivation in $\mathbb{LK}$:

(i)	$\Box_i^K\neg\varphi \rightarrow [\theta]\hat{K}_i\neg\varphi$	`AKK`
(ii)	$[\psi]\Box_i^K\neg\varphi \rightarrow [\psi][\theta]\hat{K}_i\neg\varphi$	(i), `RM`$[\cdot]$ (Proposition 5.4)
(iii)	$[\psi]\Box_i^K\neg\varphi \rightarrow [\chi][\theta]\hat{K}_i\neg\varphi$	(ii), $(*)$
(iv)	$[\psi]\Box_i^K\neg\varphi \rightarrow [\chi]\Box_i^K\neg\varphi$	(iii), `RKb`
(v)	$\langle\chi\rangle\Diamond_i^K\varphi \rightarrow \langle\psi\rangle\Diamond_i^K\varphi$	(iv)
(vi)	$\langle\psi\rangle\Diamond_i^K\varphi \rightarrow \langle\chi\rangle\Diamond_i^K\varphi$	similar to the proof of (v)
(vii)	$\langle\psi\rangle\Diamond_i^K\varphi \leftrightarrow \langle\chi\rangle\Diamond_i^K\varphi$	(v), (vi) $\dashv$

Proof of Proposition 5.6. Assume that $\vdash \psi \leftrightarrow \chi$. Then, by induction on the complexity of φ, we show $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$. Recall that the notion of complexity is given in Definition 2.5.

• $\varphi = p$. Then $\varphi(p/\psi) = \psi$ and $\varphi(p/\chi) = \chi$. By assumption, we have immediately that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.

• $\varphi = q \neq p$. Then $\varphi(p/\psi) = \varphi(p/\chi) = q$. It is then clear that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.

• $\varphi = \neg\theta$. Then $\varphi(p/\psi) = \neg\theta(p/\psi)$ and $\varphi(p/\chi) = \neg\theta(p/\chi)$. Since θ is less complex than φ (Proposition 2.6), by induction hypothesis (IH), $\vdash \theta(p/\psi) \leftrightarrow \theta(p/\chi)$. Then $\vdash \neg\theta(p/\psi) \leftrightarrow \neg\theta(p/\chi)$.

• $\varphi = \varphi_1 \wedge \varphi_2$. Then $\varphi(p/\psi) = \varphi_1(p/\psi) \wedge \varphi_2(p/\psi)$ and $\varphi(p/\chi) = \varphi_1(p/\chi) \wedge \varphi_2(p/\chi)$. Since both φ_1 and φ_2 are less complex than φ (Proposition 2.6), by induction hypothesis (IH), $\vdash \varphi_1(p/\psi) \leftrightarrow \varphi_1(p/\chi)$ and $\vdash \varphi_2(p/\psi) \leftrightarrow \varphi_2(p/\chi)$. Then $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.

• $\varphi = K_i\theta$. Then $\varphi(p/\psi) = K_i\theta(p/\psi)$ and $\varphi(p/\chi) = K_i\theta(p/\chi)$. Since θ is less complex than φ, by induction hypothesis (IH), $\vdash \theta(p/\psi) \leftrightarrow \theta(p/\chi)$. Then using `NECK`, `K` and `MP`, we obtain that $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.

• $\varphi = \langle\varphi_1\rangle\varphi_2$. Then $\varphi(p/\psi) = \langle\varphi_1(p/\psi)\rangle\varphi_2(p/\psi)$ and $\varphi(p/\chi) = \langle\varphi_1(p/\chi)\rangle\varphi_2(p/\chi)$. Since both φ_1 and φ_2 are less complex than φ (Proposition 2.6), by induction hypothesis (IH), $\vdash \varphi_1(p/\psi) \leftrightarrow \varphi_1(p/\chi)$ and $\vdash \varphi_2(p/\psi) \leftrightarrow \varphi_2(p/\chi)$. From the former and Lemma 5.5, it follows that $\vdash \langle\varphi_1(p/\psi)\rangle\varphi_2(p/\psi) \leftrightarrow \langle\varphi_1(p/\chi)\rangle\varphi_2(p/\psi)$; from the latter and $\texttt{RM}\langle\cdot\rangle$, it follows that $\vdash \langle\varphi_1(p/\chi)\rangle\varphi_2(p/\psi) \leftrightarrow \langle\varphi_1(p/\chi)\rangle\varphi_2(p/\chi)$. Then $\vdash \varphi(p/\psi) \leftrightarrow \varphi(p/\chi)$.

