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S5-Style Non-Standard Modalities in a Hypersequent Framework

Abstract. The aim of the paper is to present some non-standard modalities (such as non-contingency, contingency, essence and accident) based on **S5**-models in a framework of cut-free hypersequent calculi. We also study negated modalities, i.e. negated necessity and negated possibility, which produce paraconsistent and paracomplete negations respectively. As a basis for our calculi, we use Restall's cut-free hypersequent calculus for **S5**. We modify its rules for the above-mentioned modalities and prove strong soundness and completeness theorems by a Hintikka-style argument. As a consequence, we obtain a cut admissibility theorem. Finally, we present a constructive syntactic proof of cut elimination theorem.

Keywords: hypersequent calculus; cut elimination; proof theory; modal logic; contingency logic; essence logic; accident logic

1. Introduction

Modal logic is usually formulated in a language containing a necessity operator (denoted as $\Box$) and/or possibility operator (denoted as $\Diamond$). However, in the literature one may find other modalities (and not just temporal, deontic, epistemic, etc. ones). One of them is the non-contingency operator (following Zolin [57], we denote it as $\triangleright$) which can be defined as follows: $\triangleright A = \Box A \vee \Box\neg A$. Thus, a proposition is non-contingent iff it is necessary or its negation is necessary. A contingency operator ($\blacktriangleright$ in our notation) is defined as follows: $\blacktriangleright A = \neg\triangleright A = \Diamond A \wedge \Diamond\neg A$, i.e. a proposition is contingent iff it is possibly true and also possibly false.

Contingency may be also understood as 'ignorance' in epistemic logic [53] or a 'knowing whether' operator [15] which may be used to formalise

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some problems in AI [44] or microeconomics [21]. Among other interpretations of contingency are doxastic ('no belief' or 'undecided' [36]), deontic ('(moral) indifference' [55]), spatial ('topological border' [51]), and provability ('undecidable (in Peano Arithmetic)' [56]).

Although $\triangleright$ and $\blacktriangleright$ are expressed in the standard modal language, starting with Montgomery and Routley [40, 41, 42] logicians have studied non-contingent and contingent versions of well-known modal logics, i.e. that ones which contain $\triangleright$ and $\blacktriangleright$ as primitive operators instead of $\Box$ and $\Diamond$. Their languages in many cases are less expressive than the standard one that makes the problem of their axiomatization non-trivial. Montgomery and Routley themselves formalised via Hilbert-style calculi contingent and non-contingent logics based on **T**, **S4**, and **S5**. The basic logics, contingent and non-contingent versions of **K**, have various axiomatizations developed by Humberstone [22], Kuhn [30], Zolin [57, 58], van der Hoek and Lomuscio [53]. Transitive and Euclidian contingent and non-contingent logics were formalised by Kuhn [30], Zolin [58], Steinsvold [51]. Fan [12, 13] paid special attention to symmetric logics. Probably, the most impressive results were obtained in the case of reflexive non-contingent logics: Zolin [56] formulated a general method of constructing Hilbert-style calculi for them, using the fact that $\Box A = A \wedge \triangleright A$.[1] However, the non-reflexive case is still non-trivial.[2] Surprisingly, from a proof-theoretic perspective even the relatively simple reflexive case is problematic. Zolin [56, 57] developed sequent calculi for many non-contingent logics, including the **S5**-based one, but none of them is cut-free. This fact has inspired us to try to present cut-free calculi for these logics, but using a more general framework of hypersequents instead of ordinary sequents. This paper is supposed to be a starting point in solving this task and we choose **S5**-style modal logics, since **S5** is known for having plenty of cut-free hypersequent calculi. We choose Restall's [48] hypersequent calculus for **S5**, since it is one of the simplest calculi for this logic.[3]

[1] It is clear that his method may be adapted for contingent logics, since in reflexive logics it holds that $\Box A = A \wedge \neg\blacktriangleright A$ as well as $\Diamond A = \triangleright A \to A$ and $\Diamond A = \neg\blacktriangleright A \to A$.

[2] There are also studies of neighbourhood frames in the contingent language [10]. Among other variants of contingent logic let us mention its combination with public announcement logic [7].

[3] Pioneers in the development of hypersequent calculi as such and for **S5** in particular are Mints [39], Pottinger [47], and Avron [2]. Later on various hypersequent calculi for **S5** were presented by Poggiolesi [45], Lahav [34], Kurokawa [31], Restall

Aside from contingency and non-contingency, there are concepts of essence and accident. A sentence is essentially true iff it is either false or necessarily true, i.e. if it is true, then it is necessarily true. A sentence is accidentally true iff it is true, but not necessarily true (i.e. its falsity is possible). Thus, the operators of essential and accidental truth (we denote them as $\circ$ and $\bullet$, following Marcos [37]) are defined as follows: $\circ A = \neg A \vee \Box A = A \to \Box A$ and $\bullet A = \neg \circ A = A \wedge \neg \Box A = A \wedge \Diamond \neg A$. We should emphasize two points here. First, we follow Marcos' approach to essence and accidence which is the de dicto one, while Fine [16, 17, 18] developed a de re approach to essence. Second, as Gilbert and Venturi [19, p. 888] note, it is important to not conflate 'the notion of being accidentally/essentially true and the notion of being accidental/essential in the sense of being *mutable* or *immutable*'. They argue that Marcos deals with 'accidentally/essentially true', although he himself does not emphasize it. They introduce their own accident and essence operators: $\mathsf{A}A = \bullet A \vee \bullet \neg A$ and $\mathsf{E}A = \neg \mathsf{A}A = \circ A \vee \circ \neg A$.

> As for 'accidentally true' and 'essentially true', these can now be given straightforward formalizations as $A \wedge \mathsf{A}A$ and $A \wedge \mathsf{E}A$, respectively.' <...> 'As a final remark, one might wonder what the logic of these new operators is. But the logic of A and E is the logic of $\circ$ and $\bullet$, because all four of these operators are interdefinable (one can define $\bullet A$ as $A \wedge \mathsf{A}A$, as we mentioned above). Therefore, our ultimate claim is that the formal framework for exploring notions of essence and accident proposed by Marcos in [37] is a good one, but more precision is required to separate, and formalize, all of the desirable concepts within this sphere. [19, p. 890, notation adjusted]

We consider the modalities $\circ$ and $\bullet$, since the logic of A and E is reducible to them. Additionally, we consider 'accidentally/essentially *false*' modalities, denoting as $\widetilde{\circ}$ and $\widetilde{\bullet}$, respectively, and defining them as $\widetilde{\circ} A = \circ \neg A = \neg A \to \Box \neg A$ and $\widetilde{\bullet} A = \bullet \neg A = \neg A \wedge \Diamond A$. So a proposition is essentially false iff its falsity implies the necessity of its falsity and a proposition is accidently false iff it is false, but its truth is possible.

In accidentally/essentially true logics, we have the following equalities in the case of serial frames: $\Box A = A \wedge \neg \bullet A$, $\Diamond A = \neg \bullet A \to A$, $\Box A = A \wedge \circ A$, and $\Diamond A = \circ A \to A$ [see 8]. It simplifies the task of providing an axiomatization of these logics, but the non-serial case is non-trivial as

[48], Bednarska and Indrzejczak [3], and Indrzejczak [27] himself. See [3, 28] for a survey and comparison of these calculi.

well as well-behaved (hyper)sequent calculi have not been developed for these logics (to the best of our knowledge). As for Hilbert-style calculi for them [see 8, 11, 50]. Papers [11, 14] suggest a combination of accident and contingent logics. At that labeled (i.e. using explicit semantic elements) analytic tableaux were developed by Venturi and Yago [54] for essence and contingent logics. Notice that our calculi do not have any explicit semantic elements (see [46] for the advantages of such calculi). Let us also mention that the very notion of accidental truth was used by Small [49] in the context of Gödel's ontological proof.

The idea to formulate a paraconsistent logic over **S5** is due to Jaśkowski [29]. Taking his inspiration from Jaśkowski's work, Béziau [4] presented a paraconsistent logic **Z** which is the result of the replacement of Boolean negation in classical logic with a paraconsistent one defined as negated necessity. Thus, we have $\sim A = \neg\Box A$, where $\sim$ is paraconsistent negation, as well as $\Box A = \neg\sim A$ and $\Diamond A = \sim\neg A$. Marcos [37] generalized Béziau's approach: he considered paraconsistent logics based on modal logics which are weaker than as **S5**, and investigated a paracomplete negation (we denote it as $\dot\sim$) defined as $\dot\sim A = \neg\Diamond A$ (hence, $\Box A = \dot\sim\neg A$ and $\Diamond A = \neg\dot\sim A$). Avron and Lahav [1] developed a cut-free hypersequent calculus for **Z** which is similar to Restall's one for **S5**. We mention their calculus in the next sections (and additionally present a constructive cut elimination proof for it) to make our study more complete. Moreover, we introduce a related calculus for a paracomplete version of **Z** which we call $\dot{\mathbf{Z}}$ and understood as the result of the replacement of Boolean negation in classical logic with the paracomplete negation $\dot\sim$. For a systematic study of proof systems for the logics with negative modalities see the paper [33] by Lahav, Marcos and Zohar. However, notice that not all the calculi presented there are cut-free. In particular, the case of **S5**-based ones is problematic.

The structure of the paper is as follows. In Section 2, we describe the semantics of **S5** and its modifications with non-standard modalities. Section 3 is devoted to the presentation of hypersequent calculi for the logics in question. Section 4 contains proofs of strong soundness and completeness theorems. In section 5, we present a constructive proof of the cut elimination theorem. Section 6 contains concluding remarks.

