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Fidel Semantics for Propositional and First-Order Version of the Logic of $\mathbf{CG}_3'$

Abstract. Paraconsistent extensions of 3-valued Gödel logic are studied as tools for knowledge representation and nonmonotonic reasoning. Particularly, Osorio and his collaborators showed that some of these logics can be used to express interesting nonmonotonic semantics. $\mathbf{CG}_3'$ is one of these 3-valued logics. In this paper, we introduce Fidel semantics for a certain calculus of $\mathbf{CG}_3'$ by means of Fidel structures, named $\mathbf{CG}_3'$-structures. These structures are constructed from enriched Boolean algebras with a special family of sets. Moreover, we also show that the most basic $\mathbf{CG}_3'$-structures coincide with da Costa–Alves' bi-valuation semantics; this connection is displayed through a Representation Theorem for $\mathbf{CG}_3'$-structures. By contrast, we show that for other paraconsistent logics that allow us to present semantics through Fidel structures, this connection is not held. Finally, Fidel semantics for the first-order version of the logic of $\mathbf{CG}_3'$ are presented by means of adapting algebraic tools.

Keywords: paraconsistent logics; first-order logics; Fidel semantics

1. Introduction

In the area of non-monotonic reasoning, authors are interested in the use of paraconsistent logics in the study of knowledge representation. These logics provide mathematical bases to define knowledge representation semantics. For instance, the 3-valued Gödel logic called $\mathbf{G_3}$ is adequate expressing a stable model semantics, also called semantics of Answer Set Programming [29], which is one of the main semantics in non-monotonic reasoning. The paraconsistent extension of $\mathbf{G_3}$ (called $\mathbf{G}_3'$) proves to be useful expressing a p-stable semantics, an alternative to the stable

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$f_\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$f_\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$f_\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

	$f_\neg$
0	2
1	2
2	0

Table 1. Truth functions for the connectives $\vee$, $\wedge$, $\to$, and $\neg$ in $\mathbf{CG}'_\mathbf{3}$

semantics that in some sense is closer to the semantics defined in classical logic [28].

Other paraconsistent extensions of 3-valued Gödel logic were studied by Osorio and his collaborators [25, 26, 28] under the perspective of mathematical logic. In particular, a paraconsistent logic was defined by a matrix and called $\mathbf{CG}'_\mathbf{3}$ (Table 1), where 1 and 2 are the designated elements [26].

Later on, Kripke semantics for $\mathbf{CG}'_\mathbf{3}$ was presented in [3]; moreover, a Hilbert calculus for $\mathbf{CG}'_\mathbf{3}$ called $\mathbb{L}$ was presented in [30]. Other 3-valued paraconsistent logics were studied by Ciuciura in [6] and [7]. This author introduced a semantics via bi-valuation functions. Alternatively, Béziau describes a bi-valuation semantics for other paraconsistent logics [2]. Recently, a family of paraconsistent and paracomplete logics was introduced in [17]. Furthermore, new paraconsistent logics were presented in [27]; in particular, semantics for paraconsistent extension of 3-valued Gödel logics is given by means of Fidel structures.

In this paper, we introduce a Fidel semantics for $\mathbb{L}$ and prove an Adequacy Theorem. Moreover, we also show that bi-valuation semantics is the most basic of Fidel structures for $\mathbf{CG}'_\mathbf{3}$. Furthermore, we prove an Adequacy Theorem for the first-order version of $\mathbb{L}$ via first-order Fidel structures.

2. Preliminaries

Recall that a logic defined over a language $\mathcal{S}$ is a system $\mathcal{L} = \langle For, \vdash_{\boldsymbol{L}}\rangle$, where For is the set of formulas over $\mathcal{S}$ and a relation $\vdash_{\boldsymbol{L}} \subseteq \mathcal{P}(For) \times For$, where $\mathcal{P}(A)$ is the power set of A. The logic $\mathcal{L}$ is said to be Tarskian if it satisfies the following properties for any $\Gamma, \Omega \subseteq For$ and $\varphi, \beta \in For$:

(1) if $\alpha \in \Gamma$, then $\Gamma \vdash_{\boldsymbol{L}} \alpha$,

(2) if $\Gamma \vdash_{\boldsymbol{L}} \alpha$ and $\Gamma \subseteq \Omega$, then $\Omega \vdash_{\boldsymbol{L}} \alpha$,

(3) if $\Omega \vdash_{\boldsymbol{L}} \alpha$ and $\Gamma \vdash_{\boldsymbol{L}} \beta$ for every $\beta \in \Omega$, then $\Gamma \vdash_{\boldsymbol{L}} \alpha$.

A logic $\mathcal{L}$ is said to be finitary if it satisfies the following:

(4) if $\Gamma \vdash_{\boldsymbol{L}} \alpha$, then there exists a finite subset Γ_* of Γ such that $\Gamma_* \vdash_{\boldsymbol{L}} \alpha$.

Let $\mathcal{L}$ be a Tarskian logic and let Γ be a set of formulas. Then we say that Γ is a theory. Besides, Γ is said to be a *consistent theory* if there is φ such that $\Gamma \nvdash_{\boldsymbol{L}} \varphi$. We also say that Γ is non-trivial maximal consistent theory with respect to φ, if $\Gamma, \psi \vdash_{\boldsymbol{L}} \varphi$ for any $\psi \notin \Gamma$.

On the other hand, a logic is said to be standard if it is Tarskian and a finitary system. Furthermore, let $\mathcal{L}$ be a Tarskian logic, a set of formulas Γ is said to be closed in $\mathcal{L}$, or a closed theory of $\mathcal{L}$, if the following holds for any formula ψ: $\Gamma \vdash_{\boldsymbol{L}} \psi$ if and only if $\psi \in \Gamma$.

Straightforward from the very definitions we obtain:

LEMMA 2.1. *Any non-trivial maximal consistent set of formulas with respect to φ in $\mathcal{L}$ is closed, provided that $\mathcal{L}$ is Tarskian.*

From [31, Theorem 2.22] or [4, Chapter 2] we get:

LEMMA 2.2 (Lindenbaum–Łoś Lemma). *Let $\mathcal{L}$ be a standard logic and let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \nvdash_{\boldsymbol{L}} \varphi$. Then, there exists a set of formulas Ω such that $\Gamma \subseteq \Omega$ which is Ω maximal non-trivial with respect to φ in $\mathcal{L}$.*

In order to give a complete presentation of our paper and for readers not familiar with algebraic logic techniques, we briefly summarize some well-known results about Boolean algebra theory.

First, recall that a Boolean algebra $\boldsymbol{A}$ is an algebra in the signature $\{\vee, \wedge, ', 0, 1\}$. Every Boolean algebra has a $\{\vee, \wedge, '\}$-reduct of bounded distributive lattice and the unary operator $' : A \to A$ that verifies $1 = x \vee x' = 1$ and $0 = x \wedge x'$. As expected, we can define implication as follows: $x \to y = x' \vee y$.

