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## EF4, EF4-M and EF4-Ł: A Companion to BN4 and two Modal Four-Valued Systems Without Strong Łukasiewicz-Type Modal Paradoxes

**Abstract.** The logic BN4 was defined by R.T. Brady as a four-valued extension of Routley and Meyer’s basic logic B. The system EF4 is defined as a companion to BN4 to represent the four-valued system of (relevant) implication. The system Ł was defined by J. Łukasiewicz and it is a four-valued modal logic that validates what is known as strong Łukasiewicz-type modal paradoxes. The systems EF4-M and EF4-Ł are defined as alternatives to Ł without modal paradoxes. This paper aims to define a Belnap-Dunn semantics for EF4, EF4-M and EF4-Ł. It is shown that EF4, EF4-M and EF4-Ł are strongly sound and complete w.r.t. their respective semantics and that EF4-M and EF4-Ł are free from strong Łukasiewicz-type modal paradoxes.

**Keywords:** relevant logics; modal logics; many-valued logics; Belnap-Dunn semantics; modal paradoxes; 4-valued modal logics

### 1. Introduction

One of Jan Łukasiewicz’s latest works was the definition of the system Ł. This system is a four-valued modal logic that, for him, was the definitive modal and many-valued system, as it encompassed all that he sought throughout his life-long work. Despite Łukasiewicz’s efforts, the system was somewhat short-lived as multiple voices claimed that it was deeply flawed [12]. The main reason for this was that it verified what are generally known as strong Łukasiewicz-type modal paradoxes. These paradoxes include theses such as  $(MA \wedge MB) \rightarrow M(A \wedge B)$  or  $L(A \vee B) \rightarrow (LA \vee LB)$ . The system received a backlash so substantial that it has even been said that it pushes the boundaries of what the no-

tion of modality is [cf. 14]. Nevertheless, Łukasiewicz disdained all the claims that arose just by saying that those paradoxes were, in fact, characteristic of what he considered to be the perfect modality, the one of his system  $\mathbb{L}$  [15]. It is easy to see how  $\mathbb{L}$  became forgotten quite quickly after its introduction and was regarded as an extravagance at best. However, it is worth mentioning that the interest in this system has recently increased, as it can be seen in [11], where the authors explore Łukasiewicz's logic, or [18], where a new interpretation for the system is given.

It was back in [7] where Brady introduced his famous system BN4. This system is pretty well known and have been studied in depth, but yet it is of interest. In particular, the characteristic matrix of BN4 was defined as an expansion of Smiley's four-valued matrix, which in its turn is the characteristic matrix for the system First Degree Entailment (FDE). Furthermore, according to [9], Smiley's matrix would be a simplification of N. Belnap's eight-valued matrix [cf. 3]. In Brady's words, the system BN4 was meant to be a four-valued extension of B, Routley-Meyer's basic logic, that would act as the four-valued logic of the relevant conditional [cf. 7]. Also, in accordance with J. Slaney, BN4 would be the adequate extension of FDE in the case that one wanted to endow it with a relevant conditional [cf. 24].

As a companion to the aforementioned BN4, the system E4 was introduced by G. Robles and J. M. Méndez in [22]. According to them, this system is related to BN4 just as E, the system of (relevant) entailment, relates to R. Therefore, E4 is a four-valued extension of reductioless E. Even though the importance of E4 is out of the question, it was Slaney himself who mentioned to Robles and Méndez that E4 might not be the perfect companion to BN4. This is mostly because the characteristic matrix of E4 cannot be divided into two three-valued matrices unlike the characteristic matrix of BN4. Additionally, Slaney wonders if the logic defined by the matrix M4 could be a better companion to BN4 than E4 [cf. 22]. And EF4 is the system determined by the matrix M4.

This matrix M4 has the same source as the characteristic matrix of BN4, except for the conditional function. This matrix would support EF4 as the four-valued system of (relevant) entailment. Thus, if BN4 is the four-valued system of relevant conditional, EF4 is its companion, the four-valued system of (relevant) entailment; just as E acts with respect to R. It is also interesting to mention that BN4 can be seen as a four-valued contractionless version of R whereas EF4 has the attribute of being a four-valued reductioless version of E.

At the same time, EF4 will act as the base system to introduce two different modal systems: EF4-M and EF4-Ł. These two systems are defined as modal expansions of EF4. The first one, EF4-M, is built upon the modal notions that A. Monteiro introduced. These notions were investigated by J.M. Font and M. Rius in [10], and have more recently been brought again into the spotlight by J.Y. Beziau in [6]. The second system, EF4-Ł, is based upon the inherent modal notion of EF4, that is to say, the modality inherent to E [cf. 1]. Furthermore, in the case of EF4-Ł, the inherent modality of E happens to be equivalent to the notion of modality that Łukasiewicz and A. Tarski developed for the many-valued systems of the former [cf. 15], thus granting EF4-Ł an exclusive perspective on the modality of E and the modality of Ł.

The main aim of this paper is to introduce the systems EF4, EF4-M, EF4-Ł. To do so, we will introduce the logical matrix M4, the characteristic matrix of EF4, and its inherent semantics. Also, we will endow the systems with a Belnap-Dunn bivalent semantics. This semantics, while one might say that this semantics provides a motivation for BN4 [cf. 21], it is also a really good tool for addressing the implicative expansions of FDE, as was done in [16]. Therefore, it is appealing to use it in order to facilitate our investigation of the systems. The last result of this paper is to show that both EF4-M and EF4-Ł are free from the strong Łukasiewicz-type modal paradoxes that were at the heart of the system Ł.

To sum this up, the structure of the paper is as follows. This first introductory section is followed up by a second section focused on the logical matrix M4. In that section, the notion of a logical matrix will be defined as well as the matrix M4. Moreover, its semantics is introduced as well as the notions of consequence and validity. In Section 3 we will introduce the system EF4 in an Hilbert-style axiomatic fashion. Afterwards, the system will be endowed with the aforementioned Belnap-Dunn bivalent semantics which, in its turn, will be shown to be equivalent to the inherent semantics of the matrix M4. Once the equivalence between both semantics has been proven, we can easily show that EF4 is strongly sound w.r.t. the aforementioned semantics. The final part of that section is meant to show that the system is complete in a strong sense. In order to do this, we will introduce an extension lemma and the canonical model, based on the Belnap-Dunn bivalent semantics. As for the last and fourth section, we will introduce the logical matrices  $\mathcal{M}M4$  and  $\mathcal{L}M4$ , the characteristic matrices of EF4-M and EF4-Ł. Obviously, the systems will be defined too and endowed with Belnap-Dunn bivalent

semantics. We will show how, similarly to EF4, both systems are sound and complete in a strong sense. To conclude, we will show how none of the strong Łukasiewicz-type modal paradoxes holds for any of the modal systems defined. There is a section for conclusions at the end of the paper to sum up all that has been done.

Before getting into the second section, we will define a series of basic concepts:

**DEFINITION 1.1** (Basic concepts). A propositional language is a enumerable set of propositional variables  $p_0, p_1, \dots, p_n, \dots$ , and every or a few of the following connectives  $\rightarrow$  (entailment),  $\wedge$  (conjunction),  $\vee$  (disjunction),  $\neg$  (negation),  $\mathbf{M}$  (possibility) and  $\mathbf{L}$  (necessity). Well-formed formulae (wffs) and sets of them are defined in the customary way.

## 2. The matrix M4 and its semantics

This second section introduces the matrix M4 and its inherent semantics. First of all we introduce the notion of a logical matrix and afterwards we define the matrix M4.

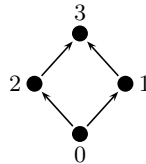
### 2.1. The matrix M4

**DEFINITION 2.1.** A logical matrix  $M$  is a structure  $\langle K, T, F, f \rangle$ , where:

- (i)  $K$  is a non-empty set;
- (ii)  $T$  (the set of designated elements) and  $F$  (the set of non-designated elements) are two non-empty subsets of  $K$  such that  $T \cup F = K$  and  $T \cap F = \emptyset$ ;
- (iii)  $f$  is the set of  $n$ -ary functions on  $K$  which are used to interpret the connectives in  $M$  such that for each  $n$ -ary connective  $c$  there is a function  $f_c \in f$  such that  $f_c: K^n \rightarrow K$ .

**DEFINITION 2.2.** The matrix M4 is a logical matrix of the form  $\langle K_{M4}, T_{M4}, F_{M4}, f_{M4} \rangle$ , where:

- (i)  $K_{M4} = \{0, 1, 2, 3\}$  and is partially ordered as follows:



- (ii)  $T_{M4} = (2, 3)$  and  $F_{M4} = (0, 1)$ ;
- (iii)  $f_{M4} = \{f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\neg}\}$ , where for all  $a, b \in K$ :  $f_{\wedge} := \min(a, b)$ ;  
 $f_{\vee} := \max(a, b)$ ; and

$$f_{\rightarrow}(a, b) := \begin{cases} 3 & \text{if either } a = 1 = b \text{ or both } a = 0 \text{ and } b = 3 \\ 2 & \text{if } a = 2 = b \\ 0 & \text{in any other cases} \end{cases}$$

$$f_{\neg}(a) := \begin{cases} 3 & \text{if } a = 0 \\ 2 & \text{if } a = 2 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } a = 3 \end{cases}$$

Thus, the connectives are expressed in the following truth-tables:

$\wedge$	0	1	2	3	$\vee$	0	1	2	3	$\rightarrow$	0	1	2	3	$\neg$	
0	0	0	0	0	0	0	1	2	3	0	3	3	3	3	0	3
1	0	1	0	1	1	1	1	3	3	1	0	3	0	3	1	1
2	0	0	2	2	2	2	3	2	3	2	0	0	2	3	2	2
3	0	1	2	3	3	3	3	3	3	3	0	0	0	3	3	0

*Remark 2.3.* For the matrix M4, the truth value 0 can be understood as false, the truth value 1 can be understood as neither true nor false, the truth value 2 can be understood as true and false at the same time, and the truth value 3 can be understood as true.