• $\varphi = \Diamond_i^K\theta$. Then $\varphi(p/\psi) = \Diamond_i^K\theta(p/\psi)$ and $\varphi(p/\chi) = \Diamond_i^K\theta(p/\chi)$. Let η be any **EL**-formula. By Proposition 2.6, $\langle\eta\rangle K_i\theta$ is less complex than φ, so is $[\eta]\hat{K}_i\neg\theta$. Then by induction hypothesis (IH), $\vdash [\eta]\hat{K}_i\neg\theta(p/\psi) \leftrightarrow [\eta]\hat{K}_i\neg\theta(p/\chi)$. We then have the following derivation in $\mathbb{LK}$:

(i)	$\Box_i^K\neg\theta(p/\chi) \to [\eta]\hat{K}_i\neg\theta(p/\chi)$	`AKK`
(ii)	$\Box_i^K\neg\theta(p/\chi) \to [\eta]\hat{K}_i\neg\theta(p/\psi)$	(i), IH
(iii)	$\Box_i^K\neg\theta(p/\chi) \to \Box_i^K\neg\theta(p/\psi)$	(ii), `RKb`
(iv)	$\neg\Box_i^K\neg\theta(p/\psi) \to \neg\Box_i^K\neg\theta(p/\chi)$	(iii)
(v)	$\Diamond_i^K\theta(p/\psi) \to \Diamond_i^K\theta(p/\chi)$	(iv), `Dual`
(vi)	$\Diamond_i^K\theta(p/\chi) \to \Diamond_i^K\theta(p/\psi)$	similar to the proof of (v)
(vii)	$\Diamond_i^K\theta(p/\psi) \leftrightarrow \Diamond_i^K\theta(p/\chi)$	(v),(vi) ⊣

Proof of Proposition 5.7. We need only show the reduction axioms of **PA** are derivable in $\mathbb{LK}$:

$$
\begin{aligned}
[\varphi]p &\leftrightarrow \neg\langle\varphi\rangle\neg p && \text{Def. } [\cdot] \\
&\leftrightarrow \neg(\varphi \wedge \neg\langle\varphi\rangle p) && \texttt{!NEG} \\
&\leftrightarrow \neg(\varphi \wedge \neg(\varphi \wedge p)) && \texttt{!ATOM} \\
&\leftrightarrow (\varphi \to p) && \texttt{TAUT}
\end{aligned}
$$

$$
\begin{aligned}
[\varphi]\neg\psi &\leftrightarrow \neg\langle\varphi\rangle\neg\neg\psi && \text{Def. } [\cdot] \\
&\leftrightarrow \neg(\varphi \wedge \neg\langle\varphi\rangle\neg\psi) && \texttt{!NEG} \\
&\leftrightarrow \neg(\varphi \wedge [\varphi]\psi) && \text{Def. } [\cdot] \\
&\leftrightarrow (\varphi \to \neg[\varphi]\psi) && \texttt{TAUT}
\end{aligned}
$$

$$
\begin{aligned}
[\varphi](\psi \wedge \chi) &\leftrightarrow \neg\langle\varphi\rangle\neg(\psi \wedge \chi) && \text{Def. } [\cdot] \\
&\leftrightarrow \neg(\varphi \wedge \neg\langle\varphi\rangle(\psi \wedge \chi)) && \texttt{!NEG} \\
&\leftrightarrow \neg(\varphi \wedge \neg(\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)) && \texttt{!CON} \\
&\leftrightarrow \neg((\varphi \wedge \neg\langle\varphi\rangle\psi) \vee (\varphi \wedge \neg\langle\varphi\rangle\chi)) && \texttt{TAUT} \\
&\leftrightarrow \neg(\langle\varphi\rangle\neg\psi \vee \langle\varphi\rangle\neg\chi) && \texttt{!NEG} \\
&\leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi) && \texttt{TAUT}, \text{Def. } [\cdot]
\end{aligned}
$$