For a given Boolean algebra $\boldsymbol{A}$ and $D \subseteq A$, we say that D is a deductive system of $\boldsymbol{A}$ if: (D1) $1 \in D$; and (D2) if $x, x \to y \in D$ then $y \in D$. Deductive systems play the same role that filters. $\mathcal{D}(\boldsymbol{A})$ denotes the set of all deductive systems of A, this ordered set by the inclusion is a bounded distributive lattice. Then, we have the following lemma.

LEMMA 2.3. *Let $\boldsymbol{A}$ be a Boolean algebra and let $Con(\boldsymbol{A})$ be the set of all congruences of $\boldsymbol{A}$. Then, there is a lattice-homomorphism between the lattice of $\mathcal{D}(\boldsymbol{A})$ and $Con(\boldsymbol{A})$, where $R(D) = \{(x, y) \in A \times A : x \to y,$*

$y \to x \in D\}$ is the associated relation to the deductive system D and the class of 1 by the congruence Θ verifies 1_Θ is a deductive system.

Monteiro presented several techniques to study the congruence for a given algebra by means of their deductive systems. In particular, he presented a characterization of maximal congruences in certain semisimple varieties via deductive systems tied to some element in the following way.

Definition 2.4 ([21]). Let $\boldsymbol{A}$ be a Boolean algebra, $D \in \mathcal{D}(\boldsymbol{A})$ and $p \in A$. We say that D is a deductive system tied to p if $p \notin D$ and for any $D' \in \mathcal{D}(\boldsymbol{A})$ such that $D \subsetneq D'$, then $p \in D'$.

Now, for a given Boolean algebra $\boldsymbol{A}$, a deductive system D of $\boldsymbol{A}$ is said to be maximal if for every deductive system M such that $D \subseteq M$, then $M = A$ or $M = D$. Thus, we have the following lemma:

Lemma 2.5 ([21]). *Let $\boldsymbol{A}$ be a Boolean algebra and let D be a deductive system of $\boldsymbol{A}$. Then, D is a maximal if and only if there exists $p \notin D$ such that D is a deductive system tied to p.*

From universal algebras results, we have that for a given maximal deductive system D of a Boolean algebra $\boldsymbol{A}$, there is a homomorphism $h\colon \boldsymbol{A} \to \mathbf{2}$ such that $h^{-1}(\{1\}) = D$, where $\mathbf{2}$ is the two-chain Boolean algebra. Moreover, the class of Boolean algebras is a variety which is generated by $\mathbf{2}$.

It is worth mentioning that Definition 2.4 and Lemma 2.5 will be used in the completeness proof for the propositional and the first-order logic w.r.t. Fidel semantics. In this proof, we will adapt algebraic techniques provide in [15] and [16] to the setting of Fidel structures. It is important to note that Fidel structures are not algebras in the universal algebra sense. Recently, the mentioned algebraic techniques were also applied to the first-order version of the logic $\mathbf{G_3'}$, but in this case, Coniglio *et al.* presented algebraic semantics for this quantified logic [9].

3. Fidel semantics for the calculus $\mathbb{L}$ of $\mathbf{CG_3'}$

In this section, we will present a new semantics for the logic $\mathbb{L}$ of $\mathbf{CG_3'}$. Firstly, recall that logic $\mathbb{L}$ was introduced by Pérez-Gaspar *et al.* in [30] as a formal axiomatic theory for $\mathbf{CG_3'}$ formed by the primitive logical connectives: $\neg$, $\to$ and $\wedge$. Some logical connectives defined in terms of the primitive ones are:

$$\begin{array}{rcl} \sim\varphi & := & \varphi \to (\neg\varphi \wedge \neg\neg\varphi) \\ \nabla\varphi & := & \sim\sim\varphi \wedge \neg\varphi \\ \varphi \vee \psi & := & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\ \varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi) \end{array}$$

Well-formed formulas are constructed as usual. The axiom schemas are:

`Pos1`: $\varphi \to (\psi \to \varphi)$
`Pos2`: $(\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))$
`Pos3`: $(\varphi \wedge \psi) \to \varphi$
`Pos4`: $(\varphi \wedge \psi) \to \psi$
`Pos5`: $\varphi \to (\psi \to (\varphi \wedge \psi))$
`Peirce's law`: $((\varphi \to \psi) \to \varphi) \to \varphi$
`Cw1`: $\varphi \vee \neg\varphi$
`E2`: $\neg\neg(\varphi \to \psi) \leftrightarrow ((\varphi \to \psi) \wedge (\neg\neg\varphi \to \neg\neg\psi))$
`E3`: $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$
`WE`: $\neg\neg\varphi \to (\neg\varphi \to \psi)$

and Modus Ponens (`MP`) is only inference rule defines the logic $\mathbb{L}$. We use $\Lambda \vdash_{\mathbb{L}} \lambda$ to mean that there exists a deduction of λ in $\mathbb{L}$ having as hypothesis a set of formulas Λ.

As we can notice, the list of axioms given above has only the first five axioms of the positive fragment of intuitionistic logic as well as `Peirce's law`, `Cw1`, `E2`, `E3` and the axiom `WE`. The next theorem sates some basic but useful properties of the formal theory $\mathbb{L}$. The proof of the following theorem is straightforward from the very definition of $\Lambda \vdash_{\mathbb{L}} \lambda$.

Theorem 3.1 ([30]). *For any sets Γ, Δ of formulas and any formulas φ, ψ, the following properties hold in $\mathbb{L}$:*

Monotonicity	*If $\Gamma \vdash_{\mathbb{L}} \varphi$ then $\Gamma \cup \Delta \vdash_{\mathbb{L}} \varphi$*
Deduction Theorem	*$\Gamma, \varphi \vdash_{\mathbb{L}} \psi$ if and only if $\Gamma \vdash_{\mathbb{L}} \varphi \to \psi$*
Rules-AND	*$\Gamma \vdash_{\mathbb{L}} \varphi \wedge \psi$ if and only if $\Gamma \vdash_{\mathbb{L}} \varphi$ and $\Gamma \vdash_{\mathbb{L}} \psi$*
Cut	*If $\Gamma \vdash_{\mathbb{L}} \varphi$ and $\Delta, \varphi \vdash_{\mathbb{L}} \psi$ then $\Gamma \cup \Delta \vdash_{\mathbb{L}} \psi$*

Using `Pos1`–`Pos5`, `MP` and Deduction Theorem we obtain:

Lemma 3.2. ([30]) *For any formulas φ, ψ, σ and ξ, the following are theorems in $\mathbb{L}$:*