2. Semantics

Let $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathscr{P}$ be a set $\{p, q, r, s, p_1, \ldots\}$ of propositional variables, $\neg, \wedge, \vee, \rightarrow$ be classical truth-value connectives, $\mathscr{A}$ be the alphabet $\langle \mathscr{P}, \clubsuit, \neg, \wedge, \vee, \rightarrow, (,) \rangle$. We fix a modal language $\mathscr{L}_\clubsuit$ with the alphabet $\mathscr{A}$ which forms the set $\mathscr{F}_\clubsuit$ of all $\mathscr{L}_\clubsuit$-formulas in a standard inductive way. In some cases we use bimodal languages, e.g. $\mathscr{L}_{\Box\Diamond}$, defined in an analogous way.

The modal logic **S5** is usually built in one of the following languages: $\mathscr{L}_\Box$, $\mathscr{L}_\Diamond$, and $\mathscr{L}_{\Box\Diamond}$. Let us consider the latter variant. A pair $\langle W, \vartheta \rangle$ is said to be an **S5**-model iff $W \neq \emptyset$ and ϑ is a mapping from $W \times \mathscr{F}_{\Box\Diamond}$ to $\{1, 0\}$ such that it preserves classical conditions for truth-value connectives and for any $A \in \mathscr{F}_{\Box\Diamond}$ and $x \in W$ we have:[4]

- $\vartheta(\Box A, x) = 1$ iff $\forall_{y \in W} \vartheta(A, y) = 1$,
- $\vartheta(\Diamond A, x) = 1$ iff $\exists_{y \in W} \vartheta(A, y) = 1$.

A formula A is true in a world $w \in W$ iff $\vartheta(A, w) = 1$. A formula A follows from the set of formulas Γ ($\Gamma \models_{\mathbf{S5}} A$) iff for every **S5**-model $\langle W, \vartheta \rangle$ and every $w \in W$, if any $B \in \Gamma$ is true in w, then A is true in w. A formula is **S5**-valid iff it follows from the empty set of formulas.

We write $\mathbf{S5}^{\rhd}$ for the non-contingency version of **S5**, i.e. the logic over **S5**-models in the language $\mathscr{L}_\rhd$. Thus, we have the case for $\rhd$ instead of the cases for $\Box$ and $\Diamond$:

- $\vartheta(\rhd A, x) = 1$ iff $\forall_{y \in W} \vartheta(A, y) = 1$ or $\forall_{y \in W} \vartheta(A, y) = 0$.

The contingency version of **S5** is built in the language $\mathscr{L}_\blacktriangleright$ over **S5**-frames and is denoted as $\mathbf{S5}^{\blacktriangleright}$. A semantic condition for contingency operator is presented below:

- $\vartheta(\blacktriangleright A, x) = 1$ iff $\exists_{y \in W} \vartheta(A, y) = 1$ and $\exists_{y \in W} \vartheta(A, y) = 0$.

The essentially and accidentally true versions of **S5**, $\mathbf{S5}^{\circ}$ and $\mathbf{S5}^{\bullet}$, respectively, are built over **S5**-frames in languages $\mathscr{L}_\circ$ and $\mathscr{L}_\bullet$. The appropriate semantic conditions are as follows:

- $\vartheta(\circ A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\forall_{y \in W} \vartheta(A, y) = 1$.
- $\vartheta(\bullet A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\exists_{y \in W} \vartheta(A, y) = 0$.

The essentially and accidentally false versions of **S5**, $\mathbf{S5}^{\widetilde{\circ}}$ and $\mathbf{S5}^{\widetilde{\bullet}}$, respectively, are built over **S5**-frames in languages $\mathscr{L}_{\widetilde{\circ}}$ and $\mathscr{L}_{\widetilde{\bullet}}$. We have the following semantic conditions:

[4] One can consider models of the form $\langle W, R, \vartheta \rangle$ as well, where $R = W \times W$. Then one needs to postulate some more complicated conditions for ϑ, e.g. $\vartheta(\Box A, x) = 1$ iff $\forall_{y \in W}(R(x, y)$ implies $\vartheta(A, y) = 1)$.

- $\vartheta(\tilde{\circ}A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\forall_{y \in W} \vartheta(A, y) = 0$,
- $\vartheta(\tilde{\bullet}A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\exists_{y \in W} \vartheta(A, y) = 1$.

The logic $\mathbf{S5}^{\sim}$ is built over **S5**-frames in the language $\mathscr{L}_{\sim}$. Its $\neg$-free fragment is Béziau's [4] paraconsistent logic **Z**. The logic $\mathbf{S5}^{\dot\sim}$ is built over **S5**-frames in the language $\mathscr{L}_{\dot\sim}$. We introduce a paracomplete companion of **Z** as the $\neg$-free fragment of $\mathbf{S5}^{\dot\sim}$ and call it $\dot{\mathbf{Z}}$. Semantic conditions for paraconsistent and paracomplete negations are as follows:

- $\vartheta({\sim}A, x) = 1$ iff $\exists_{y \in W} \vartheta(A, y) = 0$,
- $\vartheta(\dot\sim A, x) = 1$ iff $\forall_{y \in W} \vartheta(A, y) = 0$.

The notion of the entailment relation in the logics with non-standard modalities is defined in the same manner as in **S5**.

3. Hypersequent calculi

An ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas (of one of the languages considered in the paper), is a *sequent*. A finite multiset of sequents written as $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is a *hypersequent*. Let $\langle W, \vartheta \rangle$ be an **S5**-model. A sequent $\Gamma \Rightarrow \Delta$ is true in a world $w \in W$ iff $\vartheta(A, w) = 0$ (for some $A \in \Gamma$) or $\vartheta(B, w) = 1$ (for some $B \in \Delta$). A sequent is valid in $\langle W, \vartheta \rangle$ iff it is true in any $w \in W$. A sequent S follows from the set of sequents $\mathscr{S}$ iff for every **S5**-model $\langle W, \vartheta \rangle$, if any $S' \in \mathscr{S}$ is valid $\langle W, \vartheta \rangle$, then S is valid in it as well. A sequent is **S5**-valid iff it is valid in any **S5**-model. A hypersequent H is valid in $\langle W, \vartheta \rangle$ (or $\langle W, \vartheta \rangle$ is a model of H) iff at least one of the components of H is valid in $\langle W, \vartheta \rangle$. A hypersequent H follows from the set of hypersequents $\mathscr{H}$ ($\mathscr{H} \models_{\mathbf{S5}} H$) iff every model of $\mathscr{H}$ is a model of H as well. These notions are defined for the logics with non-standard modalities in a similar way.

Consider Restall's [48] hypersequent calculus $\mathbb{H}\mathbf{S5}$ for **S5**. It has the following axiom: (Ax) $A \Rightarrow A$. Its structural rules are presented below:

$$(\mathrm{EW}' \Rightarrow)\ \frac{H}{A \Rightarrow\ \mid H} \qquad (\Rightarrow \mathrm{EW}')\ \frac{H}{\Rightarrow A \mid H}$$

$$(\mathrm{IC}\Rightarrow)\ \frac{A, A, \Gamma \Rightarrow \Delta \mid H}{A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\mathrm{IC})\ \frac{\Gamma \Rightarrow \Delta, A, A \mid H}{\Gamma \Rightarrow \Delta, A \mid H}$$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad A, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G}$$

$$\text{(Merge)}\ \frac{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H}$$

In contrast to Restall, we will use a more general version of external weakening which allows to add not only a sequent of the form $A \Rightarrow$ or $\Rightarrow A$, but any hypersequent (including empty). The latter issue is important for a constructive cut elimination proof.

$$\text{(EW}\Rightarrow\text{)}\ \frac{G}{G \mid H} \qquad \text{(}\Rightarrow\text{EW)}\ \frac{H}{G \mid H}$$

One can add internal weakening and external contraction rules:

$$\text{(IW}\Rightarrow\text{)}\ \frac{\Gamma \Rightarrow \Delta \mid H}{A, \Gamma \Rightarrow \Delta \mid H} \qquad \text{(}\Rightarrow\text{IW)}\ \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, A \mid H}$$

$$\text{(EC)}\ \frac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid H}$$

However, it is not necessary to postulate them as primitive rules:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta \mid H}{\Rightarrow A \mid \Gamma \Rightarrow \Delta \mid H}\ \text{(EW}\Rightarrow\text{)}}{A, \Gamma \Rightarrow \Delta \mid H}\ \text{(Merge)} \qquad \cfrac{\cfrac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow A \mid H}\ \text{(}\Rightarrow\text{EW)}}{\Gamma \Rightarrow \Delta, A \mid H}\ \text{(Merge)}$$

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma, \Gamma \Rightarrow \Delta, \Delta \mid H}\ \text{(Merge)}}{\Gamma \Rightarrow \Delta \mid H}\ \text{(IC}\Rightarrow\text{), (}\Rightarrow\text{IC)}$$

The rules for truth-value connectives are as follows:

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\neg A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg A \mid H}$$

$$(\wedge \Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta \mid H}{A \wedge B, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad \Gamma \Rightarrow \Delta, B \mid G}{\Gamma \Rightarrow \Delta, A \wedge B \mid H \mid G}$$

$$(\vee \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad B, \Gamma \Rightarrow \Delta \mid G}{A \vee B, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, A, B \mid H}{\Gamma \Rightarrow \Delta, A \vee B \mid H}$$

$$(\rightarrow \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad B, \Theta \Rightarrow \Lambda \mid G}{A \rightarrow B, \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow \rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta, B \mid H}{\Gamma \Rightarrow \Delta, A \rightarrow B \mid H}$$