*Remark 2.4* (Divisibility of M4). The conditional truth-table of the matrix M4 can be divided into two different three-valued truth-tables. The first one is built using the truth values 0, 1 and 3, obtaining the truth-table corresponding to the conditional of the matrix MS5<sub>3</sub><sup>L</sup> [17]. The second one uses truth values 0, 2 and 3, generating the truth table of the conditional of the matrix MRM3 [7]. Each matrix is characteristic of the system whose name it bears. These truth-tables are as follows:

MS5 <sub>3</sub> <sup>L</sup>	0	1	3	MRM3	0	2	3
0	3	3	3	0	3	3	3
1	0	3	3	2	0	2	3
3	0	0	3	3	0	0	3

*Remark 2.5.* For all wffs  $A$  and  $B$ , the following are the characteristic theses and the rule of the matrix M4. Let it be noted that, here and on

the rest of the paper, we use the dot ( $\bullet$ ) to point out the main conditional of the wff:

- t1.  $A \wedge B \rightarrow A \quad A \wedge B \rightarrow B$
- t2.  $\neg(A \wedge B) \rightarrow \bullet \neg A \vee \neg B$
- t3.  $A \rightarrow A \vee B \quad B \rightarrow A \vee B$
- t4.  $\neg(A \vee B) \rightarrow \neg A \quad \neg(A \vee B) \rightarrow \neg B$
- t5.  $\neg A \wedge \neg B \rightarrow \bullet \neg(A \vee B)$
- t6.  $\neg A \rightarrow \neg A$
- t7.  $\neg\neg A \rightarrow A$
- t8.  $A \rightarrow \neg\neg A$
- t9.  $(A \vee \neg B) \vee (A \rightarrow B)$
- t10.  $\neg A \rightarrow \bullet A \vee (A \rightarrow B)$
- t11.  $B \rightarrow \bullet \neg B \vee (A \rightarrow B)$
- t12.  $\neg A \wedge B \rightarrow \bullet A \rightarrow B$
- t13.  $\neg(A \rightarrow B) \wedge \neg A \rightarrow \bullet A$
- t14.  $(\neg(A \rightarrow B) \wedge \neg A) \wedge B \rightarrow \bullet A$
- t15.  $\neg(A \rightarrow B) \rightarrow \bullet \neg B \vee A$
- t16.  $\neg(A \rightarrow B) \wedge B \rightarrow \bullet \neg B$
- t17.  $\neg B \rightarrow \bullet \neg A \vee \neg(A \rightarrow B)$
- t18.  $A \rightarrow \bullet B \vee \neg(A \rightarrow B)$
- r1.  $A, \neg B \Rightarrow \neg(A \rightarrow B)$ .

## 2.2. M4-semantics

After defining the matrix M4, now we can introduce its intrinsic four-valued semantics.

**DEFINITION 2.6** (M4-interpretation). An M4-interpretation,  $I_{M4}$ , is defined as a function from the set of wffs to  $K_{M4}$ , following Definition 2.2.

**DEFINITION 2.7** (M4-consequence and M4-validity). For any set of wffs  $\Gamma$  and any wff  $A$ ,  $A$  is a consequence of  $\Gamma$  within the M4-semantics, in symbols  $\Gamma \models_{M4} A$ , iff  $2$  or  $3 \in I_{M4}(A)$  as long as  $2$  or  $3 \in I_{M4}(\Gamma)$  for every M4-interpretation  $I_{M4}$ ,  $I_{M4}(\Gamma) = \inf\{I_{M4}(B) \mid B \in \Gamma\}$ . Specifically,  $A$  is valid within the M4-semantics, in symbols  $\models_{M4} A$ , iff  $2$  or  $3 \in I_{M4}(A)$  for every M4-interpretation  $I_{M4}$ . By  $\models_{M4}$  we refer to the relation we have just defined.

### 3. The four-valued system of (relevant) entailment: EF4

In this section we introduce the system EF4 and endow it with a bivalent Belnap-Dunn (BD) semantics. We will prove that the M4-semantics is equivalent to the BD-semantics for EF4. Afterwards we will give a soundness proof in the strong sense and introduce the canonical models for the BD-semantics. Then we will prove an extension lemma that is necessary for us to obtain completeness of the system EF4 in a strong sense.

#### 3.1. The system EF4

Firstly, we define what a logic is, then we introduce the axiomatization of the system EF4 and display some basic definitions related to it.

DEFINITION 3.1. A logic  $S$  is a structure  $\langle \mathcal{L}, \vdash_S \rangle$ , where  $\mathcal{L}$  is a propositional language (cf. Definition 1.1) and  $\vdash_S$  is a consequence relationship defined over  $\mathcal{L}$  by a set of axioms and derivation rules. The notions of proof and theorem are the customary ones for axiomatic Hilbert-style systems; i.e., for any set of wffs  $\Gamma$  and any wff  $A$ ,  $\Gamma \vdash_S A$  means that  $A$  follows from  $\Gamma$  in  $S$  and  $\vdash_S A$  means that  $A$  is a theorem of  $S$ .

The axiomatization of EF4 is as follows:

- A1.  $A \rightarrow A$
- A2.  $A \wedge B \rightarrow A \quad A \wedge B \rightarrow B$
- A3.  $A \rightarrow A \vee B \quad B \rightarrow A \vee B$
- A4.  $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow \bullet A \rightarrow (B \wedge C)$
- A5.  $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow \bullet (A \vee B) \rightarrow C$
- A6.  $A \wedge (B \vee C) \rightarrow \bullet (A \wedge B) \vee (A \wedge C)$
- A7.  $A \rightarrow B \rightarrow \bullet (B \rightarrow C) \rightarrow (A \rightarrow C)$
- A8.  $A \rightarrow B \rightarrow \bullet (C \rightarrow A) \rightarrow (C \rightarrow B)$
- A9.  $A \rightarrow \neg B \rightarrow \bullet B \rightarrow \neg A$
- A10.  $\neg\neg A \rightarrow A$
- A11.  $A \rightarrow (A \rightarrow B) \rightarrow \bullet A \rightarrow B$
- A12.  $[(A \rightarrow A) \wedge (B \rightarrow B)] \rightarrow C \rightarrow \bullet C$
- A13.  $(A \vee \neg B) \vee (A \rightarrow B)$
- A14.  $B \wedge \neg(A \rightarrow B) \rightarrow \bullet \neg B$
- A15.  $\neg(A \rightarrow B) \rightarrow \bullet A \vee \neg B$
- R1.  $A, B \Rightarrow A \wedge B$
- R2.  $A \rightarrow B, A \Rightarrow B$
- R3.  $C \vee A, C \vee \neg B \Rightarrow C \vee \neg(A \rightarrow B)$

This axiomatization can be seen as a four-valued extension of reductioless E. Let it be noted that the Reductio axiom is  $(A \rightarrow \neg A) \rightarrow \neg A$ . The inclusion of the disjunctive version of Counterexample, R3, is necessary for the proof of the extension lemma as can be seen later. The axioms of EF4 are independent of each other with the exception of A13. This can be verified thanks to [23].

As an alternative to E4, it is worth noting that this axiomatization mainly differs from the one of E4 in A15, a characteristic thesis of the matrix M4. On the other hand, all of E4 axioms hold in EF4 with the exception of  $\neg(A \rightarrow B) \wedge (A \wedge \neg B) \rightarrow (A \rightarrow B)$ . This is due to the fact that both systems are based upon E. It is also interesting to mention that both EF4 and E4 (and for what matters, most four-valued systems of any interest like BN4) validate theses of the form  $A \rightarrow \neg A \rightarrow A$ , either as theorems or as derivation rules. When these theses are validated as theorems, they are considered to be modal fallacies by Anderson and Belnap as they violate the Ackermann Property [1]. The main reason for these theses to appear in EF4 as theorems is the fact that it verifies the famous Mingle axiom,  $A \rightarrow A \rightarrow A$ , while E4 does not.

Now we define a series of notions that are basic for the system.

**DEFINITION 3.2 (EF4-derivability).** For any set of wffs  $\Gamma$  and any wff  $A$ ,  $A$  follows from  $\Gamma$  in EF4, in symbols  $\Gamma \vdash_{\text{EF4}} A$ , iff there is a finite sequence of wffs  $B_1, \dots, B_n$  such that  $B_n$  is  $A$  and for every other  $B_i$  such that  $1 \leq i \leq n$  it corresponds to one of the following cases: (i)  $B_i \in \Gamma$ ; (ii)  $B_i$  is one of the axioms of EF4; (iii)  $B_i$  is the outcome of applying one derivation rule to one or more of the previous wff.

**DEFINITION 3.3 (Disjunctive EF4-derivability).** For any sets of wffs  $\Gamma$  and  $\Theta$ ,  $\Theta$  is disjunctively EF4-derivable from  $\Gamma$ , in symbols  $\Gamma \vdash_{\text{EF4}}^d \Theta$ , iff  $A_1 \wedge \dots \wedge A_m \vdash_{\text{EF4}} B_1 \vee \dots \vee B_n$  for some wffs  $A_1, \dots, A_m \in \Gamma$  and  $B_1, \dots, B_n \in \Theta$ , as long as  $m, n \geq 1$ .

**Some theorems of EF4.** The following wffs are EF4-theorems as they follow from the axiomatization or, in some cases, from other EF4-theorems:

- Idempotency of Disjunction,  $A \vee A \rightarrow A$ , follows from A1 and A5;
- Transitivity,  $A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$ , follows from A7 and R2;
- Associativity of Conjunction,  $(A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$ , follows from A2, A4, and Transitivity;
- Associativity of Disjunction,  $(A \vee B) \vee C \rightarrow A \vee (B \vee C)$ , follows from A3, A5 and Transitivity;



- Distribution (I),  $(A \vee B) \wedge C \rightarrow \bullet A \vee (B \wedge C)$ , follows from A2, A3, A4, A5, A6, and Transitivity;
- Distribution (II),  $(A \vee B) \wedge (A \vee C) \rightarrow \bullet A \vee (B \wedge C)$ , follows from A2, A3, A4, A5, A6, Transitivity and Distribution (I);
- Distribution (III),  $A \vee (B \wedge C) \rightarrow \bullet (A \vee B) \wedge (A \vee C)$ , follows from A2, A3, A4, A5, and Transitivity;
- Import,  $A \rightarrow (B \rightarrow C) \rightarrow \bullet (A \wedge B) \rightarrow C$ , follows from A2, A7, A8, A11 and Transitivity;
- Modus Ponens,  $(A \rightarrow B) \wedge A \rightarrow \bullet B$ , follows from A1 and Import;
- Modus Tollens,  $(A \rightarrow B) \wedge \neg B \rightarrow \bullet \neg A$ , follows from A9 and Import;
- Double Negation,  $A \rightarrow \neg \neg A$ , follows from A1 and R2;
- Contraposition,  $\neg A \rightarrow B \rightarrow \bullet \neg B \rightarrow A$ , follows from A8, A9, A10 and Transitivity;
- De Morgan (I),  $\neg(A \vee B) \rightarrow \bullet \neg A \wedge \neg B$ , follows from A3, A4 and A9;
- De Morgan (II),  $\neg(A \wedge B) \rightarrow \bullet \neg A \vee \neg B$ , follows from A3, A4, A10 and Contraposition;
- Summation,  $A \rightarrow B \Rightarrow C \vee A \rightarrow \bullet C \vee B$ , follows from A3, A5 and Transitivity;
- Disjunctive Modus Ponens,  $(C \vee (A \rightarrow B)) \wedge (C \vee A) \rightarrow \bullet C \vee B$ , follows from Modus Ponens, Summation and Distribution (III);
- Counterexample,  $A, \neg B \Rightarrow \neg(A \rightarrow B)$ , follows from A3, R2, R3 and Idempotency of Disjunction;
- Product,  $A \rightarrow B \Rightarrow C \wedge A \rightarrow \bullet C \wedge B$ , follows from A2, A4, and Transitivity;
- Conjunction's Commutative Property,  $A \wedge B \rightarrow \bullet B \wedge A$ , follows from A2 and A4;
- Disjunction's Commutative Property,  $A \vee B \rightarrow \bullet B \vee A$ , follows from A3 and A5.