(a) $\vdash_{\mathbb{L}} \varphi \to \varphi$
(b) $\varphi \to \psi, \psi \to \sigma \vdash_{\mathbb{L}} \varphi \to \sigma$
(c) $\varphi \to \psi, \sigma \to \xi \vdash_{\mathbb{L}} (\varphi \wedge \sigma) \to (\psi \wedge \xi)$
(d) $\vdash_{\mathbb{L}} (\varphi \to (\psi \to \gamma)) \to ((\varphi \wedge \psi) \to \gamma)$

(e) $\varphi \to (\psi \to \gamma) \vdash_{\mathbb{L}} \psi \to (\varphi \to \gamma)$
(f) $\vdash_{\mathbb{L}} (\varphi \wedge \psi) \leftrightarrow (\psi \wedge \varphi)$
(g) $\vdash_{\mathbb{L}} \varphi \to \big((\varphi \to \psi) \to \psi\big)$
(h) $\varphi \to \sigma, (\varphi \to \psi) \to \sigma, \sigma \to \psi \vdash_{\mathbb{L}} \sigma$

From [30] we obtain that the following results hold in $\mathbb{L}$.

`Pos6`: $\varphi \to (\varphi \vee \psi)$
`Pos7`: $\psi \to (\varphi \vee \psi)$
`Pos8`: $(\varphi \to \sigma) \to \big((\psi \to \sigma) \to (\varphi \vee \psi \to \sigma)\big)$
`Cw2`: $\neg\neg\varphi \to \varphi$
`CG′3`: $\nabla\varphi \to \varphi$
`E1`: $(\neg\varphi \to \neg\psi) \leftrightarrow (\neg\neg\psi \to \neg\neg\varphi)$
`ON`: $\neg\varphi \leftrightarrow \neg\neg\neg\varphi$
`SPC`: $\Gamma, \varphi \vdash_{\mathbb{L}} \psi$ and $\Gamma, \neg\varphi \vdash_{\mathbb{L}} \psi$ if and only if $\Gamma \vdash_{\mathbb{L}} \psi$

Theorem 3.3 (Soundness-Completeness, [30]). *For any formula φ, φ is a theorem of $\mathbb{L}$ if and only if φ is a tautology of $\mathbf{CG'_3}$.*

3.1. A new semantics

In [14], Fidel presented for the first time semantics for several da Costa's logics, namely, C_n and C_ω, [10]. This kind of semantics was also presented for Paraconsistent Nelson's logic by Odintsov as we can see in Section 3 of his paper [24]. In the sequel, we will introduce Fidel structures for $\mathbb{L}$ called, for short, $\mathbf{CG'_3}$-structures.

Definition 3.4. A $\mathbf{CG'_3}$-structure is a system $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ where $\boldsymbol{A}$ is a Boolean algebra and $\{N_x\}_{x\in A}$ is a family of subset of A such that the following conditions hold for any $x \in A$:

(i) $x \vee x' = 1$ for each $x' \in N_x$,
(ii) for any $x' \in N_x$ there is $x'' \in N_{x'}$ such that $x'' \wedge x' = 0$,
(iii) for any $x, y \in A$, there are $x' \in N_x$, $y' \in N_y$, $x'' \in N_{x'}$, $y'' \in N_{y'}$, $z' \in N_{x\wedge y}$ and $z'' \in N_{z'}$ such that $z'' = x'' \wedge y''$.
(iv) for any $x, y \in A$, there are $z' \in N_{x\to y}$, $z'' \in N_{z'}$, $x' \in N_x$, $y' \in N_y$, $x'' \in N_{x'}$, $y'' \in N_{y'}$ such that $z'' = (x \to y) \wedge (x'' \to y'')$.

As an example of $\mathbf{CG'_3}$-structure, we can take a Boolean algebra $\boldsymbol{A}$ and the set $N^s_x = \{y \in A : x \vee y = 1\}$. Then, the structure $\langle \boldsymbol{A}, \{N^s_x\}_{x\in A}\rangle$ will be called a saturated $\mathbf{CG'_3}$-structure. For the sake of brevity, we denote $\langle \boldsymbol{A}, N\rangle$ and $\langle \boldsymbol{A}, N^s\rangle$ instead of $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ and $\langle \boldsymbol{A}, \{N^s_x\}_{x\in A}\rangle$.

On the other hand, we denote by $\mathfrak{Fm}$ the absolutely free algebra of formulas and, as usual, we can define a congruence relation over $\mathfrak{Fm}$ as follows: $\alpha \equiv \beta$ iff $\vdash_{\mathbb{L}} (\alpha \to \beta) \wedge (\beta \to \alpha)$. Since $\mathbb{L}$ has the axiom of positive classical propositional calculus, we have that $\equiv$ is a congruence w.r.t. the connectives $\vee$, $\wedge$ and $\to$. Now, we denote by $\mathcal{L}_\omega$ the Lindenbaum-Tarski algebra $\mathfrak{Fm}|_\equiv$ where $|\alpha| \wedge |\beta| = |\alpha \wedge \beta|$, $|\alpha| \vee |\beta| = |\alpha \vee \beta|$, $|\alpha| \to |\beta| = |\alpha \to \beta|$ and we have that $|\alpha| \leq |\beta|$ iff $\vdash_{\mathbb{L}} \alpha \to \beta$. Thus, $\langle \mathcal{L}_\omega, \wedge, \vee, \to, 1\rangle$ is the free Boolean algebra taking 0 as $|\neg(\alpha \to \alpha)|$. Now, let us define $N_{|\alpha|}$ for each formula α as follows: $N_{|\alpha|} = \{|\neg\beta| : \beta \equiv \alpha\}$ where the negation $\neg$ is given by the language of $\mathbf{CG'_3}$ and it is clear that the relation $\equiv$ is not compatible with $\neg$.

From the latter, we have that $\langle \mathcal{L}_\omega, \{N_{|\alpha|}\}_{\alpha\in\mathfrak{Fm}}\rangle$ is a $\mathbf{CG'_3}$-structure. This idea of taking the Lindenbaum-Tarski algebra of a fragment of the logic was developed by Fidel in [14] for the calculus C_ω and C_n $(n < \omega)$. Later on, this was adapted to non-algebraizable extensions of C_ω in [27].

Definition 3.5. The $\mathbf{CG'_3}$-structure $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ is said to be a substructure of the $\mathbf{CG'_3}$-structure $\langle \boldsymbol{B}, \{N'_x\}_{x\in B}\rangle$ if $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$ and $N_x \subseteq N'_x$ holds for $x \in A$.

It is easy to see that each $\mathbf{CG'_3}$-structure $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ is a substructure of the saturated structure $\langle \boldsymbol{A}, \{N^s_x\}_{x\in A}\rangle$ defined before.