The rules for necessity and possibility operators are as follows:[5]

[5] Restall's original formulation of his calculus [48] does not have the rules for $\vee$, $\rightarrow$, and $\Diamond$. They were added to it in [20].

$$(\Box\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H}{\Box A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\Box)\ \frac{\Rightarrow A\mid H}{\Rightarrow\Box A\mid H}$$

$$(\Diamond\Rightarrow)\ \frac{A\Rightarrow\ \mid H}{\Diamond A\Rightarrow\ \mid H}\qquad(\Rightarrow\Diamond)\ \frac{\Gamma\Rightarrow\Delta,A\mid H}{\Gamma\Rightarrow\Delta\mid\ \Rightarrow\Diamond A\mid H}$$

Let us formulate the rules for the non-standard modalities (all these rules are new except $(\sim\Rightarrow)$ and $(\Rightarrow\sim)$ which were introduced in [1][6]).

$$(\triangleright\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\triangleright A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\triangleright)\ \frac{\Rightarrow A\mid A\Rightarrow\ \mid H}{\Rightarrow\ \triangleright A\mid H}$$

$$(\blacktriangleright\Rightarrow)\ \frac{\Rightarrow A\mid A\Rightarrow\ \mid H}{\blacktriangleright A\Rightarrow\ \mid H}\qquad(\Rightarrow\blacktriangleright)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Rightarrow\blacktriangleright A\mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\circ\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\circ A,\Theta\Rightarrow\Lambda\mid\Gamma\Rightarrow\Delta\mid H\mid G}\qquad(\Rightarrow\circ)\ \frac{\Rightarrow A\mid A,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta,\circ A\mid H}$$

$$(\bullet\Rightarrow)\ \frac{\Rightarrow A\mid A,\Gamma\Rightarrow\Delta\mid H}{\bullet A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\bullet)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Theta\Rightarrow\Lambda,\bullet A\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

$$(\tilde{\circ}\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\tilde{\circ}A,\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\tilde{\circ})\ \frac{\Gamma\Rightarrow\Delta,A\mid A\Rightarrow\ \mid H}{\Gamma\Rightarrow\Delta,\tilde{\circ}A\mid H}$$

$$(\tilde{\bullet}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,A\mid A\Rightarrow\ \mid H}{\tilde{\bullet}A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\tilde{\bullet})\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Gamma\Rightarrow\Delta,\tilde{\bullet}A\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\sim\Rightarrow)\ \frac{\Rightarrow A\mid H}{\sim A\Rightarrow\ \mid H}\qquad(\Rightarrow\sim)\ \frac{A,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta\mid\ \Rightarrow\sim A\mid H}$$

$$(\dot{\sim}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,A\mid H}{\dot{\sim}A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\dot{\sim})\ \frac{A\Rightarrow\ \mid H}{\Rightarrow\dot{\sim}A\mid H}$$

Let $\clubsuit\in\{\triangleright,\blacktriangleright,\circ,\bullet,\tilde{\circ},\tilde{\bullet},\sim,\dot{\sim}\}$. A hypersequent calculus $\mathbb{H}\mathbf{S5}^{\clubsuit}$ for the logic $\mathbf{S5}^{\clubsuit}$ is obtained from Restall's one for **S5** by the replacement of the rules for $\Box$ and $\Diamond$ with the ones for $\clubsuit$. Hypersequent calculi $\mathbb{H}\mathbf{Z}$ and $\mathbb{H}\dot{\mathbf{Z}}$, respectively, for logics **Z** and $\dot{\mathbf{Z}}$ are $\neg$-free versions of $\mathbb{H}\mathbf{S5}^{\sim}$ and $\mathbb{H}\mathbf{S5}^{\dot{\sim}}$. The notion of a proof in hypersequent calculi in question is defined in the standard way. We write $\mathbb{H}\boldsymbol{L}\vdash H$ iff there is a proof of a hypersequent H in the hypersequent calculus for a given logic $\boldsymbol{L}$. Similarly, $\mathscr{H}\vdash_{\mathbb{H}\boldsymbol{L}} H$ means that there is a proof of a hypersequent

[6] Avron and Lahav's [1] original version of a hypersequent for **Z** is a bit different from the one which we present here. They understand hypersequents as finite sets of sequents which are understood themselves as pairs of finite sets of formulas. They use internal weakening rules, but do not use (Merge) and (IC).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}\,(\Rightarrow \neg) \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,(\neg \Rightarrow)}{\triangleright p \Rightarrow \mid \Rightarrow \neg p \mid \neg p \Rightarrow}\,(\triangleright \Rightarrow)}{\triangleright p \Rightarrow \mid \Rightarrow \triangleright \neg p}\,(\Rightarrow \triangleright)}{\triangleright p \Rightarrow \triangleright \neg p}\,(\text{Merge})}{\Rightarrow \triangleright p \to \triangleright \neg p}\,(\Rightarrow\to)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,(\neg \Rightarrow) \qquad \dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}\,(\Rightarrow \neg)}{\triangleright \neg p \Rightarrow \mid p \Rightarrow \mid \Rightarrow p}\,(\triangleright \Rightarrow)}{\triangleright \neg p \Rightarrow \mid \Rightarrow \triangleright p}\,(\Rightarrow \triangleright)}{\triangleright \neg p \Rightarrow \triangleright p}\,(\text{Merge})}{\Rightarrow \triangleright \neg p \to \triangleright p}\,(\Rightarrow\to)
\qquad
\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad p \Rightarrow p}{\triangleright p \Rightarrow \mid \Rightarrow p \mid p \Rightarrow}\,(\triangleright \Rightarrow)}{\triangleright p \Rightarrow \mid \Rightarrow \triangleright p}\,(\Rightarrow \triangleright)}{\Rightarrow \triangleright \triangleright p}\,(\Rightarrow \triangleright)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p, p \to q \Rightarrow q}\,(\to\Rightarrow) \qquad p \Rightarrow p}{\triangleright p \Rightarrow \mid p \to q \Rightarrow q \mid p \Rightarrow}\,(\cdot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{q \Rightarrow q}{p, q \Rightarrow q}\,(\text{IW}\Rightarrow)}{p, p, q \Rightarrow q}\,(\text{IW}\Rightarrow)}{p, q \Rightarrow p \to q}\,(\Rightarrow\to) \qquad p \Rightarrow p}{\triangleright p \Rightarrow \mid q \Rightarrow p \to q \mid p \Rightarrow}\,(\cdot)}{\triangleright(p \to q) \Rightarrow \mid \triangleright p \Rightarrow \mid p \Rightarrow \mid q \Rightarrow \mid \Rightarrow q}\,(\cdot)}{\triangleright(p \to q) \Rightarrow \mid \triangleright p \Rightarrow \mid p \Rightarrow \mid \Rightarrow \triangleright q}\,(\Rightarrow \triangleright)}{p, \triangleright(p \to q), \triangleright p \Rightarrow \triangleright q}\,(\text{Merge})3\text{x}}{\Rightarrow p \to \big(\triangleright(p \to q) \to (\triangleright p \to \triangleright q)\big)}\,(\Rightarrow\to)3\text{x}
$$

Figure 1. Examples of proofs in $\mathbb{H}\mathbf{S5}^{\triangleright}$, where $(\cdot)$ stands for $(\triangleright \Rightarrow)$.

H from a finite set of hypersequents $\mathscr{H}$ in $\mathbb{H}\boldsymbol{L}$. If in this proof each cut is on a formula $A \in \Gamma \cup \Delta$ for some component $\Gamma \Rightarrow \Delta$ of some hypersequent in $\mathscr{H}$, then we write $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H$. Four examples of proofs in $\mathbb{H}\mathbf{S5}^{\triangleright}$ are presented in Figure 1 (the formulas are taken from Zolin's [56] Hilbert-style axiomatization of $\mathbf{S5}^{\triangleright}$).

4. Soundness and completeness

Theorem 4.1 (Strong soundness). *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, *if* $\mathscr{H} \vdash_{\mathbb{H}\boldsymbol{L}} H$, *then* $\mathscr{H} \models_{\boldsymbol{L}} H$.

Proof. Consider the rule $(\Rightarrow \circ)$. Suppose that $\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H$ is valid in an arbitrary $\mathbf{S5}^{\circ}$-model $\langle W, \vartheta \rangle$. Then at least one of the components of this hypersequent is valid in $\langle W, \vartheta \rangle$. If $\Rightarrow A$ is valid in $\langle W, \vartheta \rangle$, then $\vartheta(A, w) = 1$ for all $w \in W$. Then $\Rightarrow \circ A$ is valid in $\langle W, \vartheta\rangle$, and hence $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in this model as well. If $A, \Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta\rangle$, then $\vartheta(A, w) = 0$ for some $w \in W$ or $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta\rangle$. Hence, $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in $\langle W, \vartheta\rangle$. Obviously, if a component of H is valid in $\langle W, \vartheta\rangle$, then $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in it as well.

Consider the rule $(\circ \Rightarrow)$. Suppose that $A, \Gamma \Rightarrow \Delta \mid H$ and $\Theta \Rightarrow \Lambda, A \mid G$ are valid in $\mathbf{S5}^{\circ}$-model $\langle W, \vartheta\rangle$. If H or G is valid in it, then $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid as well. Suppose that $A, \Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda, A$ are valid in $\langle W, \vartheta\rangle$. Then $\vartheta(A, x) = 0$ for some $x \in W$ or $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta\rangle$. Also, $\Theta \Rightarrow \Lambda$ is valid in $\langle W, \vartheta\rangle$ or $\vartheta(A, y) = 1$ for some $y \in W$. Assume that $\vartheta(A, x) = 0$ for some $x \in W$. Suppose that $\Theta \Rightarrow \Lambda$ is valid in $\langle W, \vartheta\rangle$. Then $\circ A, \Theta \Rightarrow \Lambda$ is valid in $\langle W, \vartheta\rangle$. Hence, $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid in $\langle W, \vartheta\rangle$ as well. Suppose that $\vartheta(A, y) = 1$ for some $y \in W$. Then $\circ A \Rightarrow$ is valid in $\langle W, \vartheta\rangle$, and hence $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid. Assume that $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta\rangle$. Then $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid as well in the model in question.