In order to conclude the introduction of the system EF4, we will prove that the characteristic theses of matrix M4 are part of the axiomatization of EF4 that we have displayed above.

**THEOREM 3.4** (the matrix M4 and the axiomatization of EF4). *The characteristic theses of the matrix M4 from Remark 2.5 are part of the axiomatization of EF4.*

**PROOF.** We have to show that all the theorems and the rule of Remark 2.5 follow from the axiomatization of EF4: t1 follows from A2; t2 from De Morgan (II); t3 from A3; t4 from A3 and A9; t5 from De Morgan

(I); t6 from A1; t7 from A10; t8 from Double Negation; t9 from A13; t10 from A10, t13, A9 and De Morgan (II); t11 from A14, A9, Double Negation and De Morgan (II); t12 from A10, A15, A9, Double Negation and De Morgan (I); t13 from A10, A14, A9, Product and Transitivity; t14 from A2, t13 and Transitivity; t15 from A15; t16 from A14; t17 from A9, Modus Ponens and De Morgan (II); t18 from A10, A9, Double Negation, De Morgan (II) and Modus Tollens; Finally, r1 follows from R3.  $\square$

### 3.2. Bivalent Belnap-Dunn Semantics

Right now, the goal is to introduce the BD-semantics for EF4. It is important to note that the BD-semantics does not have a clause for entailment per se, so we add one in order to be able to deal with the entailment of EF4. For more on this semantics the reader should check out [8].

DEFINITION 3.5 (BD-models). A BD-model is a structure  $\langle K_{\text{BD}}^4, I_{\text{BD}} \rangle$ , where  $K_{\text{BD}}^4 = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$  and  $I_{\text{BD}}$  is an interpretation from the set of wffs to  $K_{\text{BD}}^4$ . Each  $I_{\text{BD}}$  assigns one element of  $K_{\text{BD}}^4$  to each propositional variable. Wff meet the following clauses:

- (I) Conjunction:
  - (a)  $F \in I_{\text{BD}}(A \wedge B)$  iff  $F \in I_{\text{BD}}(A)$  or  $F \in I_{\text{BD}}(B)$ ;
  - (b)  $T \in I_{\text{BD}}(A \wedge B)$  iff  $T \in I_{\text{BD}}(A)$  and  $T \in I_{\text{BD}}(B)$ ;
- (II) Disjunction:
  - (a)  $F \in I_{\text{BD}}(A \vee B)$  iff  $F \in I_{\text{BD}}(A)$  and  $F \in I_{\text{BD}}(B)$ ;
  - (b)  $T \in I_{\text{BD}}(A \vee B)$  iff  $T \in I_{\text{BD}}(A)$  or  $T \in I_{\text{BD}}(B)$ ;
- (III) Negation:
  - (a)  $F \in I_{\text{BD}}(\neg A)$  iff  $T \in I_{\text{BD}}(A)$ ;
  - (b)  $T \in I_{\text{BD}}(\neg A)$  iff  $F \in I_{\text{BD}}(A)$ .

DEFINITION 3.6 (BD-consequence and BD-validity).  $A$  follows from  $\Gamma$  in a BD-model  $M$ , in symbols  $\Gamma \models_{\text{BD-}M} A$ , iff  $T \in I_{\text{BD}}(A)$  as long as  $T \in I_{\text{BD}}(B)$  for any  $B \in \Gamma$ . Particularly,  $A$  is true in  $M$ , in symbols  $\models_{\text{BD-}M} A$ , iff  $T \in I_{\text{BD}}(A)$ . Then,  $A$  follows from  $\Gamma$  in BD-semantics, in symbols  $\Gamma \models_{\text{BD}} A$ , iff  $\Gamma \models_{\text{BD-}M} A$  in any BD-model  $M$ . In particular,  $A$  is valid in the BD-semantics, in symbols  $\models_{\text{BD}} A$ , iff  $\models_{\text{BD-}M} A$  for any BD-model  $M$ . By  $\models_{\text{BD}}$  we are referring to the relation we have just defined.

DEFINITION 3.7 (Entailment clause). The following is the corresponding clause for the entailment of EF4:

- (i)  $F \in I_{\text{BD}}(A \rightarrow B)$  iff either  $T \in I_{\text{BD}}(A)$ ,  $F \in I_{\text{BD}}(B)$ , or  $F \notin I_{\text{BD}}(A)$ ,  $F \in I_{\text{BD}}(B)$ , or  $T \in I_{\text{BD}}(A)$ ,  $T \notin I_{\text{BD}}(B)$ ;

- (ii)  $T \in I_{\text{BD}}(A \rightarrow B)$  iff both either  $T \notin I_{\text{BD}}(A)$  or  $T \in I_{\text{BD}}(B)$ , and either  $F \notin I_{\text{BD}}(B)$  or  $F \in I_{\text{BD}}(A)$ .

It is necessary to make clear that, despite appearing individually, the clause for entailment of Definition 3.7 is part of the BD-models introduced in Definition 3.5.

### 3.3. The equivalence of the BD- and M4-semantics

In order to prove the equivalence of the BD and M4-semantics, we need to introduce a series of definitions and lemmas.

DEFINITION 3.8. Let  $I_{\text{M4}}$  be an M4-interpretation. Then we define a corresponding BD-interpretation  $I_{\text{BD}}$  as follows: for every propositional variable  $p_i$ ,

- (i)  $I_{\text{M4}}(p_i) = 0$  iff  $I_{\text{BD}}(p_i) = \{F\}$  and  $I_{\text{BD}}(p_i) \neq \{T\}$ ;
- (ii)  $I_{\text{M4}}(p_i) = 1$  iff  $I_{\text{BD}}(p_i) = \emptyset$ ;
- (iii)  $I_{\text{M4}}(p_i) = 2$  iff  $I_{\text{BD}}(p_i) = \{T, F\}$ ;
- (iv)  $I_{\text{M4}}(p_i) = 3$  iff  $I_{\text{BD}}(p_i) = \{T\}$  and  $I_{\text{BD}}(p_i) \neq \{F\}$ .

LEMMA 3.9 (Correspondence of  $I_{\text{M4}}$  with respect to  $I_{\text{BD}}$ ). *Given Definition 3.8, we extend the equivalence of propositional variables to wffs. For any wff  $A$ , it follows:*

- (i)  $I_{\text{M4}}(A) = 0$  iff  $I_{\text{BD}}(A) = \{F\}$  and  $I_{\text{BD}}(A) \neq \{T\}$ ;
- (ii)  $I_{\text{M4}}(A) = 1$  iff  $I_{\text{BD}}(A) = \emptyset$ ;
- (iii)  $I_{\text{M4}}(A) = 2$  iff  $I_{\text{BD}}(A) = \{T, F\}$ ;
- (iv)  $I_{\text{M4}}(A) = 3$  iff  $I_{\text{BD}}(A) = \{T\}$  and  $I_{\text{BD}}(A) \neq \{F\}$ .

PROOF. The proof goes by induction on the complexity of wffs as it is shown in [16, 22]. Additionally, the correspondence for wffs can be easily extended to sets of wff.  $\square$

PROPOSITION 3.10 (Correspondence of  $I_{\text{BD}}$  with respect to  $I_{\text{M4}}$ ). *Given a BD-interpretation,  $I_{\text{BD}}$ , the corresponding M4-interpretation,  $I_{\text{M4}}$ , can be defined similarly as in Definition 3.8. Then, the correspondence between  $I_{\text{BD}}$  and  $I_{\text{M4}}$  can be proved similarly as in Lemma 3.9.*

THEOREM 3.11 (Equivalence of M4-validity and BD-validity). *The concepts of BD-validity, from Definition 3.6, and M4-validity, from Definition 2.7, are equivalent.*

PROOF. For any wff  $A$ , we assume  $\models_{\text{M4}} A$ . Necessarily, we have  $\models_{\text{BD}} A$ , as, otherwise, if we had  $\not\models_{\text{BD}} A$ , there would be a BD-interpretation

$I_{\text{BD}}$  such that  $I_{\text{BD}}(A) \neq T$  and, by Proposition 3.10 there would be an M4-interpretation,  $I_{\text{M4}}$ , such that  $I_{\text{M4}}(A) = 0$  or  $1$ , contradicting our assumption. For the case where we assume  $\models_{\text{BD}} A$ , the proof goes similarly, applying Lemma 3.9 instead of Proposition 3.10. Furthermore, for any set of wffs  $\Gamma$ , we assume  $\Gamma \models_{\text{BD}} A$  and need to show  $\Gamma \models_{\text{M4}} A$ . Let  $I_{\text{M4}}$  be an M4-interpretation such that  $I_{\text{M4}}(\Gamma) = 2$  or  $3$  from which we need to show that  $I_{\text{M4}}(A) = 2$  or  $3$ . Then we define a corresponding BD-interpretation to the previous  $I_{\text{M4}}$ . Since we have that  $I_{\text{M4}}(\Gamma) = 2$  or  $3$ , then  $I_{\text{BD}}(\Gamma) = T$  necessarily follows by Lemma 3.9. And, from there,  $I_{\text{BD}}(A) = T$ . Finally, by Proposition 3.10 we have  $I_{\text{M4}}(A) = 2$  or  $3$ . For the case where we assume  $\Gamma \models_{\text{BD}} A$ , Lemma 3.16 is used instead of Lemma 3.18 and vice versa.  $\square$

After proving that the M4-semantics and the BD-semantics are equivalent, by induction on the length of formulas, we can now show that EF4 is a sound system in the strong sense.