Definition 3.6. Let $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ and $\langle \boldsymbol{B}, \{N'_y\}_{y\in B}\rangle$ be two $\mathbf{CG'_3}$-structures. The function $h\colon A \to B$ is said to be a $\mathbf{CG'_3}$-homomorphism if h is a Boolean homomorphism such that $h(N_x) \subseteq N'_{h(x)}$ for every $x \in A$. Besides, we say that h is onto $\mathbf{CG'_3}$-homomorphism if $h(A) = B$ such that $h(N_x) = N'_{h(x)}$, and we say h is a $\mathbf{CG'_3}$-isomorphism if h is a bijective $\mathbf{CG'_3}$-homomorphism;i.e., h is bijective function and, at the same time, homomorphism.

The following proposition immediately follows from Birkhoff's subdirectly irreducible representation for Boolean algebras see for instance [see, e.g., 1] and Definitions 3.5 and 3.6:

Proposition 3.7 (Representation Theorem). *Any $\mathbf{CG'_3}$-structures $\langle \boldsymbol{A}, \{N_x\}_{x\in A}, \rangle$ is $\mathbf{CG'_3}$-isomorphic to a $\mathbf{CG'_3}$-substructure of $\langle \mathbf{2}^X, N^s\rangle$ for a certain non-empty set X, where $\mathbf{2}^X$ is the functional algebra, which is defined in a standard way.*

Definition 3.8. We say that a function $v\colon \mathfrak{Fm} \to \langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ is a $\mathbf{CG_3'}$-valuation, if for any formulas φ and ψ the following conditions hold:

(v1) $v(\varphi) \in A$ where φ is atomic formula,
(v2) $v(\varphi\#\psi) = v(\alpha)\#v(\psi)$ where $\# \in \{\wedge, \vee, \to\}$,
(v3) $v(\neg\varphi) \in N_{v(\varphi)}$,
(v4) $v(\neg\neg(\varphi \to \psi)) = v(\varphi \to \psi) \wedge v(\neg\neg\varphi \to \neg\neg\psi)$,
(v5) $v(\neg\neg(\varphi \wedge \psi)) = v(\neg\neg\varphi \wedge \neg\neg\psi)$

For us, a formula α will be semantically valid, written $\vDash \alpha$, if, for each $\mathbf{CG_3'}$-structure $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ and for every $\mathbf{CG_3'}$-valuation v, we have that $v(\alpha) = 1$. Moreover, we write $\Gamma \vDash \alpha$ if, for each $\mathbf{CG_3'}$-structure $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ and for each $\mathbf{CG_3'}$-valuation v, $v(\gamma) = 1$ for every $\gamma \in \Gamma$, then $v(\alpha) = 1$.

We say that φ is derivable from Γ in $\mathbb{L}$, denoted as $\Gamma \vdash \varphi$, if there exists a derivation of φ from Γ in $\mathbb{L}$; and, it is defined in the usual way. Moreover, it is not hard to see that $\langle \mathbb{L}, \vdash\rangle$ is a Tarskian and finitary logic (see Section 2). Now, we can prove the following lemma.

Lemma 3.9 ([15, 16]). *Let Γ be a non-trivial maximal set with respect to φ in $\mathbb{L}$. Let $\Gamma|_\equiv = \{|\alpha| : \alpha \in \Gamma\}$ be a subset of the support of Boolean algebra $\mathfrak{Fm}|_\equiv$. Then*:

1. *If $\alpha \in \Gamma$ and $|\alpha| = |\beta|$, then $\beta \in \Gamma$,*
2. *$\Gamma|_\equiv$ is a deductive system of $\mathfrak{Fm}|_\equiv$. Also, if $|\varphi| \notin \Gamma|_\equiv$ and for any deductive system $\overline{D}$ which contains properly to $\Gamma|_\equiv$, then $|\varphi| \in \overline{D}$.*

In the next corollary, we will consider the saturated structure over $\mathbf{2}$ that we denote by $\langle \mathbf{2}, N\rangle$:

Corollary 3.10. *Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}$ and Γ is a non-trivial maximal set with respect to φ in $\mathbb{L}$. Then there exists a valuation $v\colon \mathfrak{Fm} \to \langle \mathbf{2}, N\rangle$ such that*: $v(\varphi) = 1$ *iff* $\alpha \in \Gamma$.

Proof. Taking into account Lemma 3.9, we know that $\Gamma|_\equiv$ is a maximal deductive system of $\mathfrak{Fm}|_\equiv$. Then, we have there is a Boolean homomorphism $h\colon \mathfrak{Fm}|_\mathbf{2} \to \mathbf{2}$ (see Section 2) such that $h^{-1}(\{1\}) = \Gamma|_\mathbf{2}$. Now, consider the canonical projection $\pi : \mathfrak{Fm} \to \mathfrak{Fm}|_\mathbf{2}$ defined by $\pi(\alpha) = |\alpha|$. Now, it is enough to take $v = h \circ \pi$ to end the proof. $\dashv$

We can now prove the following Adequacy Theorem.

Theorem 3.11. *for any set Γ of formulas and any formula α of $\mathbb{L}$, $\Gamma \vdash_{\mathbb{L}} \alpha$ if and only if $\Gamma \vDash \alpha$.*

Proof. Suppose $\Gamma \vdash_{\mathbb{L}} \varphi$, then there exists $\alpha_1, \ldots, \alpha_n$ a derivation of φ from Γ. If the length of the derivation is $n = 1$ then, φ is an axiom or $\varphi \in \Gamma$; in both cases, it is easy to see that $\Gamma \vDash \varphi$. Let us suppose as induction hypothesis that $\Gamma \vDash \alpha_i$ with $1 \leqslant i < n$, then there exists $\{j, k_1, \ldots, k_m\} \subseteq \{1, \ldots, i-1\}$ such that $\alpha_{k_1}, \ldots, \alpha_{k_m}$ is a derivation of $\alpha_j \to \varphi$. Besides, suppose that φ is obtained by applying (MP). From the induction hypothesis, we have $v(\alpha_j \to \varphi) = 1$ for every valuation, but fixed v and then, $v(\alpha_j) \to v(\varphi) = 1$. Since $j < i$ then $v(\alpha_j) = 1$. Therefore, $1 \to v(\varphi) = 1$ and so, $v(\varphi) = 1$. Then, $\Gamma \vDash \varphi$.