The other cases are considered similarly. $\dashv$

Theorem 4.2 (Strong completeness). *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, *if* $\mathscr{H} \models_{\boldsymbol{L}} H$, *then* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H$.

Proof. We adapt Avron and Lahav's [1] completeness proof for $\mathbf{Z}$. Suppose that $\mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H$. We construct a model of $\mathscr{H}$ which is not a model of H. Let $\mathbb{F}$ be the set of subformulas of formulas in $\mathscr{H} \cup \{H\}$. We call a hypersequent G an $\mathbb{F}$-hypersequent iff it satisfies the following conditions:

- if $A \in G$, then $A \in \mathbb{F}$, for each formula A;
- $\mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} G$;
- if $\Gamma \cup \Delta \subseteq \mathbb{F}$, then either $\Gamma \Rightarrow \Delta \in G$ or $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} G \mid \Gamma \Rightarrow \Delta$.

Since $\mathscr{H}$ is finite, $\mathbb{F}$ is finite as well. Let $S_1, \ldots, S_n$ be an enumeration of all the sequents $\Gamma \Rightarrow \Delta$ such that $\Gamma \cup \Delta \subseteq \mathbb{F}$. We put, $H_0 = H$, for each $1 \leqslant i \leqslant n$:

$$H_i = \begin{cases} H_{i-1} \mid S_i & \text{if } \mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H_{i-1} \mid S_i \\ H_{i-1} & \text{otherwise,} \end{cases}$$

and $H^* = H_n$. Then H^* is an $\mathbb{F}$-hypersequent such that $H \subseteq H^*$. A component $\Gamma^* \Rightarrow \Delta^*$ of H^* is said to be *maximal* iff it has no proper extension in H^* (i.e. if $\Gamma^{**} \Rightarrow \Delta^{**} \in H^*$, $\Gamma^* \subseteq \Gamma^{**}$, and $\Delta^* \subseteq \Delta^{**}$, then $\Gamma^* = \Gamma^{**}$ and $\Delta^* = \Delta^{**}$). Let W be the set of all maximal components of H^*. We write Γ_w and Δ_w (where $w \in W$), respectively, for Γ^* and Δ^* iff $w = \Gamma^* \Rightarrow \Delta^*$. Let ϑ be the valuation such that $\vartheta(p, w) = 1$ iff $p \in \Gamma_w$, for each $p \in \mathcal{P}$.

We need to prove that for each $A \in \mathbb{F}$ and each maximal component w of H^* it holds that:

(a) $A \in \Gamma_w$ implies $\vartheta(A, w) = 1$,
(b) $A \in \Delta_w$ implies $\vartheta(A, w) = 0$.

The proof is by induction on the complexity of A. The basic case (i.e. $A \in \mathcal{P}$) follows from the definition of ϑ. The proof for $\wedge$ and $\sim$ one may find in [1]. Other propositional connectives are considered similarly.

Let A be $\rhd B$. Suppose that $A \in \Gamma_w$. Assume that there is $y \in W$ such that $B \notin \Gamma_y$ and there is a $z \in W$ such that $B \notin \Delta_z$. Since y and z are maximal, $B, \Gamma_y \Rightarrow \Delta_y \notin H^*$ and $\Gamma_z \Rightarrow \Delta_z, B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_z \Rightarrow \Delta_z, B$. By the rule $(\rhd \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \rhd B \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$, i.e. $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A \Rightarrow \mid y \mid z$. By the rule (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A \Rightarrow$. Since $A \in \Gamma_w$, by (EW) and (Merge), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Hence, for each $x \in W$, $B \in \Gamma_x$, or for each $x \in W$, $B \in \Delta_x$. By the induction hypothesis for B, for each $x \in W$, $\vartheta(B, x) = 1$ or for each $x \in W$, $\vartheta(B, x) = 0$. Thus, $\vartheta(A, w) = 1$.

Suppose that $A \in \Delta_w$. Assume that $B \Rightarrow\, \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B \Rightarrow$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B \Rightarrow$. By $(\Rightarrow \rhd)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow \rhd B$, i.e. $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow A$. Since $A \in \Delta_w$, by (EC) and (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Hence, $B \Rightarrow\, \in H^*$. Therefore, there is a $y \in W$ such that $B \in \Gamma_y$. By the induction hypothesis for B, there is a $y \in W$ such that $\vartheta(B, y) = 1$. Assume that $\Rightarrow B \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B \Rightarrow$, and by $(\Rightarrow \rhd)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow A$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Hence, $\Rightarrow B \in H^*$. Thus, there is a $z \in W$ such that $B \in \Delta_z$. By the induction hypothesis for B, there is a $z \in W$ such that $\vartheta(B, z) = 1$. Therefore, $\vartheta(A, w) = 0$.

Let A be $\blacktriangleright B$. Suppose that $A \in \Gamma_w$. We show that there is an $x \in W$ such that $B \in \Delta_x$ and there is an $x \in W$ such that $B \in \Gamma_x$. Assume that $\Rightarrow B \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B \Rightarrow$ and by $(\blacktriangleright \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \blacktriangleright B \Rightarrow$, i.e. $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A \Rightarrow$ which gives us due to the fact that $A \in \Gamma_w$

and the rules (EW) and (Merge) that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$, and so that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Thus, $\Rightarrow B \in H^*$. Hence, there is a maximal component x of H^* that extends it, i.e. $B \in \Delta_x$. Therefore, the induction hypothesis for B implies that there is an $x \in W$ such that $\vartheta(B,x) = 0$. Assume that $B \Rightarrow \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B \Rightarrow$, and so that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B \Rightarrow$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Thus, $B \Rightarrow \in H^*$. Hence, there is a maximal component x of H^* such that $B \in \Gamma_x$. Therefore, the induction hypothesis for B implies that there is an $x \in W$ such that $\vartheta(B,x) = 1$. Thus, $\vartheta(A,x) = 1$.

Suppose that $A \in \Delta_w$. We show that for each $x \in W$, $B \in \Gamma_x$, or for each $x \in W$, $B \in \Delta_x$. Assume the converse, i.e. that there is a $y \in W$ such that $B \notin \Gamma_y$ and there is a $z \in W$ such that $B \notin \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_z \Rightarrow \Delta_z, B$. By ($\Rightarrow \blacktriangleright$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow A \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow A \mid y \mid z$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow A$. Since $A \in \Delta_w$, by (EW) and (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$, and so $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. By the induction hypothesis for B, for each $x \in W$, $\vartheta(B,x) = 1$ or for each $x \in W$, $\vartheta(B,x) = 0$. Hence, $\vartheta(A,w) = 0$.

Let A be $\circ B$. Suppose that $A \in \Gamma_w$. Suppose that $\{B\} \cup \Delta_w = \emptyset$ and for some maximal $x \in W$, $B \notin \Gamma_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_x \Rightarrow \Delta_x$. By the rule ($\circ \Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \circ B, \Gamma_w \Rightarrow \Delta_w, \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w \mid x$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Thus, $B \in \Delta_w$ and for each $x \in W$, $B \in \Gamma_x$. It follows by the induction hypothesis for B that $\vartheta(B,w) = 0$ and for each maximal $x \in W$, $\vartheta(B,x) = 1$. Hence, $\vartheta(A,w) = 0$.

Suppose that $A \in \Delta_w$. Assume that $B \notin \Gamma_w$ or $\Rightarrow B \notin H^*$. Suppose that $B \notin \Gamma_w$. Then since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$. Since $B \notin \Gamma_w$ and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule ($\Rightarrow \circ$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \circ B$. Since $A \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$. Using (Merge), we have $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Suppose that $\Rightarrow B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. Using ($\Rightarrow \circ$) and (Merge), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Hence, $B \in \Gamma_w$ and $\Rightarrow B \in H^*$. Then by the induction hypothesis for B, $\vartheta(B,w) = 1$ and for some $x \in W$, $\vartheta(B,x) = 0$. Hence, $\vartheta(A,w) = 1$.

Let A be $\bullet B$. Suppose that $A \in \Gamma_w$. Assume that $B \notin \Gamma_w$ or $\Rightarrow B \notin H^*$. Suppose that $B \notin \Gamma_w$. Then since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$.

Since $B \notin \Gamma_w$ and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By (EW), $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule $(\bullet \Rightarrow)$, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \bullet B, \Gamma_w \Rightarrow \Delta_w$, i.e. $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A, \Gamma_w \Rightarrow \Delta_w$. Since $A \in \Gamma_w$, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$. Using (Merge), we have $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Suppose that $\Rightarrow B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B$. By (EW), $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. Using $(\bullet \Rightarrow)$ and (Merge), we get $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Hence, $B \in \Gamma_w$ and $\Rightarrow B \in H^*$. Then by the induction hypothesis for B, $\vartheta(B, w) = 1$ and for some $x \in W$ $\vartheta(B, x) = 0$. Hence, $\vartheta(A, w) = 1$.