**THEOREM 3.12** (Strong soundness for EF4). *For any set of wffs  $\Gamma$  and any wff  $A$ : If  $\Gamma \vdash_{\text{EF4}} A$ , then  $\Gamma \models_{\text{M4}} A$  and, accordingly,  $\Gamma \models_{\text{BD}} A$ .*

### 3.4. Canonical BD-models

Now we introduce the canonical EF4-models for the BD-semantics. We firstly define the EF4-theories, and then move onto the canonical models themselves.

**DEFINITION 3.13** (EF4-theories). An EF4-theory  $a$  is a set of wffs closed under EF4-entailment, Adjunction and Disjunctive Counterexample; i.e.,  $a$  is an EF4-theory iff for all wffs  $A$ ,  $B$  and  $C$ :

- (i) if  $A \rightarrow B$  is an EF4 theorem,  $\vdash_{\text{EF4}} A \rightarrow B$ , and  $A \in a$ , then  $B \in a$ ;
- (ii) if  $A \in a$  and  $B \in a$ , then  $A \wedge B \in a$ ;
- (iii) if  $C \vee A \in a$  and  $C \vee \neg B \in a$ , then  $C \vee \neg(A \rightarrow B) \in a$ .

**DEFINITION 3.14.** For any EF4-theory  $a$ :

- (i)  $a$  is prime iff for all wffs  $A$ ,  $B$ , if  $A \vee B \in a$ , then  $A \in a$  or  $B \in a$ ;
- (ii)  $a$  is regular iff all EF4 theorems belong to  $a$ ;
- (iii)  $a$  is a-consistent iff  $a$  is non-trivial; i.e.,  $a$  does not have all wffs.

Since all theories are closed under EF4-entailment and the system possesses the theorematic versions of Modus Ponens and Modus Tollens (see Subsection 3.1), we obtain:

PROPOSITION 3.15. *Let  $a$  be an EF4-theory. Then  $a$  is closed under Modus Ponens and Modus Tollens; i.e., for all wffs  $A, B$ , if  $A \rightarrow B \in a$  and  $A \in a$ , then  $B \in a$ ; and if  $A \rightarrow B \in a$  and  $\neg B \in a$ , then  $\neg A \in a$ , respectively.*

Moreover, by A10 and Double Negation (see Subsection 3.1), we have:

LEMMA 3.16 (EF4-Theories and Double Negation). *In any EF4-theory  $a$ , for any wff  $A$ :  $A \in a$  iff  $\neg\neg A \in a$ .*

LEMMA 3.17 (Conjunction and Disjunction in prime EF4-theories). *In any prime EF4-theory  $a$ , for all wffs  $A, B$ :*

- (i)  $A \wedge B \in a$  iff  $A \in a$  and  $B \in a$ ;
- (ii)  $\neg(A \wedge B) \in a$  iff  $\neg A \in a$  or  $\neg B \in a$ ;
- (iii)  $A \vee B \in a$  iff  $A \in a$  or  $B \in a$ ;
- (iv)  $\neg(A \vee B) \in a$  iff  $\neg A \in a$  and  $\neg B \in a$ .

PROOF. (i) Follows by A2 and the fact that  $a$  is closed under Adjunction. (ii) Follows by De Morgan (II) and the primeness of  $a$ . (iii) Follows by A3 and the fact that  $a$  is prime. (iv) Follows by De Morgan (I) and the closure of  $a$  under Adjunction.  $\square$

LEMMA 3.18 (Entailment in prime and regular EF4-theories). *In any prime and regular EF4-theory  $a$ , for all wffs  $A, B$ :*

- (i)  $A \rightarrow B \in a$  iff both either  $A \notin a$  or  $B \in a$ , and either  $\neg A \in a$  or  $\neg B \notin a$ ;
- (ii)  $\neg(A \rightarrow B) \in a$  iff either  $A \in a$ ,  $\neg B \in a$ , or  $\neg A \notin a$ ,  $\neg B \in a$ , or  $A \in a$ ,  $B \notin a$ .

PROOF. The properties of the EF4-theory  $a$ , such as its closure under EF4-entailment, will be used along the proof. Also, the theses t10–t12, t16–t18 from Remark 2.5, axioms A13–A15 of EF4, and the derivation rule Counterexample from Subsection 3.1 will be used.

(i) From left to right: We assume  $A \rightarrow B \in a$  and by reductio ad absurdum we have  $(\alpha)$   $A \in a$  and  $B \notin a$ , and  $(\beta)$   $\neg A \notin a$  and  $\neg B \in a$ , but both lead us to a contradiction, as  $a$  is closed under Modus Ponens and Modus Tollens, as it was shown back in Proposition 3.15.

From right to left we have four different cases:  $(\alpha)$   $A \notin a$  and  $\neg A \in a$ ,  $(\beta)$   $A \notin a$  and  $\neg B \notin a$ ,  $(\gamma)$   $B \in a$  and  $\neg A \in a$ , and  $(\delta)$   $B \in a$  and  $\neg B \notin a$ .  $(\alpha)$  follows by t10;  $(\beta)$  follows by A13;  $(\gamma)$  follows by t12;  $(\delta)$  follows by t11.

(ii) From left to right: We assume  $\neg(A \rightarrow B) \in a$ , and by reductio we have:  $(\alpha)$   $A \notin a$  and  $\neg A \in a$ ,  $(\beta)$   $A \notin a$ ,  $\neg A \in a$  and  $B \in a$ ,  $(\gamma)$   $A \notin a$  and  $\neg B \notin a$ ,  $(\delta)$   $A \notin a$ ,  $\neg B \notin a$  and  $B \in a$ ,  $(\varepsilon)$   $\neg B \notin a$ ,  $\neg A \in a$  and  $A \notin a$ ,  $(\zeta)$   $\neg B \notin a$ ,  $\neg A \in a$  and  $B \in a$ ; and  $(\eta)$   $\neg B \notin a$  and  $B \in a$ .  $(\alpha)$  follows by t16;  $(\beta)$  follows similarly to  $\alpha$ ;  $(\gamma)$  follows by A15;  $(\delta)$  follows by A14;  $(\varepsilon)$  follows similarly to the first two cases;  $(\zeta)$  follows similarly to  $\delta$ ;  $(\eta)$  follows similarly to the fourth case.

From right to left we have three different cases:  $(\alpha)$   $A \in a$  and  $\neg B \in a$ ,  $(\beta)$   $\neg A \notin a$  and  $\neg B \in a$ , and  $(\gamma)$   $A \in a$  and  $B \notin a$ .  $(\alpha)$  follows by the derivation rule Counterexample;  $(\beta)$  follows by t17;  $(\gamma)$  follows by t18.  $\square$

DEFINITION 3.19 ( $\tau$ -interpretation). Let  $K^{4c}$  be a set equivalent to  $K_{BD}^4$  (from Definition 3.11) and  $\tau$  a regular and prime EF4-theory. Then, a  $\tau$ -interpretation is a function from the set of wffs to  $K^{4c}$  such that for any propositional variable  $p_i$ :

- (i)  $F \in I_\tau(p_i)$  iff  $\neg p_i \in \tau$ ;
- (ii)  $T \in I_\tau(p_i)$  iff  $p_i \in \tau$ .

Additionally, for any wff  $A$ , a  $\tau$ -interpretation assigns an element from  $K^{4c}$  accordingly to Definitions 3.5 and 3.7.

DEFINITION 3.20 (BD-semantics canonical model). The BD-semantics canonical model is a structure  $\langle K, I_\tau^c \rangle$ , where  $K^{4c}$  is the set from Definition 3.19 and  $I_\tau^c$  is a  $\tau$ -interpretation.

The canonical model is a particular instance of the general structure BD-models constitute.

DEFINITION 3.21 (Canonical relation  $\models_\tau$ ). For any set  $\Gamma$  of wffs and any wff  $A$ , the canonical relation  $\models_\tau$  is defined as follows:  $\Gamma \models_\tau A$  iff  $T \in I_\tau(A)$ , as long as  $T \in I_\tau(B)$  for any  $B \in \Gamma$ . In particular,  $\models_\tau A$ ,  $A$  is valid in the canonical model, iff  $T \in I_\tau(A)$ .

Directly from Definitions 3.5, 3.7 and 3.20 we have:

THEOREM 3.22 (The canonical model is a BD-model). *The canonical model is a BD-model.*

Finally, we show that the propositional variables clauses can be extended into wffs  $\tau$ -interpretations.

**THEOREM 3.23** (Clauses extension into wffs  $\tau$ -interpretations). *In any prime and regular EF4-theory  $\tau$  (see Definition 3.19), for any wff  $A$ :*

- (i)  $F \in I_\tau(A)$  iff  $\neg A \in \tau$ ;
- (ii)  $T \in I_\tau(A)$  iff  $A \in \tau$ .

**PROOF.** The proof goes by induction on the length of the wffs. The proof is by Proposition 3.15 and Lemmas 3.16, 3.17, 3.18. Also, Lemma 7.5 from [22] can be consulted for an in-depth look at this proof.  $\square$

### 3.5. EF4 extension and primeness lemmas

The following are the extension and primeness lemmas that are going to be used for the completeness of EF4. But before we include an useful proposition from Anderson and Belnap's FDE.