Conversely, let us suppose $\Gamma \nvdash_{\mathbb{L}} \varphi$. So, from Lindenbaum–Łoś lemma, there exists a Ω maximal consistent theory such that $\Gamma \subseteq \Omega$ and $\Omega \nvdash_{\mathbb{L}} \varphi$. Let us consider the quotient algebra $\boldsymbol{A} := \mathfrak{Fm}|_{\Omega}$, where $[\alpha]_{\Omega} = \{\beta \in \mathfrak{Fm} : \Omega \vdash_{\mathbb{L}} \alpha \to \beta, \Omega \vdash_{\mathbb{L}} \beta \to \alpha\}$ is the class of α by Ω. According to Lemma 3.9, it is not hard to see that A is a Boolean algebra, and the canonical projection $q\colon \mathfrak{Fm} \to A$ such that $q(\alpha) = [\alpha]_{\Omega}$ is a homomorphism of algebras. Consider now the saturated $\mathbf{CG_3'}$-structure $\langle \boldsymbol{A}, \{N_x\}_{x \in A}\rangle$, and so the function $v\colon \mathfrak{Fm} \to \langle \boldsymbol{A}, \{N_x\}_{x \in A}\rangle$ defined by $v(\alpha) = [\psi]_{\Omega}$ is a $\mathbf{CG_3'}$-valuation. Thus, we have $[\psi]_{\Omega} = 1$ iff $\Omega \vdash_{\mathbb{L}} \psi$; but $v(\varphi) = [\varphi]_{\Omega} \neq 1$, which is a contradiction. ⊣

3.2. Bi-valuation semantics as the most basic Fidel structure

The bi-valuation semantics was considered for the first time in [11] in order to present a semantics for da Costa's logics C_n $(n < \omega)$ [see 10]. More recently, Ciuciura presented this kind of semantics for Sette's P_1 logic and a new hierarchy of the paraconsistent calculi introduced by him in [7, 8]. In this part of the paper, we firstly present this kind of semantics for the logic $\mathbb{L}$; and secondly, we will show that it is a special case of a Fidel structure.

Definition 3.12. Let v be a function from *Var* into $\{0, 1\}$, the function v is said to be a $\mathbf{CG_3'}$-bi-valuation if the following conditions hold:

(i) $v(\alpha \wedge \beta) = 1$ iff $v(\alpha) = 1$ and $v(\beta) = 1$,
(ii) $v(\alpha \vee \beta) = 1$ iff $v(\alpha) = 1$ or $v(\beta) = 1$,
(iii) $v(\alpha \to \beta) = 1$ iff $v(\alpha) = 0$ or $v(\beta) = 1$,
(iv) $v(\alpha) = 0$ implies $v(\neg\alpha) = 1$,
(v) $v(\neg\alpha) = 1$ implies $v(\neg\neg\alpha) = 0$,

(vi) $v(\neg\neg(\alpha \wedge \beta)) = 1$ iff $v(\neg\neg\alpha) = 1$ and $v(\neg\neg\beta) = 1$,
(vii) $v(\neg\neg(\alpha \to \beta)) = 1$ iff $v(\alpha \to \beta) = 1$ and $v(\neg\neg\alpha \to \neg\neg\beta) = 1$.

It is worth noting that conditions (i) to (iv) were first considered in [11, Lemma 3] to describe a bi-valuation semantics for C_n. Now, we will see that a $\mathbf{CG'_3}$-bi-valuation has only one $\mathbf{CG'_3}$-structure associated.

Lemma 3.13. *Any $\mathbf{CG'_3}$-bi-valuation v is $\{\vee, \wedge, \to\}$-homomorphism from Var into $\mathbf{2}$. moreover, v is a $\mathbf{CG'_3}$-valuation over the saturated and, structure $\langle \mathbf{2}, N\rangle$.*

Proof. Taking a $\mathbf{CG'_3}$-bi-valuation v, it is not hard to see that from conditions (i), (ii) and (iii) of Definition 3.12, we have that h is a Boolean homomorphism. Now, from conditions (iv) to (viii) and taking $N = \{0, 1\}$, in view Definition 3.4, we have that v is a $\mathbf{CG'_3}$-valuation. ⊣

Conversely, let us consider a $\mathbf{CG'_3}$-bi-valuation v. It follows immediately from the very definitions that v is a $\mathbf{CG'_3}$-bi-valuation. To prove the Adequacy Theorem, we need to define semantical consequence for this case as follows: a formula α will be semantically valid, written $\vDash \alpha$, if, for every $\mathbf{CG'_3}$-bi-valuation v, we have that $v(\alpha) = 1$. Moreover, we write $\Gamma \vDash_{BV} \alpha$ if, for each $\mathbf{CG'_3}$-bi-valuation v, $v(\beta) = 1$ for every $\beta \in \Gamma$, then $v(\alpha) = 1$. Then, we have the following Adequacy Theorem:

Theorem 3.14. *Foe any set Γ of formulas and any formula α of $\mathbb{L}$, $\Gamma \vdash_{\mathbb{L}} \alpha$ if only if $\Gamma \vDash_{BV} \alpha$.*

Proof. The necessary condition follows immediately from Theorem 3.11 and Lemma 3.13.

Conversely, suppose $\Gamma \vDash_{BV} \alpha$. Now, let us consider $\langle \boldsymbol{M}, N\rangle$ a fixed $\mathbf{CG'_3}$-structure and an arbitrary $\mathbf{CG'_3}$-valuation $v' \colon Var \to \langle \boldsymbol{M}, N\rangle$. Thus, from Proposition 3.7, there is one-to-one $\mathbf{CG'_3}$-homomorphism $e \colon \langle \boldsymbol{M}, N\rangle \to \langle \mathbf{2}^X, N^s\rangle$ for some non-empty set X. Taking $v = e \circ v'$, we have a valuation $v(i) \colon Var \to \{1, 0\}$ such that $v(\alpha)(i) = (e \circ v')(\alpha)(i)$, for every $i \in X$. Hence, by hypothesis we infer that $v(\gamma)(i) = 1$ with $\gamma \in \Gamma$, then $v(\alpha)(i) = 1$, for every $i \in X$. From the latter and the fact that e is a one-to-one algebraic homomorphism, we have that $v'(\gamma) = 1$ with $\gamma \in \Gamma$, then $v'(\alpha) = 1$. Therefore, according to Theorem 3.11, we have $\Gamma \vdash_{\mathbb{L}} \alpha$. ⊣

It is important to note that Proposition 3.7 allows us to give a bi-valuation semantics. In contrast, the Representation Theorems given in

[27] do not permit us to display bi-valuation semantics because the representations are built on Heyting algebras that are not Boolean algebras. On the other hand, bi-valuation semantics [19] and Fidel semantics [14], were given for da Costa's logic C_ω, for this case the relation showed in this subsection does not hold either. On the contrary, it is possible to see that the most basic Fidel structure for C_n given in [14] is in fact the da Costa–Alves' bi-valuations semantics given in [11].