Suppose that $A \in \Delta_w$. Suppose that $\{B\} \cup \Delta_w = \emptyset$ and for some maximal $x \in W$, $B \notin \Gamma_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid B, \Gamma_x \Rightarrow \Delta_x$. By the rule $(\Rightarrow \bullet)$, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \bullet B \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w \mid x$. By (Merge), $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Thus, $B \in \Delta_w$ and for each maximal $x \in W$, $B \in \Gamma_x$. It follows by the induction hypothesis for B that $\vartheta(B, w) = 0$ and for each maximal $x \in W$, $\vartheta(B, x) = 1$. Hence, $\vartheta(A, w) = 0$.

Let A be $\sim B$. This case is considered in [1] by Avron and Lahav.

The other cases are similar to the previous ones.

Now we show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$, but not for H. Let $H' \in H^*$. If its every component is a subsequent of some component of H^*, then due to (EW) and (Merge) H^* is derivable from H' and hence from $\mathscr{H}$ which contradicts to the fact that $\mathscr{H} \nvdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. Hence, there is a component $\Gamma \Rightarrow \Delta$ of H' which is not a subsequent of any component of H^*. Let us prove that $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$. Let $w \in W$. Then either $\Gamma \nsubseteq \Gamma_w$ or $\Delta \nsubseteq \Delta_w$. Suppose that $\Gamma \nsubseteq \Gamma_w$ (the case of $\Delta \nsubseteq \Delta_w$ is similar). Then for some $A \in \Gamma$, $A \notin \Gamma_w$. Since $A \in \mathbb{F}$, w is maximal, and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid A, \Gamma_w \Rightarrow \Delta_w$. Assume that $A \notin \Delta_w$. Then $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, A$. By (Cut), $A \notin \Delta_w$. Then $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w$, i.e. $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^* \mid w$. Hence, $\mathscr{H} \vdash^{\text{cf}}_{\mathbb{H}\boldsymbol{L}} H^*$. A contradiction. Then $A \in \Delta_w$ which implies, by proposition (b) and the maximality of w, that $\vartheta(A, w) = 0$. Since $A \in \Gamma$, $\Gamma \Rightarrow \Delta$ is true in a world w. Since w is an arbitrary world, $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$ which implies that $\mathscr{H}$ is valid in $\langle W, \vartheta \rangle$ as well.

Assume that $\Gamma \Rightarrow \Delta$ is some component of H. Since $H \subseteq H^*$, there is a maximal component w of H^* such that $\Gamma \subseteq \Gamma_w$ and $\Delta \subseteq \Delta_w$. By propositions (a) and (b), we obtain that $A \in \Gamma$ implies $\vartheta(A, w) = 1$ as well as $A \in \Delta$ implies $\vartheta(A, w) = 0$. Thus, $\Gamma \Rightarrow \Delta$ is not true at w. Hence, it is not valid in $\langle W, \vartheta \rangle$. Therefore, $\langle W, \vartheta \rangle$ is not a model for H. $\dashv$

Corollary 4.1. *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, $\mathscr{H} \vdash_{\mathbb{H}\boldsymbol{L}} H$ *iff* $\mathscr{H} \models_{\boldsymbol{L}} H$.

Proof. Follows from Theorems 4.1 and 4.2. ⊣

Corollary 4.2. *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$, *and* $\mathscr{H} \cup \{H\}$ *be a finite set of hypersequents. Then* $\mathscr{H} \vdash_{\mathbb{H}\boldsymbol{L}} H$ *implies* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{H}\boldsymbol{L}} H$.

Proof. Follows from Theorem 4.2. Notice that in the proof of this theorem (Cut) is used only once in order to show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$ and is applied only to formulas which belongs to $\mathscr{H}$. ⊣

Corollary 4.3 (Cut admissibility). *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$, *and* H *be a hypersequent. Then* $\vdash_{\mathbb{H}\boldsymbol{L}} H$ *implies that there is a cut-free proof of* H *in* $\mathbb{H}\boldsymbol{L}$.

Proof. Put $\mathscr{H} = \emptyset$ in the proof of Theorem 4.2. Then the only application of (Cut) in the proof of this Theorem disappears. ⊣

Corollary 4.4 (Subformula property). *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For every hypersequent which is provable in* $\mathbb{H}\boldsymbol{L}$ *there is a proof such that each formula which occurs in it is a subformula of the formulas which occur in the conclusion.*

Proof. Follows from Corollary 4.3 and the fact that in any of the rules of $\mathbb{H}\boldsymbol{L}$ each formula which occurs in the premises is a subformula of the formulas which occur in the conclusion. ⊣

Let us recall that strong soundness and completeness as well as cut admissibility theorems are proven for $\mathbf{Z}$ by Avron and Lahav [1] (however, constructive cut elimination was not proven). As we know from Corollary 4.3, in any of the hypersequent calculi in question if we have a proof of a hypersequent H, then we can be sure that there exists a cut-free proof of the same hypersequent. However, the problem is how to find such a cut-free proof. Constructive cut elimination will give us an answer.

5. Constructive cut elimination

We use the strategy originally introduced by Metcalfe, Olivetti, and Gabbay [38] for fuzzy logics and further developed by Ciabattoni, Metcalfe, and Montagna [6]. It was adapted for modal logics by Kurokawa [31], Indrzejczak [23, 24, 25, 26, 28], Lellmann [35], Kuznets and Lellmann [32]. We use the following version of the cut rule for our syntactic proof.

$$\text{(Multi-Cut)}\ \frac{\Gamma \Rightarrow \Delta, A^i \mid H \qquad A^j, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G},$$

where $i, j > 0$ and A^i (resp. A^j) denotes i (resp. j) occurrences of A. Similarly, Γ^i denotes i occurrences of Γ.

Let us recall that a formula introduced by the application of a logical rule is said to be *principal formula*, formulas used for the proof of the principal formula are said to be *side formulas*, all other elements of the hypersequent are said to be *parametric formulas*. We say that a hypersequent which contains a principal formula is an *active hypersequent*.

The length $\mathfrak{l}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is (the maximal number of applications of inference rules) plus 1 occurring on any branch of $\mathfrak{D}$. The complexity $\mathfrak{c}(A)$ of a formula A is the number of occurrences of its connectives. The cut rank $\mathfrak{r}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is the maximal complexity of cut formulas in $\mathfrak{D}$ plus 1. Thus, a cut-free derivation $\mathfrak{D}$ has $\mathfrak{r}(\mathfrak{D}) = 0$.

We need to prove two lemmas.

LEMMA 5.1 (Right reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that*:

(1) *$\mathfrak{D}_1$ is a derivation of $\Gamma \Rightarrow \Delta, A \mid H$,*
(2) *$\mathfrak{D}_2$ is a derivation of $A^{i_1}, \Theta_1 \Rightarrow \Lambda_1 \mid \ldots \mid A^{i_n}, \Theta_n \Rightarrow \Lambda_n \mid G$,*
(3) *$\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$,*
(4) *A is a principal formula of a logical rule in $\mathfrak{D}_1$.*

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma^{i_1}, \Theta_1 \Rightarrow \Lambda_1, \Delta^{i_1} \mid \ldots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Lambda_n, \Delta^{i_n} \mid H \mid G$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

PROOF. By induction on $\mathfrak{l}(\mathfrak{D}_2)$. Basic case is easy and left for the reader.

Inductive case. We have different cases depending on the last rule applied to $\mathfrak{D}_2$.

Case 1. The last rule is applied on only side sequents G. Left for the reader.

Case 2. The last rule is any non-modal rule that does not have A as the principal formula.

Let us use the following abbreviations, for any $1 \leq l \leq n$ and $1 \leq k \leq n$, where $x, y \in \{l, i_l, j_l\}$ and $\Phi, \Psi, \Upsilon, \Omega \in \{\Gamma, \Delta, \Theta, \Lambda, \Pi, \Sigma\}$:

- $\mathfrak{A}_l^{\Phi\Psi} = \Phi_l \Rightarrow \Psi_l$,
- $\mathfrak{A}_{i_l}^{\Phi\Psi} = \Phi^{i_l} \Rightarrow \Psi^{i_l}$,

- $\mathfrak{A}^{\Phi\Psi}_{l} \times \mathfrak{A}^{\Upsilon\Omega}_{k} = \Phi_l, \Upsilon_k \Rightarrow \Psi_l, \Omega_k$,
- $\mathfrak{A}^{\Phi\Psi}_{i_l} \times \mathfrak{A}^{\Upsilon\Omega}_{k} = \Phi^{i_l}, \Upsilon_k \Rightarrow \Psi^{i_l}, \Omega_k$.