**PROPOSITION 3.24.** *For arbitrary wffs  $B_1, \dots, B_n, B', C_1, \dots, C_m, C'$  and  $D$  we consider the following abbreviations:  $B := B_1 \wedge \dots \wedge B_n$ ,  $C := C_1 \vee \dots \vee C_m$ ,  $B'' := B \wedge B'$ ,  $C'' := C \vee C'$ . If  $B \vdash_{\text{EF4}} C \vee D$  and  $B' \wedge D \vdash_{\text{EF4}} C'$ , then  $B'' \wedge D \vdash_{\text{EF4}} C''$ ,  $B'' \vdash_{\text{EF4}} C'' \vee D$ ,  $B'' \vdash_{\text{EF4}} C'' \vee (B'' \wedge D)$ ,  $C'' \vee (B'' \wedge D) \vdash_{\text{EF4}} C'' \vee C$ ,  $C'' \vee (B'' \wedge D) \vdash_{\text{EF4}} C$  and  $B'' \vdash_{\text{EF4}} C''$ .*

**LEMMA 3.25.** *If  $B_1, \dots, B_n \vdash_{\text{EF4}} A$ , then  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee A$  for any wff  $C$ .*

**PROOF.** Assuming the hypothesis of the lemma, we proceed by induction on the length of the derivation of  $A$  from  $\{B_1, \dots, B_n\}$  in EF4. There are five different options: (i)  $A \in \{B_1, \dots, B_n\}$ : By A2 together with the commutative and associative properties of  $\wedge$  we have  $\vdash_{\text{EF4}} (B_1 \wedge \dots \wedge B_n) \rightarrow A$  and, by Summation and Proposition 3.24, we get  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee A$ . (ii)  $A$  is an axiom: Since  $A$  is an axiom we already have  $\vdash_{\text{EF4}} A$ . By A3 we get  $\vdash_{\text{EF4}} A \rightarrow C \vee A$  and, applying R2, we have  $\vdash_{\text{EF4}} C \vee A$ . Then, by Proposition 3.24,  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee A$ . (iii)  $A$  is derived using R1:  $A$  has the pattern  $D \wedge E$  for some wffs  $D, E$ . By the induction hypothesis we have  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee D$  and  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee E$ . By using R1, we get  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} (C \vee D) \wedge (C \vee E)$ , whence by Distribution over Conjunction we have:  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee (D \wedge E)$ . (iv)  $A$  is derived using R2: By the induction hypothesis,  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee (D \rightarrow A)$  and  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee D$  for some wff  $D$ . By using Disjunctive Modus Ponens, we have  $C \vee (B_1 \wedge \dots \wedge B_n) \vdash_{\text{EF4}} C \vee A$ . (v)  $A$  is derived

using R3:  $A$  has the pattern  $D \vee \neg(E \rightarrow F)$  for some wffs  $D$ ,  $E$  and  $F$ . By the induction hypothesis we have  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} C \vee (D \vee E)$  and  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} C \vee (D \vee \neg F)$ . By Associativity of Disjunction we get  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} (C \vee D) \vee E$  and  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} (C \vee D) \vee \neg F$ , and now, using R3, we get  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} (C \vee D) \vee \neg(E \rightarrow F)$ , and using again Associativity of Disjunction we get  $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_{\text{EF4}} C \vee (D \vee \neg(E \rightarrow F))$ .  $\square$

DEFINITION 3.26.  $\Gamma$  is a maximal set iff  $\Gamma \not\vdash_{\text{EF4}}^{\text{d}} \bar{\Gamma}$ , where  $\bar{\Gamma}$  is the complementary set to  $\Gamma$ .

LEMMA 3.27 (EF4 extension lemma). *Let  $\Gamma$  and  $\Theta$  be sets of wffs such that  $\Gamma \not\vdash_{\text{EF4}}^{\text{d}} \Theta$ . Then, there are sets of wffs  $\Gamma'$  and  $\Theta'$  such that  $\Gamma \subseteq \Gamma'$ ,  $\Theta \subseteq \Theta'$ ,  $\Theta' = \bar{\Gamma}'$  and  $\Gamma' \not\vdash_{\text{EF4}}^{\text{d}} \Theta'$ .*

PROOF. Let  $A_1, \dots, A_n, \dots$  be an enumeration of the set of wffs. Assuming the hypothesis of the lemma, the sets  $\Gamma'$  and  $\Theta'$  are defined as follows:  $\Gamma' = \bigcup_{k \in \mathbb{N}} \Gamma_k$  and  $\Theta' = \bigcup_{k \in \mathbb{N}} \Theta_k$ , where  $\Gamma_0 = \Gamma$  and  $\Theta_0 = \Theta$ . For every  $k \in \mathbb{N}$ ,  $\Gamma_{k+1}$  and  $\Theta_{k+1}$  are built according to one of the following options.

(I) If  $\Gamma_k \cup \{A_{k+1}\} \vdash_{\text{EF4}}^{\text{d}} \Theta_k$ , then  $\Gamma_{k+1} = \Gamma_k$  and  $\Theta_{k+1} = \Theta_k \cup \{A_{k+1}\}$ .  
 (II) If  $\Gamma_k \cup \{A_{k+1}\} \not\vdash_{\text{EF4}}^{\text{d}} \Theta_k$ , then  $\Gamma_{k+1} = \Gamma_k \cup \{A_{k+1}\}$  and  $\Theta_{k+1} = \Theta_k$ . As a consequence we have  $\Gamma \subseteq \Gamma'$ ,  $\Theta \subseteq \Theta'$  and  $\Gamma' \cup \Theta'$  is the set of wffs. We prove: (a)  $\Gamma_k \not\vdash_{\text{EF4}}^{\text{d}} \Theta_k$ , for every  $k \in \mathbb{N}$ . By reductio ad absurdum, we assume that, for any  $i \in \mathbb{N}$ , it follows: (b)  $\Gamma_i \not\vdash_{\text{EF4}}^{\text{d}} \Theta_i$  but  $\Gamma_{i+1} \vdash_{\text{EF4}}^{\text{d}} \Theta_{i+1}$ . Now we have to consider two different possibilities, the two possible ways of building  $\Gamma_{i+1}$  and  $\Theta_{i+1}$  from (I) and (II): (i)  $\Gamma_i \cup \{A_{i+1}\} \not\vdash_{\text{EF4}}^{\text{d}} \Theta_i$ . By (II) we have  $\Gamma_{i+1} = \Gamma_i \cup \{A_{i+1}\}$  and  $\Theta_{i+1} = \Theta_i$ . By (b) we have  $\Gamma_i \cup \{A_{i+1}\} \vdash_{\text{EF4}}^{\text{d}} \Theta_i$ , which leads us to a contradiction. (ii)  $\Gamma_i \cup \{A_{i+1}\} \vdash_{\text{EF4}}^{\text{d}} \Theta_i$ . By (I) we have  $\Gamma_{i+1} = \Gamma_i$  and  $\Theta_{i+1} = \Theta_i \cup \{A_{i+1}\}$ . By (b) we have: (1).  $\Gamma_i \vdash_{\text{EF4}}^{\text{d}} \Theta_i \cup \{A_{i+1}\}$ . Now we assume  $\Gamma_i$  and  $\Theta_i$ . The wffs in this derivation are named as  $B_1, \dots, B_m$  and  $C_1, \dots, C_n$ , where  $m, n \geq 1$ . We will use  $B$  for the conjunction of  $B_1, \dots, B_m$ , i.e.,  $B_1 \wedge \cdots \wedge B_m$ . We will use  $C$  for the disjunction of  $C_1, \dots, C_n$ , i.e.,  $C_1 \vee \cdots \vee C_n$ . Thanks to this, now we can rewrite (1) as follows: (2).  $B \vdash_{\text{EF4}} C \vee A_{i+1}$ . On the other hand, given hypothesis (ii), there is a conjunction of elements of  $\Gamma_i$  named  $B'$ , and a disjunction of elements of  $\Theta_i$  named  $C'$ , such that: (3).  $B' \wedge A_{i+1} \vdash_{\text{EF4}} C'$ . By  $B''$  we will refer to the conjunction of  $B$  and  $B'$ ,  $B \wedge B'$ . By  $C''$  to the disjunction of  $C$  and  $C'$ ,  $C \vee C'$ . Our intention is to prove: (4).  $B'' \vdash_{\text{EF4}} C''$ , i.e.,  $\Gamma_i \vdash_{\text{EF4}}^{\text{d}} \Theta_i$ , contradicting reductio hypothesis, (b), and thus proving (a).



By Proposition 3.24 and (3) we have: (5).  $B'' \wedge A_{i+1} \vdash_{\text{EF4}} C''$ . By (2) and Proposition 3.24 we get: (6).  $B'' \vdash_{\text{EF4}} C'' \vee A_{i+1}$ . Now, thanks to (6) and Proposition 3.24 we get: (7).  $B'' \vdash_{\text{EF4}} C'' \vee (B'' \wedge A_{i+1})$ . And, by (5) and Lemma 3.25, we get: (8).  $C'' \vee (B'' \wedge A_{i+1}) \vdash_{\text{EF4}} C'' \vee C''$ . Which, simplified, is: (9).  $C'' \vee (B'' \wedge A_{i+1}) \vdash_{\text{EF4}} C''$ . Lastly, thanks to (7) and (9) we have:  $B'' \vdash_{\text{EF4}} C''$ . That, as we had pointed out above, is equivalent to:  $\Gamma_i \vdash_{\text{EF4}}^d \Theta_i$ , contradicting the reductio hypothesis (b). Consequently (a),  $\Gamma_k \not\vdash_{\text{EF4}}^d \Theta_k$ , for every  $k \in \mathbb{N}$ , is proven. Thus, we have sets of wff  $\Gamma'$  and  $\Theta'$  such that  $\Gamma \subseteq \Gamma'$ ,  $\Theta \subseteq \Theta'$ ,  $\Gamma' \not\vdash_{\text{EF4}}^d \Theta'$ , and  $\Theta = \bar{\Gamma}$ . Finally, let it be noted that  $\Gamma'$  is a maximal set since  $\Gamma' \vdash_{\text{EF4}}^d \bar{\Gamma}'$ . Therefore, Lemma 3.27 or EF4 extension lemma is proven.  $\square$

LEMMA 3.28 (EF4 primeness lemma). *If  $\Gamma$  is an EF4 maximal set of wffs, then  $\Gamma$  is a prime theory.*

PROOF. Let  $\Gamma$  be an EF4 maximal set of wffs. First we prove that  $\Gamma$  is a theory. For this purpose we will show that is closed under (I) Adjunction, (II) EF4-entailment, and (III) Disjunctive Counterexample: (I): We assume  $A \in \Gamma$  and  $B \in \Gamma$ , and  $A \wedge B \notin \Gamma$  as our reductio hypothesis. Then we have  $\Gamma \vdash_{\text{EF4}} A$  and  $\Gamma \vdash_{\text{EF4}} B$ , which by R1 leads us to  $\Gamma \vdash_{\text{EF4}} A \wedge B$ , and so contradicting  $\Gamma$  maximality. (II): We assume  $\vdash_{\text{EF4}} A \rightarrow B$  and  $A \in \Gamma$ , and  $B \notin \Gamma$  as our reductio hypothesis. We have  $\Gamma \vdash_{\text{EF4}} A \rightarrow B$  and  $\Gamma \vdash_{\text{EF4}} A$ , and, thanks to R2,  $\Gamma \vdash_{\text{EF4}} B$ , which contradicts  $\Gamma$  maximality. (III): We assume  $C \vee A \in \Gamma$  and  $C \vee \neg B \in \Gamma$ , and  $C \vee \neg(A \rightarrow B) \notin \Gamma$  as our reductio hypothesis. We have  $\Gamma \vdash_{\text{EF4}} C \vee A$  and  $\Gamma \vdash_{\text{EF4}} C \vee \neg B$ . By R3, we get  $\Gamma \vdash_{\text{EF4}} C \vee \neg(A \rightarrow B)$ , which contradicts  $\Gamma$  maximality. Thus, it is proven that  $\Gamma$  is a theory. Now we prove that is actually a prime theory: We assume  $A \vee B \in \Gamma$ ,  $A \notin \Gamma$ , and  $B \notin \Gamma$  as the reductio hypothesis. Then  $\Gamma \vdash_{\text{EF4}}^d A \vee B$ , contradicting the maximality of  $\Gamma$ .  $\square$

DEFINITION 3.29 (The set of consequences from set  $\Gamma$  with respect to EF4). Let  $\Gamma$  be an EF4 set of wffs. Then, the set of consequences, in symbols  $\text{Cn}\Gamma[\text{EF4}]$ , is defined as follows:  $\text{Cn}\Gamma[\text{EF4}] = \{A \mid \Gamma \vdash_{\text{EF4}} A\}$ .