$$
\begin{aligned}
[\varphi]K_i\psi &\leftrightarrow \neg\langle\varphi\rangle\neg K_i\psi && \text{Def. } [\cdot] \\
&\leftrightarrow \neg(\varphi \wedge \neg\langle\varphi\rangle K_i\psi) && \texttt{!NEG} \\
&\leftrightarrow \neg(\varphi \wedge \neg(\varphi \wedge K_i[\varphi]\psi)) && \texttt{!K} \\
&\leftrightarrow (\varphi \to \varphi \wedge K_i[\varphi]\psi) && \texttt{TAUT} \\
&\leftrightarrow (\varphi \to K_i[\varphi]\psi) && \texttt{TAUT}
\end{aligned}
$$

$$
\begin{aligned}
[\varphi][\psi]\chi &\leftrightarrow \neg\langle\varphi\rangle\neg\neg\langle\psi\rangle\neg\chi && \text{Def. } [\cdot] \\
&\leftrightarrow \neg\langle\varphi\rangle\langle\psi\rangle\neg\chi && \texttt{RM}\langle\cdot\rangle \\
&\leftrightarrow \neg\langle\langle\varphi\rangle\psi\rangle\neg\chi && \texttt{!!} \\
&\leftrightarrow \neg\langle\varphi \wedge [\varphi]\psi\rangle\neg\chi && \\
&\leftrightarrow [\varphi \wedge [\varphi]\psi]\chi && \text{Def. } [\cdot]
\end{aligned}
$$

where the penultimate '$\leftrightarrow$' follows from $\vdash \langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge [\varphi]\psi)$ and Lemma 5.5. The proof for $\vdash \langle\varphi\rangle\psi \leftrightarrow (\varphi \wedge [\varphi]\psi)$ is as follows:

$$
\begin{aligned}
\langle\varphi\rangle\psi &\leftrightarrow \langle\varphi\rangle\neg\neg\psi && \texttt{TAUT}, \texttt{RM}\langle\cdot\rangle \\
&\leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\neg\psi) && \texttt{!NEG} \\
&\leftrightarrow (\varphi \wedge [\varphi]\psi) && \text{Def. } [\cdot]
\end{aligned}
$$

⊣

Proof of Proposition 5.9. By induction on the complexity of **LK**-formulas φ (recall the notion of complexity of a formula is given in Definition 2.5).

Case p.

$$\begin{array}{lll} \text{(i)} & \langle\top\rangle p \leftrightarrow (\top \wedge p) & \texttt{!ATOM} \\ \text{(ii)} & (\top \wedge p) \leftrightarrow p & \texttt{TAUT} \\ \text{(iii)} & \langle\top\rangle p \leftrightarrow p & \text{(i), (ii)} \end{array}$$

Case $\neg\varphi$. Recall that φ is less complex than $\neg\varphi$, that is, $\varphi <^S_\Diamond \neg\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\top\rangle\varphi \leftrightarrow \varphi$

$$\begin{array}{lll} \text{(i)} & \langle\top\rangle\neg\varphi \leftrightarrow (\top \wedge \neg\langle\top\rangle\varphi) & \texttt{!NEG} \\ \text{(ii)} & (\top \wedge \neg\langle\top\rangle\varphi) \leftrightarrow \neg\langle\top\rangle\varphi & \texttt{TAUT} \\ \text{(iii)} & \langle\top\rangle\neg\varphi \leftrightarrow \neg\langle\top\rangle\varphi & \text{(i), (ii)} \\ \text{(iv)} & \langle\top\rangle\varphi \leftrightarrow \varphi & \text{IH} \\ \text{(v)} & \langle\top\rangle\neg\varphi \leftrightarrow \neg\varphi & \text{(iii), (iv)} \end{array}$$

Case $\varphi \wedge \psi$. Recall that both φ and ψ are less complex than $\varphi \wedge \psi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\top\rangle\varphi \leftrightarrow \varphi$ and $\vdash \langle\top\rangle\psi \leftrightarrow \psi$.

$$\begin{array}{lll} \text{(i)} & \langle\top\rangle(\varphi \wedge \psi) \leftrightarrow (\langle\top\rangle\varphi \wedge \langle\top\rangle\psi) & \texttt{!CON} \\ \text{(ii)} & \langle\top\rangle\varphi \leftrightarrow \varphi & \text{IH} \\ \text{(iii)} & \langle\top\rangle\psi \leftrightarrow \psi & \text{IH} \\ \text{(iv)} & \langle\top\rangle(\varphi \wedge \psi) \leftrightarrow (\varphi \wedge \psi) & \text{(i)–(iii)} \end{array}$$