Subcase 2.1. The rule of last inference of $\mathfrak{D}_2$ is (Merge).

$$\dfrac{\Theta \Rightarrow \Lambda, A^j \mid G \qquad \dfrac{A^{i_1}, \Gamma_1 \Rightarrow \Delta_1 \mid A^{i_2}, \Gamma_2 \Rightarrow \Delta_2 \mid A^{i_3}, \mathfrak{A}^{\Gamma\Delta}_3 \mid \ldots \mid A^{i_n}, \mathfrak{A}^{\Gamma\Delta}_n \mid H}{A^{i_1+i_2}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid A^{i_3}, \mathfrak{A}^{\Gamma\Delta}_3 \mid \ldots \mid A^{i_n}, \mathfrak{A}^{\Gamma\Delta}_n \mid H}}{\Gamma_1, \Gamma_2, \Theta^{i_1+i_2} \Rightarrow \Delta_1, \Delta_2, \Lambda^{i_1+i_2} \mid \mathfrak{A}^{\Gamma\Delta}_3 \times \mathfrak{A}^{\Theta\Lambda}_{i_3} \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_n \times \mathfrak{A}^{\Theta\Lambda}_{i_n} \mid H \mid G}$$

We transform the derivation as follows: we first apply cut and then (Merge).

$$\dfrac{\dfrac{\Theta \Rightarrow \Lambda, A^j \mid G \qquad A^{i_1}, \Gamma_1 \Rightarrow \Delta_1 \mid A^{i_2}, \Gamma_2 \Rightarrow \Delta_2 \mid A^{i_3}, \mathfrak{A}^{\Gamma\Delta}_3 \mid \ldots \mid A^{i_n}, \mathfrak{A}^{\Gamma\Delta}_n \mid H}{\Gamma_1, \Theta^{i_1} \Rightarrow \Delta_1, \Lambda^{i_1} \mid \Gamma_2, \Theta^{i_2} \Rightarrow \Delta_2, \Lambda^{i_2} \mid \mathfrak{A}^{\Gamma\Delta}_3 \times \mathfrak{A}^{\Theta\Lambda}_{i_3} \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_n \times \mathfrak{A}^{\Theta\Lambda}_{i_n} \mid H \mid G}}{\Gamma_1, \Gamma_2, \Theta^{i_1+i_2} \Rightarrow \Delta_1, \Delta_2, \Lambda^{i_1+i_2} \mid \mathfrak{A}^{\Gamma\Delta}_3 \times \mathfrak{A}^{\Theta\Lambda}_{i_3} \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_n \times \mathfrak{A}^{\Theta\Lambda}_{i_n} \mid H \mid G}$$

The other cases are considered similarly.

Case 3. The last inference is an application of non-modal left introduction rule whose principal formula is A. Left for the reader.

Case 4. The rule of last inference of $\mathfrak{D}_2$ is $(\triangleright \Rightarrow)$.

Subcase 4.1. A is principal in $\mathfrak{D}_2$ and $A = \triangleright B$. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\dfrac{B, \triangleright B^{i_1}, \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \triangleright B^{i_n}, \mathfrak{A}^{\Theta\Lambda}_n \mid G_1 \qquad \triangleright B^{i_1}, \mathfrak{A}^{\Pi\Sigma}_1, B \mid \ldots \mid \triangleright B^{i_n}, \mathfrak{A}^{\Pi\Sigma}_n \mid G_2}{\triangleright B \Rightarrow \mid \triangleright B^{i_1}, \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \triangleright B^{i_n}, \mathfrak{A}^{\Theta\Lambda}_n \mid \triangleright B^{i_1}, \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \triangleright B^{i_n}, \mathfrak{A}^{\Pi\Sigma}_n \mid G_1 \mid G_2}$$

Since $\mathfrak{D}_1$ ends as the condition (4) states, the last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{\Rightarrow B \mid B \Rightarrow \mid H}{\Rightarrow \triangleright B \mid H}$$

We have

$$\dfrac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\Rightarrow \mid \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2}$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}^{\Theta\Lambda}_2 \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n \mid H \mid G_1.$$

$$\mathfrak{A}^{\Pi\Sigma}_1, B \mid \mathfrak{A}^{\Pi\Sigma}_2 \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_2.$$

Using these hypersequents and $\Rightarrow B \mid B \Rightarrow \mid H$, by (Cut), (Merge) with (IC) (or just (EC)) as well as (EW), we get

$$
\dfrac{\dfrac{\mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2 \qquad \dfrac{\Rightarrow B \mid B \Rightarrow \mid H \qquad B, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1}{B \Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid H \mid G_1}}{\dfrac{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid H \mid H \mid G_1 \mid G_2}{\overline{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}}}}{\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}
$$

Subcase 4.2. The rule of the last inference of $\mathfrak{D}_2$ is $(\rhd \Rightarrow)$ and the principal formula in $\mathfrak{D}_2$ is not A. Then the last inference of $\mathfrak{D}_2$ looks as follows.

$$
\dfrac{B, A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\rhd B \Rightarrow \mid A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}
$$

$\mathfrak{D}_1$ ends with the hypersequent $\Gamma \Rightarrow \Delta, A^j \mid H$. We have:

$$
\dfrac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\rhd B \Rightarrow \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}
$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$
B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1.
$$

$$
\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2.
$$

Applying $(\rhd \Rightarrow)$, we get the required result.

Case 5. The rule of last inference of $\mathfrak{D}_2$ is $(\blacktriangleright \Rightarrow)$. Hence, $A = \blacktriangleright B$ and A is principal in $\mathfrak{D}_2$. (Cut) is applied to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ as follows:

$$
\dfrac{\dfrac{B, \Gamma \Rightarrow \Delta \mid H_1 \qquad \Theta \Rightarrow \Lambda, B \mid H_2}{\Rightarrow \blacktriangleright B \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2} \qquad \dfrac{\Rightarrow B \mid B \Rightarrow \mid G}{\blacktriangleright B \Rightarrow \mid G}}{\Rightarrow \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G}
$$

Using (Cut) and (EW), we transform this derivation as follows:

$$
\dfrac{\dfrac{\Theta \Rightarrow \Lambda, B \mid H_2 \qquad \dfrac{\Rightarrow B \mid B \Rightarrow \mid G \qquad B, \Gamma \Rightarrow \Delta \mid H_1}{B \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H_1 \mid G}}{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G}}{\Rightarrow \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G}
$$

Case 6. The rule of last inference of $\mathfrak{D}_2$ is $(\sim \Rightarrow)$. Hence, $A = {\sim} B$ and A is principal in $\mathfrak{D}_2$. Cut is applied to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ as follows:

$$\dfrac{\dfrac{B,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta\mid\Rightarrow{\sim}B\mid H}\qquad\dfrac{\Rightarrow B\mid G}{{\sim}B\Rightarrow\mid G}}{\Rightarrow\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

Using (Cut) and (EW), we transform this derivation as follows:

$$\dfrac{\dfrac{\Rightarrow B\mid G\qquad B,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta\mid H\mid G}}{\Rightarrow\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

Case 7. The rule of last inference of $\mathfrak{D}_2$ is $(\dot{\sim}\Rightarrow)$.

Subcase 7.1. A is principal in $\mathfrak{D}_2$ and $A=\dot{\sim}B$. Cut is applied to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ as follows.

$$\dfrac{\dfrac{B\Rightarrow\mid H}{\Rightarrow\dot{\sim}B\mid H}\qquad\dfrac{\dot{\sim}B^{i_1},\mathfrak{A}_1^{\Theta\Lambda},B\mid\dot{\sim}B^{i_2},\mathfrak{A}_2^{\Theta\Lambda}\mid\ldots\mid\dot{\sim}B^{i_n},\mathfrak{A}_n^{\Theta\Lambda}\mid G}{\dot{\sim}B\Rightarrow\mid\dot{\sim}B^{i_1},\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\dot{\sim}B^{i_n},\mathfrak{A}_n^{\Theta\Lambda}\mid G}}{\Rightarrow\mid\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}$$

Using the induction hypothesis, (Cut), (EC), and (EW) we obtain the required result.

$$\dfrac{\dfrac{\dfrac{\mathfrak{A}_1^{\Theta\Lambda},B\mid\mathfrak{A}_2^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G_1\qquad B\Rightarrow\mid H}{\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid H\mid G}}{\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}}{\Rightarrow\mid\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}$$

Subcase 7.2. The rule of the last inference of $\mathfrak{D}_2$ is $(\dot{\sim}\Rightarrow)$ and the principal formula in $\mathfrak{D}_2$ is not A. Then we have:

$$\dfrac{\Gamma\Rightarrow\Delta,A^j\mid H\qquad\dfrac{A^{i_1},\mathfrak{A}_1^{\Theta\Lambda},B\mid A^{i_1},\mathfrak{A}_2^{\Theta\Lambda}\mid\ldots\mid A^{i_n},\mathfrak{A}_n^{\Theta\Lambda}\mid G}{\dot{\sim}B\Rightarrow\mid A^{i_1},\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid A^{i_n},\mathfrak{A}_n^{\Theta\Lambda}\mid G}}{\dot{\sim}B\Rightarrow\mid\mathfrak{A}_{i_1}^{\Gamma\Delta}\times\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_{i_n}^{\Gamma\Delta}\times\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}$$

Using the induction hypothesis and $(\dot{\sim}\Rightarrow)$, we obtain the required result.

$$\dfrac{\mathfrak{A}_{i_1}^{\Gamma\Delta}\times\mathfrak{A}_1^{\Theta\Lambda},B\mid\mathfrak{A}_{i_2}^{\Gamma\Delta}\times\mathfrak{A}_2^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_{i_n}^{\Gamma\Delta}\times\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}{\dot{\sim}B\Rightarrow\mid\mathfrak{A}_{i_1}^{\Gamma\Delta}\times\mathfrak{A}_1^{\Theta\Lambda}\mid\ldots\mid\mathfrak{A}_{i_n}^{\Gamma\Delta}\times\mathfrak{A}_n^{\Theta\Lambda}\mid H\mid G}$$

Case 8. The rule of last inference of $\mathfrak{D}_2$ is $(\circ\Rightarrow)$.