OBSERVATION 3.30 ( $\text{Cn}\Gamma[\text{EF4}]$  is a regular theory). *Let  $\Gamma$  be a set of wffs. Then, it is obvious that  $\text{Cn}\Gamma[\text{EF4}]$  is closed under EF4 derivation rules and will contain every EF4 theorem. Consequently  $\Gamma$  is closed under EF4-entailment, too.*

### 3.6. Completeness theorem for EF4

Finally, now we can show that EF4 is a complete system in a strong sense. Also, this allows us to prove that EF4 is, actually, an axiomatization of the matrix M4.

**THEOREM 3.31** (EF4 completeness theorem). *For any EF4 wff  $A$ , if  $\Gamma \models_{\text{BD}} A$ , then  $\Gamma \vdash_{\text{EF4}} A$ .*

**PROOF.** Let  $\Gamma$  be a set of wffs and  $A$  any wff; we assume  $\Gamma \not\vdash_{\text{EF4}} A$  and from there we will show that  $\Gamma \not\models_{\text{BD}} A$ . Given the assumption we have  $A \notin \text{Cn}\Gamma[\text{EF4}]$  and, therefore,  $\text{Cn}\Gamma[\text{EF4}] \not\vdash_{\text{EF4}}^d A$ . Otherwise we will have  $B_1 \wedge \dots \wedge B_n \vdash_{\text{EF4}} A$  for wffs  $B_1, \dots, B_n$  such that  $B_1 \wedge \dots \wedge B_n \in \text{Cn}\Gamma[\text{EF4}]$  and, thus,  $A$  would be part of  $\text{Cn}\Gamma[\text{EF4}]$ . By Lemma 3.27, EF4 extension lemma, there is a maximal set  $\Gamma'$  such that  $\text{Cn}\Gamma[\text{EF4}] \subseteq \Gamma'$ ,  $\Gamma \subseteq \Gamma'$ , and  $A \notin \Gamma'$ . By Lemma 3.28  $\Gamma'$  is a prime theory and, by Observation 3.30, is a regular theory. Therefore we have a  $\tau$ -interpretation such that  $T \in I_\tau(\Gamma)$  but  $T \notin I_\tau(A)$ . Additionally, by Theorem 3.22, we have that the canonical model of Definition 3.20 is, actually, a model. Lastly, we have  $\Gamma \not\vdash_{I_\tau} A$  by Definition 3.20 and, from there, by Definition 3.6 and Theorem 3.2,  $\Gamma \not\models_{\text{BD}} A$ .  $\square$

**COROLLARY 3.32** (System EF4 is an axiomatization of the matrix M4). *The defined system EF4 is an axiomatization of the matrix M4 from Definition 2.2.*

**PROOF.** Given that EF4 is a sound and complete system in a strong sense with respect to BD-semantics by Theorems 3.12 and 3.31, and thanks to the equivalence of the concepts of validity of the BD-semantics and the M4-semantics shown in Theorem 3.19, we can conclude that the system EF4 is an axiomatization of the the matrix M4.  $\square$

### 3.7. EF4 characteristics

In order to conclude the section devoted to EF4 we show how the system does not have the Variable Sharing Property (VSP), but has the quasi-relevance property and it is a paraconsistent system.

**PROPOSITION 3.33** (EF4 does not have the VSP). *The system EF4 does not have the Variable Sharing Property (VSP), i.e., antecedent and consequent of any entailment share at least one propositional variable.*

PROOF. EF4 does not have the VSP since it does not validate the theorem  $\neg(A \rightarrow A) \rightarrow \cdot (B \rightarrow B)$ , a R-Mingle thesis [cf. 1]. This can be verified by using [13]. Nevertheless, EF4 falsifies the most important paradoxes of material implication, such as  $\neg A \rightarrow \cdot A \rightarrow B$ ,  $A \rightarrow \cdot B \rightarrow A$  or  $(A \rightarrow B) \vee (B \rightarrow A)$ .  $\square$

PROPOSITION 3.34 (EF4 has the quasi-relevance property). *For any EF4 wffs  $A$  and  $B$ , if  $\vdash_{\text{EF4}} A \rightarrow B$ , then either (a) both  $A$  and  $B$  share, at least, one propositional variable, or (b) both  $\vdash_{\text{EF4}} \neg A$  and  $\vdash_{\text{EF4}} B$ .*

PROOF. We assume the case where  $\vdash_{\text{EF4}} A \rightarrow B$  but  $A$  and  $B$  have no common propositional variable. Additionally we assume, by reductio,  $\not\vdash_{\text{EF4}} \neg A$  and  $\not\vdash_{\text{EF4}} B$ . By Theorems 3.11 and 3.12,  $\not\vdash_{\text{M4}} \neg A$  or  $\not\vdash_{\text{M4}} B$ . Then there will be M4-interpretations  $I_{\text{M4}}$  and  $I'_{\text{M4}}$  such that: (i)  $0 \in I_{\text{M4}}(\neg A)$ , or (ii)  $1 \in I_{\text{M4}}(\neg A)$ , (iii)  $0 \in I'_{\text{M4}}(B)$ , or (iv)  $1 \in I'_{\text{M4}}(B)$ . Now we shall show that if  $\vdash_{\text{EF4}} A \rightarrow B$  follows, then none of the cases (i)-(iv) is possible, thus, proving Proposition 3.34. We will show just cases (i) and (iv), as cases (ii) and (iii) are proven in a similar fashion. (i): We have  $3 \in I_{\text{M4}}(A)$  from the hypothesis. Let  $I''_{\text{M4}}$  be a M4-interpretation identical to  $I_{\text{M4}}$ , except that for every propositional variable  $p_i$  of  $B$ ,  $2 \in I''_{\text{M4}}(p_i)$ . Necessarily,  $I''_{\text{M4}}$  is consistent as  $A$  and  $B$  share no propositional variables just as we assumed. Then  $3 \in I''_{\text{M4}}(A)$  and  $2 \in I''_{\text{M4}}(B)$ , as  $\{2\}$  is closed under  $\wedge$ ,  $\vee$ ,  $\rightarrow$  and  $\neg$ . Hence,  $0 \in I''_{\text{M4}}(A \rightarrow B)$ , which leads us to  $\not\vdash_{\text{EF4}} A \rightarrow B$ , contradicting our first assumption. (iv): Let  $I'''_{\text{M4}}$  be an M4-interpretation equal to  $I'_{\text{M4}}$  with the exception that, for every propositional variable  $p_i$  that happens in  $A$ ,  $2 \in I'''_{\text{M4}}(p_i)$ . Just like the case prior to this,  $I'''_{\text{M4}}$  is consistent as  $A$  and  $B$  share no propositional variables. Then  $1 \in I'''_{\text{M4}}(B)$  and  $2 \in I'''_{\text{M4}}(A)$ , as  $\{2\}$  is closed under  $\wedge$ ,  $\vee$ ,  $\rightarrow$  and  $\neg$ . Thus  $0 \in I'''_{\text{M4}}(A \rightarrow B)$  and, then,  $\not\vdash_{\text{EF4}} A \rightarrow B$ , contradicting our very first assumption.  $\square$

PROPOSITION 3.35. *The system EF4 is paraconsistent.*

PROOF. We have to prove that *Ex Contradictione Quodlibet* (ECQ) rule of derivation,  $A \wedge \neg A \Rightarrow B$ , is not provable in EF4. We assume an M4-interpretation  $I_{\text{M4}}$  for distinct propositional variables  $p_i$  and  $p_m$  such that:  $2 \in I_{\text{M4}}(p_i)$  and  $1 \in I_{\text{M4}}(p_m)$ . Then  $p_i, \neg p_i \not\vdash_{\text{M4}} p_m$ , turning ECQ into not-provable in EF4 by virtue of Theorem 3.12.  $\square$

## 4. The four-valued modal systems of (relevant) implication

This section is devoted to the introduction of the modal systems EF4-M and EF4-L. The first one answers to the modality defined by Monteiro that, as it has been said, was reintroduced by Font and Rius in [10] and, more recently, by Beziau in [6]. The second one is the modality defined by Tarski and Łukasiewicz for their own modal systems that, in this case, is equivalent to the one that Anderson and Belnap defined for the system E. In the first place, we will introduce the system EF4-M and we will endow it with an extension of the previously defined BD-semantics so we can obtain results of soundness and completeness in a strong sense. Secondly, we will introduce EF4-L and obtain results of soundness and completeness in a strong sense too. To conclude this section, we will show how none of these systems has the modal paradoxes that were part of Łukasiewicz's system L.

### 4.1. First modality: EF4-M

In the first place we will define the characteristic matrix of EF4-M and introduce its inherent semantics. Afterwards we define the BD-semantics for EF4-M and show that it is equivalent to the matrix semantics. Then it is time to introduce the system EF4-M itself and obtain results of soundness and completeness in a strong sense.