Case $K_i\varphi$. Recall that φ is less complex than $K_i\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\top\rangle\varphi \leftrightarrow \varphi$.

$$\begin{array}{lll} \text{(i)} & \langle\top\rangle K_i\varphi \leftrightarrow (\top \wedge K_i[\top]\varphi) & \texttt{!K} \\ \text{(ii)} & \top \wedge K_i[\top]\varphi \leftrightarrow K_i\neg\langle\top\rangle\neg\varphi & \texttt{TAUT}, \text{Def. } [\cdot] \\ \text{(iii)} & K_i\neg\langle\top\rangle\neg\varphi \leftrightarrow K_i\neg(\top \wedge \neg\langle\top\rangle\varphi) & \texttt{!NEG}, \texttt{RE} \\ \text{(iv)} & K_i\neg(\top \wedge \neg\langle\top\rangle\varphi) \leftrightarrow K_i\langle\top\rangle\varphi & \texttt{TAUT}, \texttt{RE} \\ \text{(v)} & \langle\top\rangle\varphi \leftrightarrow \varphi & \text{IH} \\ \text{(vi)} & K_i\langle\top\rangle\varphi \leftrightarrow K_i\varphi & \text{(v)}, \texttt{RE} \\ \text{(vii)} & \langle\top\rangle K_i\varphi \leftrightarrow K_i\varphi & \text{(i)–(iv), (vi)} \end{array}$$

Case $\langle\psi\rangle\varphi$. Recall that ψ is less complex than $\langle\psi\rangle\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\top\rangle\psi \leftrightarrow \psi$.

$$\begin{array}{lll} \text{(i)} & \langle\top\rangle\langle\psi\rangle\varphi \leftrightarrow \langle\langle\top\rangle\psi\rangle\varphi & \texttt{!!} \\ \text{(ii)} & \langle\top\rangle\psi \leftrightarrow \psi & \text{IH} \\ \text{(iii)} & \langle\top\rangle\langle\psi\rangle\varphi \leftrightarrow \langle\psi\rangle\varphi & \text{(i), (ii)}, \texttt{RE} \end{array}$$

Case $\Diamond_i^K \varphi$. Let ψ be any **EL**-formula. Recall that $\langle\psi\rangle K_i\varphi$ is less complex than $\Diamond_i^K\varphi$ (Proposition 2.6). By induction hypothesis (IH), $\vdash \langle\top\rangle\langle\psi\rangle K_i\varphi \leftrightarrow \langle\psi\rangle K_i\varphi$.

(i)	$\Box_i^K\neg\varphi \to [\psi]\hat{K}_i\neg\varphi$	AKK
(ii)	$[\psi]\hat{K}_i\neg\varphi \leftrightarrow [\top][\psi]\hat{K}_i\neg\varphi$	IH
(iii)	$\Box_i^K\neg\varphi \to [\top][\psi]\hat{K}_i\neg\varphi$	(i), (ii)
(iv)	$\Box_i^K\neg\varphi \to [\top]\Box_i^K\neg\varphi$	(iii), RKb
(v)	$[\top]\Box_i^K\neg\varphi \to [\top][\psi]\hat{K}_i\neg\varphi$	(i), RM$[\cdot]$
(vi)	$[\top]\Box_i^K\neg\varphi \to [\psi]\hat{K}_i\neg\varphi$	(ii), (v)
(vii)	$[\top]\Box_i^K\neg\varphi \to \Box_i^K\neg\varphi$	(vi), RKb
(viii)	$[\top]\Box_i^K\neg\varphi \leftrightarrow \Box_i^K\neg\varphi$	(iv), (vii)
(ix)	$\langle\top\rangle\Diamond_i^K\varphi \leftrightarrow \Diamond_i^K\varphi$	(viii), RE ⊣

PROOF OF PROPOSITION 5.11. Assume that $\vdash \varphi \to \psi$, we have the following derivation in $\mathbb{LK}$, where χ is any **EL**-formula:

(i)	$\neg\psi \to \neg\varphi$	assumption, TAUT
(ii)	$K_i\neg\psi \to K_i\neg\varphi$	(i), NECK, K, MP
(iii)	$\langle\chi\rangle K_i\neg\psi \to \langle\chi\rangle K_i\neg\varphi$	(ii), RM$\langle\cdot\rangle$
(iv)	$\langle\chi\rangle K_i\neg\varphi \to \Diamond_i^K\neg\varphi$	AKK
(v)	$\langle\chi\rangle K_i\neg\psi \to \Diamond_i^K\neg\varphi$	(iii), (iv)
(vi)	$\neg\Diamond_i^K\neg\varphi \to \neg\langle\chi\rangle K_i\neg\psi$	(v)
(vii)	$\neg\langle\chi\rangle K_i\neg\psi \leftrightarrow \neg\langle\chi\rangle\neg\neg K_i\neg\psi$	TAUT, RM$\langle\cdot\rangle$
(viii)	$\neg\langle\chi\rangle K_i\neg\psi \leftrightarrow [\chi]\hat{K}_i\psi$	(vii), Def. $[\cdot]$, Def. $\hat{K}_i$
(ix)	$\Box_i^K\varphi \to [\chi]\hat{K}_i\psi$	(vi), (viii), Def. $\Box_i^K$
(x)	$\Box_i^K\varphi \to \Box_i^K\psi$	(ix), RKb ⊣

PROOF OF PROPOSITION 5.12. We have the following derivation in $\mathbb{LK}$, where χ is any **EL**-formula:

(i)	$\Box_i^K\hat{K}_i\neg\varphi \to [\chi]\hat{K}_i\hat{K}_i\neg\varphi$	AKK
(ii)	$\hat{K}_i\hat{K}_i\neg\varphi \to \hat{K}_i\neg\varphi$	4
(iii)	$[\chi]\hat{K}_i\hat{K}_i\neg\varphi \to [\chi]\hat{K}_i\neg\varphi$	(ii), Proposition 5.4
(iv)	$\Box_i^K\hat{K}_i\neg\varphi \to [\chi]\hat{K}_i\neg\varphi$	(i), (iii)
(v)	$\Box_i^K\hat{K}_i\neg\varphi \to \Box_i^K\neg\varphi$	(iv), RKb
(vi)	$\Box_i^K\neg\varphi \to [\top]\hat{K}_i\neg\varphi$	AKK
(vii)	$[\top]\hat{K}_i\neg\varphi \leftrightarrow \hat{K}_i\neg\varphi$	Corollary 5.10
(viii)	$\Box_i^K\neg\varphi \to \hat{K}_i\neg\varphi$	(vi), (vii)
(ix)	$\Box_i^K\Box_i^K\neg\varphi \to \Box_i^K\hat{K}_i\neg\varphi$	(viii), Proposition 5.11
(x)	$\Box_i^K\Box_i^K\neg\varphi \to \Box_i^K\neg\varphi$	(ix), (v)
(xi)	$\Diamond_i^K\varphi \to \Diamond_i^K\Diamond_i^K\varphi$	(x), RE, Dual ⊣

PROOF OF PROPOSITION 5.13. We have the following derivation in $\mathbb{LK}$, where ψ, χ are any **EL**-formulas (thus $\langle\psi\rangle\chi \in$ **PAL**).:

(i)	$\Box_i^K \neg\varphi \to \hat{K}_i \Box_i^K \neg\varphi$	T
(ii)	$[\psi]\Box_i^K \neg\varphi \to [\psi]\hat{K}_i \Box_i^K \neg\varphi$	(i), Proposition 5.4
(iii)	$\Box_i^K \neg\varphi \to [\psi \wedge [\psi]\chi]\hat{K}_i \neg\varphi$	AKK*
(iv)	$[\psi \wedge [\psi]\chi]\hat{K}_i \neg\varphi \leftrightarrow [\psi][\chi]\hat{K}_i \neg\varphi$	Proposition 5.7
(v)	$\Box_i^K \neg\varphi \to [\psi][\chi]\hat{K}_i \neg\varphi$	(iii), (iv)
(vi)	$\Box_i^K \neg\varphi \to [\psi]\Box_i^K \neg\varphi$	(v), RKb
(vii)	$\Box_i^K \neg\varphi \to [\psi]\hat{K}_i \Box_i^K \neg\varphi$	(vi), (ii)
(viii)	$\Box_i^K \neg\varphi \to \Box_i^K \Box_i^K \neg\varphi$	(vii), RKb
(ix)	$\Diamond_i^K \Diamond_i^K \varphi \to \Diamond_i^K \varphi$	(viii), RE, Dual ⊣

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