Subcase 8.1. A is principal in $\mathfrak{D}_2$ and $A=\circ B$. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\dfrac{B, \circ B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad \circ B^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\circ B^{i_1+1}, \mathfrak{A}_1^{\Pi\Sigma} \mid \circ B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid \circ B^{i_2}, \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}$$

Since $\mathfrak{D}_1$ ends as the condition (4) states, the last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{\Rightarrow B \mid B, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \circ B \mid H}$$

We have:

$$\dfrac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\mathfrak{A}_{i_1+1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \mathfrak{A}_{1,\ldots,n}^{\Gamma\Delta\Theta\Lambda} \mid \mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Pi\Sigma} \mid H \mid G_1 \mid G_2}$$

where:

- $\mathfrak{A}_{1,\ldots,n}^{\Gamma\Delta\Theta\Lambda} = \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda}$
- $\mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Pi\Sigma} = \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma}$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2$$

Let us abbreviate them as follows:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_1 \mid H$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_2 \mid H$$

Then we reason as follows, using (Cut) and (EC):

$$\dfrac{\dfrac{\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_2 \mid H \qquad \dfrac{\Rightarrow B \mid B, \Gamma \Rightarrow \Delta \mid H \qquad B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_1 \mid H}{B, \Gamma \Rightarrow \Delta \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_1 \mid H \mid H}}{\mathfrak{A}_{i_1+1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_1 \mid \mathfrak{A}_2 \mid H \mid H \mid H}}{\mathfrak{A}_{i_1+1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_1 \mid \mathfrak{A}_2 \mid H}$$

Subcase 8.2. The rule of the last inference of $\mathfrak{D}_2$ is $(\circ \Rightarrow)$ and the principal formula in $\mathfrak{D}_2$ is not A. Then the last inference of $\mathfrak{D}_2$ looks as follows.

$$\dfrac{B, A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\circ B, A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma} \mid A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid A^{i_2}, \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}$$

$\mathfrak{D}_1$ ends as follows: $\Gamma \Rightarrow \Delta, A^j \mid H$. The result of the application of (Cut) to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ is as follows:

$$\circ B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2$$

Applying $(\circ \Rightarrow)$, we get the required result.

Cases dealing with $\bullet$, $\widetilde{\circ}$, and $\widetilde{\bullet}$ are considered similarly. ⊣

LEMMA 5.2 (Left reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that*:

(1) *$\mathfrak{D}_1$ is a derivation of $\Gamma_1 \Rightarrow \Delta_1, A^{i_1} \mid \dots \mid \Gamma_n \Rightarrow \Delta_n, A^{i_n} \mid H$,*
(2) *$\mathfrak{D}_2$ is a derivation of $A, \Theta \Rightarrow \Lambda \mid G$,*
(3) *$\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$.*

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma_1, \Theta^{i_1} \Rightarrow \Lambda^{i_1}, \Delta_1 \mid \dots \mid \Gamma_n, \Theta^{i_n} \Rightarrow \Lambda^{i_n}, \Delta_n \mid H \mid G$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

PROOF. The proof is by induction on the height of $\mathfrak{D}_1$. By induction on $\mathfrak{l}(\mathfrak{D}_2)$. The basic case is easy and left for the reader.

Inductive case. We have different cases depending on the last rule applied to $\mathfrak{D}_2$. The first three cases are similar to Lemma 5.1.

Case 4. The rule of last inference of $\mathfrak{D}_1$ is $(\rhd \Rightarrow)$. In this case, A is not a principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\frac{B, \mathfrak{A}_1^{\Theta\Lambda}, A^{i_1} \mid \dots \mid \mathfrak{A}_n^{\Theta\Lambda}, A^{i_n} \mid G_1 \qquad \mathfrak{A}_1^{\Pi\Sigma}, A^{i_1}, B \mid \dots \mid \mathfrak{A}_n^{\Pi\Sigma}, A^{i_n} \mid G_2}{\rhd B \Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda}, A^{i_1} \mid \dots \mid \mathfrak{A}_n^{\Theta\Lambda}, A^{i_n} \mid \mathfrak{A}_1^{\Pi\Sigma}, A^{i_1} \mid \dots \mid \mathfrak{A}_n^{\Pi\Sigma}, A^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $A^j, \Gamma \Rightarrow \Delta \mid H$. We have:

$$\frac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\rhd B \Rightarrow \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1.$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \dots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_2.$$

Applying $(\rhd \Rightarrow)$, we get the required result.

Case 5. The rule of last inference of $\mathfrak{D}_1$ is $(\Rightarrow \triangleright)$. In this case, A has to be a principal formula. We have:

$$\dfrac{\dfrac{B \Rightarrow \mid \Rightarrow B \mid H}{\Rightarrow \triangleright B \mid H} \qquad \triangleright B, \Gamma \Rightarrow \Delta \mid G}{\Gamma \Rightarrow \Delta \mid H \mid G}$$

By Lemma 5.1, the claim holds since this case satisfies the condition of application of the Lemma.

Case 6. The rule of last inference of $\mathfrak{D}_1$ is $(\blacktriangleright \Rightarrow)$. In this case A is not principal and is contained in a side hypersequent G. The case is similar to the Case 1 of Lemma 5.1.

Case 7. The rule of last inference of $\mathfrak{D}_1$ is $(\Rightarrow \blacktriangleright)$.

Subcase 7.1. A is a principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{B, \mathfrak{A}_1^{\Theta\Lambda}, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda}, \blacktriangleright B^{i_n} \mid G_1 \qquad \mathfrak{A}_1^{\Pi\Sigma}, \blacktriangleright B^{i_1}, B \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma}, \blacktriangleright B^{i_n} \mid G_2}{\Rightarrow \blacktriangleright B \mid \mathfrak{A}_1^{\Theta\Lambda}, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda}, \blacktriangleright B^{i_n} \mid \mathfrak{A}_1^{\Pi\Sigma}, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma}, \blacktriangleright B^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $\blacktriangleright B^j, \Gamma \Rightarrow \Delta \mid H$. We have:

$$\dfrac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\Gamma \Rightarrow \Delta \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} | H | G_1 | G_2}$$

Using the inductive hypothesis and $(\Rightarrow \blacktriangleright)$, we get the following derivation $\mathfrak{D}_3$ with $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$.

$$\dfrac{\Gamma \Rightarrow \Delta \mid B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Pi\Sigma} \mid H \mid G_1 \mid G_2}{\Gamma \Rightarrow \Delta \mid \Rightarrow \blacktriangleright B \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Pi\Sigma} \mid H \mid G_1 \mid G_2}$$

where

- $\mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Theta\Lambda} = \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda}$
- $\mathfrak{A}_{2,\ldots,n}^{\Gamma\Delta\Pi\Sigma} = \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma}$

By Lemma 5.1, the claim holds since this case satisfies the condition of application of the Lemma.

Subcase 7.2. A is not a principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{B, \mathfrak{A}_1^{\Theta\Lambda}, A^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda}, A^{i_n} \mid G_1 \qquad \mathfrak{A}_1^{\Pi\Sigma}, A^{i_1}, B \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma}, A^{i_n} \mid G_2}{\Rightarrow \blacktriangleright B \mid \mathfrak{A}_1^{\Theta\Lambda}, A^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda}, A^{i_n} \mid \mathfrak{A}_1^{\Pi\Sigma}, A^{i_1} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma}, A^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $A^j, \Gamma \Rightarrow \Delta \mid H$. We have:

$$\dfrac{\mathfrak{D}_1 \qquad \mathfrak{D}_2}{\Rightarrow \blacktriangleright B \mid \mathfrak{A}_{1,\ldots,n}^{\Gamma\Delta\Theta\Lambda} \mid \mathfrak{A}_{1,\ldots,n}^{\Gamma\Delta\Pi\Sigma} \mid H \mid G_1 \mid G_2}$$

where

- $\mathfrak{A}^{\Gamma\Delta\Theta\Lambda}_{1,\dots,n} = \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_{1} \mid \dots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_{n}$
- $\mathfrak{A}^{\Gamma\Delta\Pi\Sigma}_{1,\dots,n} = \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_{1} \mid \dots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_{n}$

By the induction hypothesis and ($\blacktriangleright \Rightarrow$), we get the required result similarly to Case 4 of this Lemma.

Case 8. The rule of last inference of $\mathfrak{D}_1$ is ($\sim \Rightarrow$). In this case A is not principal and is contained in a side hypersequent G. The case is similar to the Case 1 of Lemma 5.1.

Case 9. The rule of last inference of $\mathfrak{D}_1$ is ($\Rightarrow \sim$).

Subcase 9.1. A is principal.

$$\dfrac{\dfrac{B, \mathfrak{A}^{\Theta\Lambda}_1, {\sim}B^{i_1} \mid \mathfrak{A}^{\Theta\Lambda}_2, {\sim}B^{i_2} \mid \dots \mid \mathfrak{A}^{\Theta\Lambda}_n, {\sim}B^{i_n} \mid G}{\mathfrak{A}^{\Theta\Lambda}_1, {\sim}B^{i_1} \mid \dots \mid \mathfrak{A}^{\Theta\Lambda}_n, {\sim}B^{i_n} \mid \Rightarrow {\sim}B \mid G} \qquad {\sim}B^j, \Gamma \Rightarrow \Delta \mid H}{\Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid \Gamma \Rightarrow \Delta \mid H \mid G}$$

Using the inductive hypothesis and ($\Rightarrow \sim$), we get the following derivation $\mathfrak{D}_3$ with $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$.

$$\dfrac{B, \Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid H \mid G}{\Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid \Gamma \Rightarrow \Delta \mid \Rightarrow {\sim}B \mid H \mid G}$$

By Lemma 5.1, the claim holds since this case satisfies the condition of application of the Lemma.