4. Fidel semantics for the first-order version of $\mathbb{L}$

In this section, we introduce a first-order version of $\mathbb{L}$ logic and give an Adequacy Theorem for this calculus by adapting the algebraic technique given in [15, 16]. Consider the symbols $\forall$ (universal quantifier) and $\exists$ (existential quantifier), together with commas and parenthesis as the punctuation marks. Let $Var = \{v_1, v_2, \ldots\}$ be a denumerable set of *individual* variables. A first-order signature is a triple $\Theta = \langle \mathcal{C}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \{\mathcal{P}_n\}_{n\in\mathbb{N}}\rangle$ such that:

- $\mathcal{C}$ is a set of individual constants;
- for each $n \geqslant 1$, $\mathcal{F}_n$ is a set of function symbols of arity n,
- for each $n \geqslant 1$, $\mathcal{P}_n$ is a set of predicate symbols of arity n.

As usual, it will be assumed that Θ has at least one predicate symbol in order to have a non-empty set of formulas.

The notions of bound and free variables inside a formula, closed terms, closed formulas (or sentences), and of term free for a variable in a formula are defined as usual [see 20]. We denote by Ter_Θ and $\mathfrak{Fm}_\Theta$ the set of terms and the set of first-order formulas over Θ, by using the connectives in Σ, respectively. Given a formula φ, the formula obtained from φ by substituting every free occurrence of a variable x by a term t will be denoted by $\varphi(x/t)$.

Definition 4.1. Let Θ be a first-order signature. The logic $\mathcal{Q}\mathbb{L}$ over Θ is defined by the Hilbert calculus obtained by extending $\mathbb{L}$ expressed in the language $\mathfrak{Fm}_\Theta$ by adding the following:

Axiom schemas:

(`Ax14`) $\varphi(x/t) \to \exists x\varphi$ if t is a term free for x in φ

(`Ax15`) $\forall x\varphi \to \varphi(x/t)$ if t is a term free for x in φ

Inference Rules:

($\exists$`In`) $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$ where x does not occur free in ψ

$$(\forall\mathtt{In})\quad \frac{\varphi \to \psi}{\varphi \to \forall x\psi} \text{ where } x \text{ does not occur free in } \varphi$$

A Θ-structure for $\mathcal{QL}$ is a triple $\mathfrak{A} = \langle U, \langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle, \cdot^{\mathfrak{A}}\rangle$ such that U is a non-empty set, $\langle \boldsymbol{A}, \{N_x\}_{x\in A}\rangle$ is a complete $\mathbf{CG'_3}$-structure (i.e. $\boldsymbol{A}$ is a complete Boolean algebra) and $\cdot^{\mathfrak{A}}$ is an interpretation map which assigns:

- to each individual constant $c \in \mathcal{C}$, an element $c^{\mathfrak{A}}$ of U;
- to each function symbol f of arity n, a function $f^{\mathfrak{A}}\colon U^n \to U$;
- to each predicate symbol P of arity n, a function $P^{\mathfrak{A}}\colon U^n \to A$.

Given a Θ-structure $\mathfrak{A}$ for $\mathcal{QL}$, an *assignment* over $\mathfrak{A}$ is a function $s\colon \mathit{Var} \to U$. Given s and $a \in U$, let $s[x \to a]$ be the assignment such that $s[x \to a](x) = a$ and $s[x \to a](y) = s(y)$ for every $x \neq y$. A Θ-structure $\mathfrak{A}$ and an assignment s induce an interpretation map $[\![\cdot]\!]^{\mathfrak{A}}_s$ for terms and formulas defined as follows:

$$
\begin{aligned}
&[\![x]\!]^{\mathfrak{A}}_s = s(x) \text{ if } x \in \mathit{Var},\\
&[\![c]\!]^{\mathfrak{A}}_s = c^{\mathfrak{A}} \text{ if } c \in \mathcal{C},\\
&[\![f(t_1,\dots,t_n)]\!]^{\mathfrak{A}}_s = f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_s,\dots,[\![t_n]\!]^{\mathfrak{A}}_s), \text{ if } f \in \mathcal{F}_n,\\
&[\![P(t_1,\dots,t_n)]\!]^{\mathfrak{A}}_s = P^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_s,\dots,[\![t_n]\!]^{\mathfrak{A}}_s), \text{ if } P \in \mathcal{P}_n,\\
&[\![\phi\#\varphi]\!]^{\mathfrak{A}}_s = [\![\phi]\!]^{\mathfrak{A}}_s \# [\![\varphi]\!]^{\mathfrak{A}}_s \text{ for } \# \in \{\wedge,\vee,\to\},\\
&[\![\neg\varphi]\!]^{\mathfrak{A}}_s \in N_{[\![\varphi]\!]^{\mathfrak{A}}_s},\\
&[\![\forall x\varphi]\!]^{\mathfrak{A}}_s = \bigwedge_{a\in U} [\![\varphi]\!]^{\mathfrak{A}}_{s[x\to a]},\\
&[\![\exists x\varphi]\!]^{\mathfrak{A}}_s = \bigvee_{a\in U} [\![\varphi]\!]^{\mathfrak{A}}_{s[x\to a]},\\
&[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to[\![t]\!]^{\mathfrak{A}}_s]} = [\![\alpha(x/t)]\!]^{\mathfrak{A}}_s.
\end{aligned}
$$

We call the last condition a *substitution condition* and it is clear that for the algebraic presentation of the first-order logics this condition holds. Moreover, this condition also holds for $\{\neg\}$-free formulas of $\mathcal{QL}$, but it is not the case for formulas that contain the operation $\neg$. The substitution condition is essential to prove the Soundness Theorem as we will see.

First, we say that $\mathfrak{A}$ and s *satisfy* a formula φ, denoted by $\mathfrak{A} \vDash \varphi[s]$, if $[\![\varphi]\!]^{\mathfrak{A}}_s = 1$. On the other hand, φ is *true in* $\mathfrak{A}$ if $\mathfrak{A} \vDash \varphi[s]$ for every s. We say that φ is a *semantical consequence* of Γ in $\mathcal{QL}$, denoted by $\Gamma \vDash \alpha$, if, for any structure $\mathfrak{A}$: if any $\psi \in \Gamma$ is true in $\mathfrak{A}$, then α is true in $\mathfrak{A}$.

The following proof is exactly the same for algebraizable logics but we will describe it for the paper completeness.

THEOREM 4.2 (Soundness for $\mathcal{QL}$). *For any set Γ of formulas and any formula φ of $\mathfrak{Fm}_\Theta$, if $\Gamma \vdash_{\mathbb{L}} \varphi$ then $\Gamma \vDash \varphi$.*

PROOF. Consider a given structure $\mathfrak{A} = \langle U, \langle \mathcal{A}, \{N_x\}_{x\in\mathcal{A}}\rangle, \cdot^{\mathfrak{A}}\rangle$. It is enough to prove the following facts: the axioms (`Ax14`) and (`Ax15`) are true in $\mathfrak{A}$, and the inference rules ($\exists$`In`) and ($\forall$`In`) preserve truth in $\mathfrak{A}$.