#### 4.1.1. $\mathcal{MM}4$ -semantics, $\mathcal{MBD}$ -semantics and their equivalence

We define the matrix  $\mathcal{MM}4$  and its semantics.

DEFINITION 4.1 (The matrix  $\mathcal{MM}4$ ). The matrix  $\mathcal{MM}4$  is defined as a modal expansion of the matrix M4 from Definition 2.2. The set  $f$  is modified to be as follows:  $f = \{f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\neg}\}$ ,  $f_{LM4}$  and  $f_{MM4}$ . For the already defined elements of  $f$ , Definition 2.2 applies. The newly introduced elements are defined as follows:

	L		M
0	0	0	0
1	0	1	3
2	0	2	3
3	3	3	3

PROPOSITION 4.2 (Non-definability of  $f_L$  and  $f_M$ ). *Functions  $f_L$  and  $f_{MM4}$  are non-definable from functions  $f_{\wedge}$ ,  $f_{\vee}$ ,  $f_{\rightarrow}$  and  $f_{\neg}$  in  $\mathcal{MM}4$ .*

PROOF. It suffices to show that  $f_{\wedge}(2, 2) = f_{\vee}(2, 2) = f_{\rightarrow}(2, 2) = f_{\neg}(2) = 2$ , while  $\mathbf{LA} = 2$  or  $\mathbf{MA} = 2$  are impossible by Definition 4.1 above.  $\square$

*Remark 4.3* (Characteristic theses of the matrix  $\mathcal{MM}4$ ). The following are the characteristics theses of the matrix  $\mathcal{MM}4$ . For all wffs  $A$  and  $B$ :

- t19.  $\mathbf{LA} \rightarrow A$
- t20.  $\mathbf{LA} \wedge \neg A \rightarrow \cdot A$
- t21.  $A \rightarrow \cdot \neg A \vee \mathbf{LA}$
- t22.  $\neg \mathbf{LA} \wedge A \rightarrow \cdot \neg \neg A$
- t23.  $\neg A \rightarrow \neg \mathbf{LA}$
- t24.  $\neg \mathbf{LA} \vee A$
- t25.  $\mathbf{LA} \wedge \neg \mathbf{LA} \rightarrow \cdot B$
- t26.  $\mathbf{MA} \wedge \neg A \rightarrow \cdot A$
- t27.  $A \rightarrow \mathbf{MA}$
- t28.  $\neg A \vee \mathbf{MA}$
- t29.  $\neg \mathbf{MA} \rightarrow \neg A$
- t30.  $\mathbf{MA} \wedge \neg A \rightarrow \cdot A$
- t31.  $\neg A \rightarrow \cdot A \vee \neg \mathbf{MA}$
- t32.  $\mathbf{MA} \wedge \neg \mathbf{MA} \rightarrow \cdot B$

The notions of  $\mathcal{MM}4$ -interpretation,  $\mathcal{MM}4$ -consequence and  $\mathcal{MM}4$ -validity are defined similarly to those of EF4. Therefore, the reader might refer to Definitions 2.6 and 2.7.

Next, we define an updated version of the bivalent Belnap-Dunn models to include the modal connectives:<sup>1</sup>

DEFINITION 4.4 ( $\mathcal{M}BD$ -models). An  $\mathcal{M}BD$ -model is a structure  $\langle K_{\mathbf{BD}}^4, I_{\mathcal{M}BD} \rangle$ , where  $K_{\mathbf{BD}}^4$  is the set of Definition 3.11 and  $I_{\mathcal{M}BD}$  is a function from the set of wffs to  $K_{\mathbf{BD}}^4$ . For propositional variables,  $p_i$ , one of  $K_{\mathbf{BD}}^4$  elements is assigned. For Conjunction, Disjunction, Negation and Entailment wff, the clauses from Definitions 3.11 and 3.13 follow. For the connectives of Necessity and Possibility, the following clauses apply:

- (I) Necessity:
  - (a)  $F \in I_{\mathcal{M}BD}(\mathbf{LA})$  iff either  $F \in I_{\mathcal{M}BD}(A)$  or  $T \notin I_{\mathcal{M}BD}(A)$ ;
  - (b)  $T \in I_{\mathcal{M}BD}(\mathbf{LA})$  iff  $F \notin I_{\mathcal{M}BD}(A)$ ;
- (II) Possibility:
  - (a)  $F \in I_{\mathcal{M}BD}(\mathbf{MA})$  iff both  $F \in I_{\mathcal{M}BD}(A)$  and  $T \notin I_{\mathcal{M}BD}(A)$ ;
  - (b)  $T \in I_{\mathcal{M}BD}(\mathbf{MA})$  iff  $F \notin I_{\mathcal{M}BD}(A)$ .

---

<sup>1</sup> For modal connectives in Belnap-Dunn semantics see [20].

The notions of  $\mathcal{M}BD$ -consequence and  $\mathcal{M}BD$ -validity are equal to those of Definition 3.6.

Finally, we prove that the  $\mathcal{M}M4$ -semantics and the  $\mathcal{M}BD$ -semantics are equivalent in a similar fashion of what we did back for  $EF4$ .

The proof of the equivalence of the concepts of  $\mathcal{M}M4$ -validity and  $\mathcal{M}BD$ -validity is equal to that of  $EF4$  (cf. Subsection 3.3), therefore we omit it. Let it be noted that:

**PROPOSITION 4.5** (Equivalence of  $\mathcal{M}M4$ -validity and  $\mathcal{M}BD$ -validity). *There is a strict equivalence relationship between items; e.g.,  $\mathcal{M}BD$ -validity with  $BD$ -validity.*

#### 4.1.2. $EF4$ -M axiomatization

The axiomatization of  $EF4$ -M is equal to that of  $EF4$  plus the following axioms:

- A16.  $\mathcal{L}A \rightarrow A$
- A17.  $A \rightarrow \cdot \neg A \vee \mathcal{L}A$
- A18.  $\neg \mathcal{L}A \vee A$
- A19.  $\mathcal{L}A \wedge \neg \mathcal{L}A \rightarrow \cdot B$

#### 4.1.3. Some theorems of $EF4$ -M

We will show that  $EF4$ -M is a modal expansion of  $EF4$  and, after that, we will show that the characteristic theses of the matrix  $\mathcal{M}M4$  are included in the axiomatization of  $EF4$ -M.

Immediate by Proposition 4.2 we obtain:

**THEOREM 4.6** ( $EF4$ -M is a modal expansion of  $EF4$ ). *The system  $EF4$ -M is a modal expansion of  $EF4$ .*

**THEOREM 4.7** (The matrix  $\mathcal{M}M4$  and the axiomatization of  $EF4$ -M). *Characteristic theses of the matrix  $\mathcal{M}M4$  are part of the axiomatization of  $EF4$ -M.*

**PROOF.** The theses of  $EF4$  follow automatically as  $EF4$ -M is a modal expansion of  $EF4$  by Theorem 4.7, and we have shown that  $M4$  theses are part of  $EF4$  in Theorem 3.4. Thus, we need to show that the new theses from Remark 4.3, corresponding to the modal connectives, follow: t19 follows from Conjunction's Commutative Property; t20 from A2, A7, and Conjunction's Commutative Property; t21 from A9, A10, Disjunction's Commutative Property, Double Negation, and De Morgan (II); t22 from Disjunction's Commutative Property; t23 from A9, and Conjunction's

Commutative Property; t24 from A16; t25 from A22; t26 from A17; t27 from A18; t28 from A19; t29 from A9 and A18; t30 from A2; t31 from A9, A10, A17, and De Morgan (II); t32 from A23.  $\square$

#### 4.1.4. Soundness and completeness for EF4-M

Now it is time to give a soundness proof in the strong sense for EF4-M. Since strong soundness was proven for EF4 back in Theorem 3.12, to give a strong soundness proof for EF4-M, it is enough to show that the new axioms are valid [cf. 13].

**THEOREM 4.8** (Strong soundness for EF4-M). *For any set of wffs  $\Gamma$ , and any wff  $A$ , if  $\Gamma \vdash_{\text{EF4-M}} A$ , then  $\Gamma \models_{\mathcal{MM4}} A$  and, consequently,  $\Gamma \models_{\mathcal{MBD}} A$ .*

For the completeness theorem, first we will extend the notion of EF4-theories to include the modal connectives. Afterwards we will define modal interpretations and a canonical model for the  $\mathcal{MBD}$ -semantics. All that allows us to extend the clauses of modal interpretations into wffs. Then we will be ready to give the completeness theorem. We will also show that EF4-M is an axiomatization of the matrix  $\mathcal{MM4}$ .

**LEMMA 4.9** (Necessity and possibility connectives in regular, a-consistent and prime theories). *Let  $a$  be a regular, a-consistent and prime theory, and  $A$  a wff. Unlike the case of EF4, it is worth noting that in this case, theories need to be a-consistent. The following clauses applyf:*

- (I) (a)  $LA \in a$  iff  $A \in a$  and  $\neg A \notin a$ ;
- (b)  $\neg LA \in a$  iff  $\neg A \in a$  or  $A \notin a$ ;
- (II) (a)  $MA \in a$  iff  $A \in a$  or  $\neg A \notin a$ ;
- (b)  $\neg MA \in a$  iff  $\neg A \in a$  and  $A \notin a$ .

**PROOF.** We begin with (I) from left to right: (a) We assume  $LA \in a$ ; by Conjunction's Commutative Property we have  $A \in a$ . By reductio we have  $\neg A \in a$ , and by A9 and Conjunction's Commutative Property we have  $\neg LA \in a$ , and, afterwards,  $LA \wedge \neg LA \in a$ . Lastly, by A22, we have  $B \in a$  for any wff  $B$ , contradicting the a-consistency of  $a$ ; (b) We assume  $\neg LA \in a$  and  $\neg A \notin a$ , and  $A \in a$  as our reductio hypothesis. By t21 we have  $LA \in a$ , and by t25, we have  $B \in a$  for any wff  $B$ , contradicting a a-consistency; From right to left: (a) We assume  $A \in a$  and  $\neg A \notin a$ . By t21 we get  $LA \in a$ ; (b) We assume  $\neg A \in a$  or  $A \notin a$ . For the case  $\neg A \in a$ , by t23 we get  $\neg LA \in a$ . For the case  $A \notin a$ , by A16, and since  $a$  is regular and prime, we have  $\neg LA \in a$ .