Subcase 9.2. A is not principal.

$$\dfrac{\dfrac{B, \Theta_1 \Rightarrow \Lambda_1, A^{i_1} \mid \Theta_2 \Rightarrow \Lambda_2, A^{i_2} \mid \dots \mid \Theta_n \Rightarrow \Lambda_n, A^{i_n} \mid G}{\Theta_1 \Rightarrow \Lambda_1, A^{i_1} \mid \dots \mid \Theta_n \Rightarrow \Lambda_n, A^{i_n} \mid \Rightarrow {\sim}B \mid G} \qquad A^j, \Gamma \Rightarrow \Delta \mid H}{\Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid \Rightarrow {\sim}B \mid H \mid G}$$

Using the inductive hypothesis, we get the required result.

$$\dfrac{B, \Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \Gamma^{i_2}, \Theta_2 \Rightarrow \Delta^{i_2}, \Lambda_2 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid G}{\Gamma^{i_1}, \Theta_1 \Rightarrow \Delta^{i_1}, \Lambda_1 \mid \dots \mid \Gamma^{i_n}, \Theta_n \Rightarrow \Delta^{i_n}, \Lambda_n \mid \Rightarrow {\sim}B \mid G}$$

Case 10. The rule of last inference of $\mathfrak{D}_1$ is ($\circ \Rightarrow$). In this case, A is not a principal formula. Then the last inference of $\mathfrak{D}_1$ looks as follows.

$$\dfrac{B, A^{i_1}, \mathfrak{A}^{\Theta\Lambda}_1 \mid \dots \mid A^{i_n}, \mathfrak{A}^{\Theta\Lambda}_n \mid G_1 \qquad A^{i_1}, \mathfrak{A}^{\Pi\Sigma}_1, B \mid \dots \mid A^{i_n}, \mathfrak{A}^{\Pi\Sigma}_n \mid G_2}{\circ B, A^{i_1}, \mathfrak{A}^{\Pi\Sigma}_1 \mid A^{i_1}, \mathfrak{A}^{\Theta\Lambda}_1 \mid \dots \mid A^{i_n}, \mathfrak{A}^{\Theta\Lambda}_n \mid A^{i_2}, \mathfrak{A}^{\Pi\Sigma}_2 \mid \dots \mid A^{i_n}, \mathfrak{A}^{\Pi\Sigma}_n \mid G_1 \mid G_2}$$

$\mathfrak{D}_1$ ends as follows: $\Gamma \Rightarrow \Delta, A^j \mid H$. The result of the application of (Cut) to $\mathfrak{D}_1$ and $\mathfrak{D}_2$ is as follows:

$$\circ B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \dots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Pi\Sigma}_2 \mid \dots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2$$

Applying $(\circ \Rightarrow)$, we get the required result.

Case 11. The rule of last inference of $\mathfrak{D}_1$ is $(\Rightarrow \circ)$.

Subcase 11.1. A is principal. Then we have:

$$\dfrac{\dfrac{\Rightarrow B \mid B, \mathfrak{A}_1^{\Gamma\Delta}, \circ B^{i_1} \mid \mathfrak{A}_2^{\Gamma\Delta}, \circ B^{i_2} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta}, \circ B^{i_n} \mid G}{\mathfrak{A}_1^{\Gamma\Delta}, \circ B^{i_1+1} \mid \mathfrak{A}_2^{\Gamma\Delta}, \circ B^{i_2} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta}, \circ B^{i_n} \mid G} \qquad \circ B^j, \Theta \Rightarrow \Lambda \mid H}{\mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1+1}^{\Theta\Lambda} \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}$$

By the inductive hypothesis, we have the following derivation $\mathfrak{D}_3$ with $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$.

$$\dfrac{\Rightarrow B \mid B, \mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1+1}^{\Theta\Lambda} \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}{\mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1}^{\Theta\Lambda}, \circ B \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}$$

By Lemma 5.1, the claim holds since this case satisfies the condition of application of the Lemma.

Subcase 11.2. A is not principal. Then we have:

$$\dfrac{\dfrac{\Rightarrow B \mid B, \mathfrak{A}_1^{\Gamma\Delta}, A^{i_1} \mid \mathfrak{A}_2^{\Gamma\Delta}, A^{i_2} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta}, A^{i_n} \mid G}{\mathfrak{A}_1^{\Gamma\Delta}, \circ B, A^{i_1} \mid \mathfrak{A}_2^{\Gamma\Delta}, A^{i_2} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta}, A^{i_n} \mid G} \qquad A^j, \Theta \Rightarrow \Lambda \mid H}{\mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1}^{\Theta\Lambda}, \circ B \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}$$

Then we transform this derivation as follows:

$$\dfrac{\dfrac{\Rightarrow B \mid B, \mathfrak{A}_1^{\Gamma\Delta}, A^{i_1} \mid \mathfrak{A}_2^{\Gamma\Delta}, A^{i_2} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta}, A^{i_n} \mid G \qquad A^j, \Theta \Rightarrow \Lambda \mid H}{\Rightarrow B \mid B, \mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1}^{\Theta\Lambda} \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}}{\mathfrak{A}_1^{\Gamma\Delta} \times \mathfrak{A}_{i_1}^{\Theta\Lambda}, \circ B \mid \mathfrak{A}_2^{\Gamma\Delta} \times \mathfrak{A}_{i_2}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}$$

The other cases are treated similarly. ⊣

THEOREM 5.1 (Constructive elimination of cuts). *Let* $\boldsymbol{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$, *where* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \sim, \dot{\sim}\}$. *If a derivation* $\mathfrak{D}$ *in* $\mathbb{H}\boldsymbol{L}$ *has an application of* (Cut), *then it can be transformed into a cut-free derivation* $\mathfrak{D}'$.

Proof. Assume that a derivation $\mathfrak{D}$ in $\mathbb{H}\boldsymbol{L}$ has at least one application of (Cut), i.e. $\mathfrak{r}(\mathfrak{D}) > 0$. The proof proceeds by the double induction on $\langle \mathfrak{r}(\mathfrak{D}), \mathfrak{nr}(\mathfrak{D}) \rangle$, where $\mathfrak{nr}(\mathfrak{D})$ is the number of application of (Cut) in $\mathfrak{D}$. Consider an uppermost application of (Cut) in $\mathfrak{D}$ with cut rank $\mathfrak{r}(\mathfrak{D})$. We apply Lemmas 5.1 and 5.2 to its premises and decrease either $\mathfrak{r}(\mathfrak{D})$ or $\mathfrak{nr}(\mathfrak{D})$. Then we can use the inductive hypothesis. ⊣

6. Conclusion

In this paper, we have developed cut-free hypersequent calculi for several modal logics over **S5**-frames having non-standard modalities as primitive ones. There are other modalities one may also consider. For example, Pan and Yang [43] introduced the following weak essentially true and strong accidentally true modalities:

- $\vartheta(\circledast A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\exists_{y \in W} \vartheta(A, y) = 1$,
- $\vartheta(\odot A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y \in W} \vartheta(A, y) = 0$.

Thus, $\circledast A = \neg A \vee \Diamond A = A \rightarrow \Diamond A$ and $\odot A = A \wedge \Box \neg A$. Since these modalities are quite unusual, we decided not to include them into the main part of our paper, but we can present sound, complete, and cut-free hypersequent rules for them:

$$(\circledast \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \quad A \Rightarrow \mid G}{\circledast A, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \circledast)\ \frac{A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, A \mid H}{\Gamma \Rightarrow \Delta, \circledast A \mid \Theta \Rightarrow \Lambda \mid H}$$

$$(\odot \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, A \mid H}{\odot A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H} \qquad (\Rightarrow \odot)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \quad A \Rightarrow \mid G}{\Gamma \Rightarrow \Delta, \odot A \mid H \mid G}$$

By analogy we may define weak essentially false and strong accidentally false modalities as follows:

- $\vartheta(\widetilde{\circledast} A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\exists_{y \in W} \vartheta(A, y) = 0$,
- $\vartheta(\widetilde{\odot} A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\forall_{y \in W} \vartheta(A, y) = 1$.

Hence, $\widetilde{\circledast} A = A \vee \Diamond \neg A = \neg A \rightarrow \Diamond \neg A$ and $\widetilde{\odot} A = \neg A \wedge \Box A$. The appropriate sound, complete, and cut-free hypersequent rules for them are presented below:

$$(\widetilde{\circledast} \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \quad \Rightarrow A \mid G}{\widetilde{\circledast} A, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \widetilde{\circledast})\ \frac{\Gamma \Rightarrow \Delta, A \mid A, \Theta \Rightarrow \Lambda \mid H}{\Gamma \Rightarrow \Delta, \widetilde{\circledast} A \mid \Theta \Rightarrow \Lambda \mid H}$$

$$(\widetilde{\odot} \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid A, \Theta \Rightarrow \Lambda \mid H}{\widetilde{\odot} A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H} \qquad (\Rightarrow \widetilde{\odot})\ \frac{A, \Gamma \Rightarrow \Delta \mid H \quad \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \widetilde{\odot} A \mid H \mid G}$$

Among other non-standard modalities we would like to mention the so called 'boxdot' modality $\boxdot A = \Box A \wedge A$ introduced by Boolos [5] for the needs of provability logic and being interpreted as 'provable and true' (for its use in context of essence and accident logics [see 52]):

- $\vartheta(\boxdot A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y \in W} \vartheta(A, y) = 1$.

The appropriate rules for $\boxdot$ are as follows:

$$(\boxdot \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad A, \Theta \Rightarrow \Lambda \mid G}{\boxdot A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\Rightarrow \boxdot)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \boxdot A \mid H \mid G}$$

One may also consider strong non-contingency modality $\trianglerighteq A = (A \to \Box A) \wedge (\neg A \to \Box \neg A)$ (the symbol for it is ours) introduced by Fan [9]. We believe there might be other interesting non-standard modalities in the literature. As for future research, we emphasize two tasks. First, to formulate the uniform method of generation of hypersequent calculi for the non-standard modalities defined over **S5**-models. Second, to extend our results on other modal logics, such as **S4**, **T**, **K**, etc.

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