(`Ax14`) and (`Ax15`): Suppose that φ is $\alpha(x/t) \to \exists x\alpha$, and let s be an assignment. Then, by substitution condition, $[\![\varphi]\!]^{\mathfrak{A}}_s = [\![\alpha]\!]^{\mathfrak{A}}_{s[x\to[\![t]\!]^{\mathfrak{A}}_s]} \to [\![\exists x\alpha]\!]^{\mathfrak{A}}_s$. It is clear that $[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to[\![t]\!]^{\mathfrak{A}}_s]} \leq \bigvee_{a\in U}[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]}$, hence $[\![\alpha(x/t)]\!]^{\mathfrak{A}}_s \leq [\![\exists x\alpha]\!]^{\mathfrak{A}}_s$. Therefore $[\![\alpha(x/t) \to \exists x\alpha]\!]^{\mathfrak{A}}_s = 1$. The validity of (`Ax15`) is proved analogously.

($\exists$`In`): Let $\alpha \to \beta$ such that x is not free in β, and let $\varphi = \exists x\alpha \to \beta$. Suppose that that $[\![\alpha \to \beta]\!]^{\mathfrak{A}}_s = 1$ for every s, and fix an assignment s. By definition, $[\![\varphi]\!]^{\mathfrak{A}}_s = [\![\exists x\alpha]\!]^{\mathfrak{A}}_s \to [\![\beta]\!]^{\mathfrak{A}}_s = \bigvee_{a\in U}[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \to [\![\beta]\!]^{\mathfrak{A}}_s$. By hypothesis, $[\![\alpha]\!]^{\mathfrak{A}}_{s'} \leq [\![\beta]\!]^{\mathfrak{A}}_{s'}$ for every s'. In particular, $[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \leq [\![\beta]\!]^{\mathfrak{A}}_{s[x\to a]} = [\![\beta]\!]^{\mathfrak{A}}_s$ for every $a \in U$, since x is not free in β. So, $\bigvee_{a\in U}[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \to [\![\beta]\!]^{\mathfrak{A}}_s = [\![\exists x\alpha \to \beta]\!]^{\mathfrak{A}}_s = [\![\varphi]\!]^{\mathfrak{A}}_s = 1$.

The preservation of truth by the rule ($\forall$`In`) is proved analogously. $\dashv$

Now, let us consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash_{\mathbb{L}} \alpha \to \beta$ and $\vdash_{\mathbb{L}} \alpha \to \beta$ where α and β are formulas. Then, from the axioms of $\mathbb{L}$ (see Section 3), we have that the algebra $\mathfrak{Fm}_\Theta|_\equiv$ is a Boolean algebra, the proof is exactly the same as in the propositional case. The equivalence class of a formula α w.r.t. $\equiv$ will be denoted by $\overline{\alpha}$.

It is clear that $\mathcal{QL}$ is a Tarskian logic and it is possible to consider the notion of set of maximal non-trivial formulas w.r.t some formula φ and the notion of closed theories is defined in the same way as the propositional case, see Section 2. Therefore, we have that the Lindenbaum-Łoś' Lemma holds for $\mathcal{QL}$. Then, we have the following:

LEMMA 4.3 ([15, 16]). *Let Γ be a maximal non-trivial set w.r.t. φ in $\mathcal{QL}$. Let $\Gamma|_\equiv = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the Boolean algebra $\mathfrak{Fm}_\Theta|_\equiv$. Then:*

1. *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$, then $\beta \in \Gamma$. If $\overline{\alpha} \in \Gamma|_\equiv$, then $\overline{\forall x\alpha}, \overline{\exists x\alpha} \in \Gamma|_\equiv$.*
2. *$\Gamma|_\equiv$ is a deductive system of $\mathfrak{Fm}_\Theta|_\equiv$. Also, if $\overline{\varphi} \notin \Gamma|_\equiv$ then, for any closed deductive system $\overline{D}$ containing properly to $\Gamma|_\equiv$, it is the case that $\overline{\varphi} \in \overline{D}$.*

Remark 4.4. It is worth mentioning that item 2. of last the lemma states that $\Gamma|_{\equiv}$ is a maximal deductive system. Besides, we know that $\mathfrak{Fm}_{\Theta}|_{\equiv}$ is a Boolean algebra, and for every Γ maximal non-trivial w.r.t. φ we have that $\Gamma|_{\equiv}$ is a maximal deductive system of $\mathfrak{Fm}_{\Theta}|_{\equiv}$.

From the last remark and results of Boolean algebra Theory (see Section 2), there is a homomorphism $h\colon \mathfrak{Fm}_{\Theta}|_{\equiv} \to \mathbf{2}$ such that $h^{-1}(\{1\}) = \Gamma|_{\equiv}$, see Proposition 3.7. Thus, if we consider the canonical projection $\pi\colon \mathfrak{Fm}_{\Theta} \to \mathfrak{Fm}_{\Theta}|_{\equiv}$, there is a homorphism $f\colon \mathfrak{Fm}_{\Theta} \to \mathbf{2}$ defined by $f = h \circ \pi$ such that $f^{-1}(\{1\}) = \Gamma$. Observe that $f(\alpha) = h(\overline{\alpha})$ where $\overline{\alpha}$ denotes the class of α by $\equiv$.

Theorem 4.5 (Completeness for sentences of $\mathcal{Q}\mathbb{L}$ w.r.t. the class of $\mathbf{CG_3'}$-structures). *For any set Γ of closed formulas an any closed formula φ over Θ, $\Gamma \vDash \varphi$ implies that $\Gamma \vdash_{\mathbb{L}} \varphi$.*

Proof. Let us suppose that $\Gamma \nvdash_{\mathbb{L}} \varphi$. Then, there is an M maximal non-trivial w.r.t. φ such that $\Gamma \subseteq M$. Hence, $\alpha \in M$ for every $\alpha \in \Gamma$ and $\varphi \notin M$. Now, let us consider the algebra $\mathcal{A} := \mathfrak{Fm}_{\Theta}|_{\equiv_M}$ defined by the congruence $\alpha \equiv_M \beta$ iff $(\alpha \to \beta), (\beta \to \alpha) \in M$. Hence, it is not hard to see that $\mathcal{A}$ is the two-chain Boolean algebra in virtue of Remark 4.4 and properties for Boolean algebras given in Section 2. Besides, it is easy to see that if $[\alpha]_M$ denotes the equivalence class in $\mathcal{A}$ of the formula α then: $[\alpha]_M \leq [\beta]_M$ iff $\alpha \to \beta \in M$.