For (II), from left to right: (a) We assume  $MA \in a$ , and  $A \notin a$  and  $\neg A \in a$  as reductio hypothesis. By A17 we have  $A \in a$ , which leads to a contradiction; (b) We assume  $\neg MA \in a$ , and by t29 we have  $\neg A \in a$ . We also assume  $A \in a$  by reductio. By A18 we have  $MA \in a$  and, by A23 we have  $B \in a$  for any wff  $B$ , contradicting  $a$  a-consistency; From right to left: (a) We assume  $A \in a$  or  $\neg A \notin a$ ; by A18 we have  $MA \in a$ , and by A19 and the fact that  $a$  is prime, we have  $MA \in a$ ; (b) We assume  $\neg A \in a$  and  $A \notin a$ ; by t31 we have  $\neg MA \in a$ .  $\square$

*Remark 4.10* (EF4-M axiomatization). From Lemma 4.9 it follows that EF4-M could be axiomatized by using just A16, A17, A18 and A19 as we have proposed in Subsection 4.1.2. Nevertheless, the option of axiomatizing by adding the theses corresponding to possibility also exists. Both ways of providing EF4-M axiomatization are valid.

The concepts of  $\mathcal{M}\tau$ -interpretation,  $\mathcal{M}BD$ -semantics canonical model, and canonical relation  $\models_{\mathcal{M}\tau}$  are defined similarly to those of EF4 (cf. Definitions 3.19–3.21) and therefore omitted. Furthermore, Lemma 9 takes care of the canonical interpretation of the modal operators. Finally, completeness is also proven similarly to EF4 (cf. Theorem 3.31), with the only exception that the maximal set  $\Gamma$  is a-consistent, since  $A \notin \Gamma$ .

**COROLLARY 4.11** (System EF4-M is an axiomatization of the matrix  $\mathcal{M}M4$ ). *The defined system EF4-M is an axiomatization of the matrix  $\mathcal{M}M4$  from Definition 4.1.*

**PROOF.** Given that EF4-M is a sound and complete system in a strong sense, with respect to  $\mathcal{M}BD$ -semantics, and thanks to the equivalence of the concepts of validity of  $\mathcal{M}BD$ -semantics and  $\mathcal{M}M4$ -semantics from Proposition 4.5, we can conclude that the system EF4-M is an axiomatization of the matrix  $\mathcal{M}M4$ .  $\square$

## 4.2. Second modality: EF4-Ł

For this second modality we are taking a different approach: EF4-Ł would be introduced as a definitional extension. Therefore, it is obvious that it follows automatically from and is equivalent to EF4. Nevertheless, the system still holds quite some value as the definitions used for introducing the modal connectives are of importance. On the one hand, we will use the definitions that Tarski used to modally interpret Łukasiewicz's many-valued systems that, in part, motivate EF4. On the



other hand, we will use the notion of Necessity implicit in E, the other great motivator for EF4. In this particular case, both different ways of defining the modality lead to the same results, as can be seen below. Therefore, the main aim of this system EF4-L, despite how obvious it might be, is to show explicitly how the inherent modality of EF4 works. This is especially important since EF4 can be considered as a four-valued version of E and a spiritual successor to Łukasiewicz's many-valued systems as modally interpreted by Tarski.

As we did with the first modal system, we will introduce the matrix ŁM4 first and define the axiomatization of EF4-L just after that. We will show how it is equivalent to EF4 and, therefore, the characteristics of EF4 are also the characteristics of EF4-L. Let it be noted that the name EF4-L does not derive from the system L, but rather from the fact that it is inspired by Łukasiewicz's many-valued logics. As said, we begin by introducing the matrix ŁM4.

**DEFINITION 4.12** (The matrix ŁM4). The matrix ŁM4 is defined as a modal expansion of the matrix M4 from Definition 2.2, where the set  $f$  is modified as follows:  $f = f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\neg}, f'_{\perp}$  and  $f'_{\text{M}}$ . For the already defined elements of  $f$  we will follow Definition 2.2. The new ones are defined as follow:

	L		M
0	0	0	0
1	0	1	3
2	2	2	2
3	3	3	3

Additionally, by  $I_{\text{ŁM4}}$  we will be referring to a ŁM4-interpretation built upon the matrix ŁM4, defined in a similar way to the previous cases.

*Remark 4.13* (Characteristic theses of the matrix ŁM4). The following theses and rules are the characteristic of the matrix ŁM4:

- t33.  $\text{L}A \rightarrow A$
- t34.  $\neg \text{L}A \wedge A \rightarrow \neg . A$
- t35.  $\neg A \rightarrow \neg \text{L}A$
- t36.  $\neg \text{L}A \vee A$
- t37.  $\text{M}A \wedge \neg A \rightarrow . A$
- t38.  $A \rightarrow \text{M}A$
- t39.  $\neg A \vee \text{M}A$
- t40.  $\neg \text{M}A \rightarrow \neg A$

- r2.  $A \Rightarrow LA$   
 r3.  $\neg A \Rightarrow \neg MA$

DEFINITION 4.14 (Definitional extensions and interdefinitions of  $L$  and  $M$ ). For any wff  $A$ ,  $L$  and  $M$  definitional extensions are as follow:

- (i)  $LA := \neg(A \rightarrow \neg A)$   
 (ii)  $LA := A \rightarrow A \rightarrow. A$   
 (iii)  $MA := \neg A \rightarrow A$   
 (iv)  $MA := \neg(\neg A \rightarrow \neg A) \rightarrow \neg A$

As pointed out above, in this specific case, the definitional extensions for the corresponding connectives are equivalent and, therefore, we include all of them. The interdefinitions are:

- (v)  $LA := \neg M\neg A$   
 (vi)  $MA := \neg L\neg A$

It follows automatically by the above definition that:

OBSERVATION 4.15 (Definibility of  $f'_L$  and  $f'_M$ ). *Functions  $f'_L$  and  $f'_M$  are definable from the functions  $f_{\rightarrow}$  and  $f_{\neg}$  in the matrix  $\mathbb{L}M4$  and, thus, also in the matrix  $M4$ .*

As obvious as the above observation is, it has been included to keep a parallelism with EF4-M.

#### 4.2.1. EF4- $\mathbb{L}$ axiomatization

The axiomatization of EF4- $\mathbb{L}$  is equal to that of EF4 plus the following axioms: A16, A18 and

- A17'.  $\neg LA \wedge A \rightarrow. \neg A$   
 R4.  $A \Rightarrow LA$

It follows automatically as EF4- $\mathbb{L}$  is a definitional extension of EF4 that:

OBSERVATION 4.16 (EF4- $\mathbb{L}$  properties). *EF4- $\mathbb{L}$  is a sound and complete system in a strong sense.*

### 4.3. Modal paradoxes elimination in EF4-M and EF4- $\mathbb{L}$

We will show how the main modal paradoxes that are part of Łukasiewicz's system  $\mathbb{L}$  are not valid in EF4.

THEOREM 4.17 (Modal paradoxes elimination). *For any distinct propositional variables  $p$  and  $q$ , the following Łukasiewicz-type strong modal paradoxes:*

- (i)  $Mp \wedge Mq \rightarrow \bullet M(p \wedge q)$
- (ii)  $L(p \vee q) \rightarrow \bullet Lp \vee Lq$
- (iii)  $Lp \rightarrow \bullet q \rightarrow Lq$
- (iv)  $Lp \rightarrow \bullet Mq \rightarrow q$
- (v)  $p \rightarrow Lp$
- (vi)  $LMp \rightarrow p$

are not valid in any of the defined modal systems, EF4-M and EF4-Ł.

PROOF. We will show that there is, at least, an interpretation for every modal paradox for each system that falsifies them.

In EF4-M: The interpretation  $I_{\mathcal{M}M4}(p) = 2$  and  $I_{\mathcal{M}M4}(q) = 1$  falsifies (i) and (ii). The interpretation  $I_{\mathcal{M}M4}(p) = 3$  and  $I_{\mathcal{M}M4}(q) = 1$  falsifies (iii) and (iv). The interpretation  $I_{\mathcal{M}M4}(p) = 2$  falsifies (v) and (vi).

In EF4-Ł: The interpretation  $I_{LM4}(p) = 2$  and  $I_{LM4}(q) = 1$  falsifies (i)–(iv). The interpretation  $I_{LM4}(p) = 1$  falsifies (v) and (vi).  $\square$

To conclude, it is important to mention that, while (i)–(vi) of Theorem 4.17 above are the most important modal paradoxes, there are more. For example  $A \rightarrow B \rightarrow \bullet MA \rightarrow MB$  and  $A \rightarrow B \rightarrow \bullet LA \rightarrow LB$ . These two new paradoxes are valid in both EF4-M and EF4-Ł and, while this might seem like a downside for both systems, it should only be regarded as so if the intention behind the logics was the one of establishing a classical conditional, as in the case of the system Ł. Nevertheless, since in both systems, EF4-M and EF4-Ł, the connective  $\rightarrow$  is based on EF4 and, therefore it represents an entailment, then both theses can be seen as acceptable. Finally, let us point out that, while (V) and (VI) are flagrant modal paradoxes, they are not valid in Łukasiewicz's logic as they are immediately falsified by the four-valued matrix characteristic of said logic.

## 5. Conclusions

The conclusions that we can draw from all of the previous sections are twofold. On the one hand, the system EF4 is a very interesting companion to BN4. Just as Brady's system can be seen as a four-valued extension of contractionless R, in the same sense EF4 can be seen as a four-valued extension of reductioless E. It could be argued that the system E4 is indeed the companion to BN4. Nevertheless, the same results follow for both systems (i.e., soundness and completeness in a

strong sense). Therefore, when choosing E4 or EF4 as the companion for BN4 one has to ask oneself if the divisibility of the matrix M4, the characteristic matrix of EF4, is enough to overlook the validation of the Mingle axiom or if, otherwise, it would be preferable to have a non-divisible matrix such as the one of E4.

On the other hand, the modal systems we introduced, EF4-M and EF4-Ł, are two modal systems of interest. Both of them being sound and complete in the strong sense makes them, at least, worth noting. Furthermore, EF4-M is a great representation of how the modality of four-valued systems can be expressed and provides solid ground for the ideas rekindled by Beziau. As for EF4-Ł, we are able to have a look at what a solid version of Łukasiewicz's Ł would be like, in the same sense that other four-valued logics free of strong Łukasiewicz-type modal paradoxes are. As, for example, the logics from [16]. In this case, EF4-Ł, a quasi-relevant one.

**Acknowledgments.** Sincere thanks are due to four anonymous referees for their comments on a previous version of this paper.

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