Let us consider the canonical structure $\mathfrak{A} = \langle U, \langle \mathcal{A}, \{N_x\}_{x \in \mathcal{A}}\rangle, \cdot^{\mathfrak{A}}\rangle$ such that U is the set Ter_{Θ} of terms over Θ and $\langle \mathcal{A}, \{N_x\}_{x \in \mathcal{A}}\rangle$ is the saturated $\mathbf{CG_3'}$-structure over $\mathcal{A}$. Now, for every term t consider its name $\hat{t}$ as a constant of Θ. Assume that, if $\hat{t}$ is a constant, then $\hat{t}^{\mathfrak{A}} := t$, and if $f \in \mathcal{F}_n$, then $f^{\mathfrak{A}}(t_1, \dots, t_n) := f(t_1, \dots, t_n)$. From this, it follows that, for any $t \in U$ and any assignment s, $[\![t]\!]^{\mathfrak{A}}_s = t$. On the other hand, if $P \in \mathcal{P}_n$, assume that the mapping $P^{\mathfrak{A}}$ is defined as follows: $P^{\mathfrak{A}}(t_1, \dots, t_n) = [P(t_1, \dots, t_n)]_M$. It is not hard to see that $[\![\alpha]\!]^{\mathfrak{A}}_s = [\alpha]_M$ for every closed formula α and every s. Now, if we have $[\![\psi(x/h)]\!]^{\mathfrak{A}}_s$, then we define $s(x) = [\![h]\!]^{\mathfrak{A}}_s$ where h is a term or a constant given by a term. So, it is not hard to see that $[\![\psi(x/\hat{t})]\!]^{\mathfrak{A}}_s = [\![\psi(x/t)]\!]^{\mathfrak{A}}_s$ for every formula $\psi(x)$ and every term t.

Suppose now that α is $\exists x \beta$. By axiom (`Ax14`), for every $t \in U$, $\beta(x/\hat{t}) \to \alpha \in M$ and so $[\beta(x/\hat{t})]_M \leq [\alpha]_M$. By induction hypothesis, we have that $[\beta(x/\hat{t})]_M = [\![\beta(x/\hat{t})]\!]^{\mathfrak{A}}_s = [\![\beta(x/t)]\!]^{\mathfrak{A}}_s$. Thus, $[\beta(x/t)]_M \leq [\alpha]_M$, for every $t \in U$. Now, let ψ be a sentence such that $[\beta(x/t)]_M \leq [\psi]_M$

for every term $t \in U$ and so $[\beta(x/\hat{t})]_M \leq [\psi]_M$ for every term $t \in U$. In particular, $[\beta(x/\hat{x})]_M \leq [\psi]_M$ and then, $[\beta(x)]_M \leq [\psi]_M$. This means that $\beta(x) \to \psi \in M$. Since x does not occur freely in ψ then, by ($\exists$In), $\alpha \to \psi \in M$. This means that $[\alpha]_M \leq [\psi]_M$ and so $[\alpha]_M = \bigvee_{t \in U}[\beta(x/t)]_M$. Analogously, but now by using (Ax15) and ($\forall$In), it is proved that $[\![\alpha]\!]^{\mathfrak{A}}_s = [\alpha]_M$ for $\alpha = \forall x\beta$. So, $[\![\cdot]\!]^{\mathfrak{A}}_s$ is an interpretation associated with s. Besides, $[\![\alpha]\!]^{\mathfrak{A}}_\mu = [\![\alpha]\!]^{\mathfrak{A}}_s$ for every closed formula $\alpha \in M$ and every assignment μ. Thus, $\mathfrak{A}$ is a Θ-structure for $\mathcal{Q}\mathbb{L}$ such that, for every closed formula α, α is true in $\mathfrak{A}$ iff $\alpha \in M$. From this we have that $\Gamma \nvDash \varphi$. ⊣

Given a formula α, the set of variables that occur freely in α is $\{x_1, \ldots, x_n\}$ The *universal closure* of α is the closed formula $(\forall\alpha)$ given by α, if $n = 0$, or otherwise $\forall x_1 \ldots \forall x_n \alpha$. Then, the completeness theorem of $\mathcal{Q}\mathbb{L}$ for arbitrary formulas can now be easily obtained from the last result:

Theorem 4.6 (Completeness of $\mathcal{Q}\mathbb{L}$ w.r.t. the class of $\mathbf{CG_3'}$-structures). *For any set Γ of formulas and any formula φ over Θ, $\Gamma \vDash \varphi$ implies that $\Gamma \vdash_{\mathbb{L}} \varphi$.*

Proof. By (Ax15) and ($\forall$In) it is easy to prove that $\alpha \vdash_{\mathbb{L}} (\forall)\alpha$ and $(\forall)\alpha \vdash_{\mathbb{L}} \alpha$, for every formula α. On the other hand, by definition of $\vDash$, it is not hard to see that $\alpha \vDash \alpha$ and $(\forall)\alpha \vDash \alpha$, for every formula α. Then, for every $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathbb{L}} \varphi$ iff $(\forall)\Gamma \vdash_{\mathbb{L}} (\forall)\varphi$, and $\Gamma \vDash \varphi$ iff $(\forall)\Gamma \vDash (\forall)\varphi$, where $(\forall)\Gamma = \{(\forall)\beta \ : \ \beta \in \Gamma\}$. Thus, the desired result follows immediately from Theorem 4.5. ⊣

5. Conclusions and further research perspectives

In this paper, we described a semantics by means of Fidel structures for the propositional and the first-order version of $\mathbb{L}$ showing that for the propositional case these structures are in fact a generalization of the bi-valuations semantics. This work opens the possibility of applying all the machinery developed for the bi-valuation semantics [see 5] to the scope of Fidel structures. In contrast, we have shown that for other logics as C_ω and their non-algebraizable extension given in [27], the relation between Fidel structures and bi-valuations semantics simply does not hold.

In this setting, we can say that the model theory for the first-order version of the logic **J3** was developed by D'Ottaviano in [12, 13]. This was generalized to n-valued Łukasiewicz logic in [22, 23], where one can see, among others, the proof of Craig Interpolation Theorem. These studies are given using algebraic technique tools. On the other hand, it is not hard to see that **J3** is equivalent to $\mathbb{L}$. As future work, we are interested in exploring the model theory using Fidel structures for the first-order version of $\mathbb{L}$.

Other studies of Model Theory for first-order Intuitionistic logic were given by Fitting in [18] by means of Kripke semantics; in particular, the proof of Craig Interpolation Theorem was presented. It is possible to see that $\mathbb{L}$ has a Kripke semantics applying the result from [3], then we are interested in presenting semantics for the first-order version of $\mathbb{L}$ and developing its model theory. Clearly, we first need to give a corresponding Adequacy theorem for $\mathcal{Q}\mathbb{L}$ w.r.t. Kripke-like semantics.

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