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The Rule of Existential Generalisation and Explicit Substitution

Abstract. The present paper offers the rule of existential generalisation (EG) that is uniformly applicable within extensional, intensional and hyperintensional contexts. In contradistinction to Quine and his followers, quantification into various modal contexts and some belief attitudes is possible without obstacles. The hyperintensional logic deployed in this paper incorporates explicit substitution and so the rule (EG) is fully specified inside the logic. The logic is equipped with a natural deduction system within which (EG) is derived from its rules for the existential quantifier, substitution and functional application. This shows that (EG) is not primitive, as often assumed even in advanced writings on natural deduction. Arguments involving existential generalisation are shown to be valid if the sequents containing their premises and conclusions are derivable using the rule (EG). The invalidity of arguments seemingly employing (EG) is explained with recourse to the definition of substitution.

Keywords: existential generalisation; quantifying in; explicit substitution; hyperintensional logic; natural deduction in sequent style

1. Introduction

1.1. The Rule of Existential generalisation and three types of context

The rule of *existential generalisation* (*EG*),

$$\text{(EG)}\ \frac{\varphi_{(t/x)}}{\exists x.\varphi}$$

where $\varphi_{(t/x)}$ is a formula φ in which all free occurrences of the variable x are substituted by a term t, is well-known in philosophical logic

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mostly due to Quine [e.g. 1943; 1956], who started the famous topic of *quantifying in* [see also, e.g., Kaplan, 1968]).

According to Quine and his various followers, (EG) serves without problems if it is applied within extensional contexts as in classical logic, but it fails if it is applied within (hyper)intensional contexts created by (hyper)intensional operators. Here are auxiliary definitions of the three kinds of contexts:

a. *Extensional contexts* (*E-contexts*) are contexts governed by *extensional operators* such as "+" or "*kiss [someone]*"; the operators operate on *extensions*, e.g. numbers, individuals or their sets.
b. *Intensional contexts* (*I-contexts*) are contexts governed by *intensional operators* such as "*necessarily*" or "*seek [something]*"; these operators operate on *(possible world) intensions*, e.g. propositions, properties or individual concepts of *possible world semantics* (*PWS*).
c. *Hyperintensional contexts* (*H-contexts*) are governed by *hyperintensional operators* such as "*believe*" (two types of H-contexts are distinguished in Section 4.2).

As will be apparent below, we are not in need of more precise definitions of the contexts in this paper, although it is not difficult to establish them. First, note that a context is only secondarily identified as an E-/I-/H-context by finding an occurrence of a certain term (as suggested above). Primarily — and more accurately — it is identified by its 'inferential potentiality', i.e. by the set of admissible consequences of the respective expressions. The *de re* modal contexts, for example, are well known for the possibility of substituting terms in them whose intensions differ, but which are co-extensive, since the terms refer (in a given possible world and time instant) to the same *res*; this allows certain inferences. On the other hand, in the *de dicto* modal contexts, inferences based on such substitutions would be incorrect, for it is the logical equivalence of the terms denoting intensions which is needed.

It is well known that the investigation of the inferential potential of terms in contexts led to the discovery of hyperintensional contexts. Since *hyperintensionality* is an important notion of this paper, it will be useful to take a closer look at it. Here is Cresswell's seminal definition:

> It is well known that it seems possible to have a situation in which there are two propositions p and q which are logically equivalent and yet are such that a person may believe the one but not the other. If we regard a proposition as a set of possible worlds then two logically equivalent propositions will be identical, and so if 'x believes that' is

> a genuine sentential functor, the situation described in the opening sentence could not arise. I call this the paradox of hyperintensional contexts. Hyperintensional contexts are simply contexts which do not respect logical equivalence. [Cresswell, 1975, p. 25]

Cresswell thus considered substitutability in the scope of the operator *believes that* φ. He found that even though ϕ and ψ are logically equivalent, i.e. $\phi = \psi$, they are not intersubstitutable within the *de dicto* belief sentence "*a believes that* φ". If ϕ and ψ were understood as standing for the same PWS-proposition P, the intersubstitutability of ϕ and ψ would be allowed. But such PWS's proposal validates (pre-theoretically) invalid arguments such as $\phi = \psi$, $Bel(a, \phi)$ / $Bel(a, \psi)$ since a may believe that ϕ but not that ψ even though they are extensionally but even intensionally (according to PWS) identical. PWS's explication of sentential meanings must therefore be replaced by *hyperintensional semantics* that regards ϕ and ψ as standing for distinct, though logically equivalent, *hyperintensions* H_ϕ and H_ψ. The context produced by "*believes that*" is hyperintensional for it is the sameness of ϕ's and ψ's hyperintensions which is required for correct inferences.

Now we are ready to state the main aim of this paper:

1. To provide a formulation of (EG) which is appropriate for E-, I- and H-contexts.

Our test example with an E-context even includes *partiality*, i.e. the fact that some sentences, descriptions and other terms lack a semantic value, which has an obvious relevance to the topic:

$$A_E \qquad \frac{\textit{The King of France (KF) is bald.}}{\textit{Something is such that it is bald.}}$$

The acceptance of both extensions and intensions within a system of *intensional logic* such as Montague's [1974] offers the possibility of dismissing Quine's reluctance to quantify into intensional contexts. The differentiation of various *readings* of arguments such as:

$$A_I \qquad \frac{\textit{Necessarily, the number of planets (NP) is greater than 7.}}{\textit{Something is such that necessarily, it is greater than 7.}}$$

enables one to determine their validity with reference to the (in)applicability of (EG) within particular readings. Here I will adapt and extend some results from Tichý's [1986] in-depth specialised study that employs a suitable intensional logic.

Contemporary logic also attempts to solve a number of issues related to hyperintensionality, an application of (EG) in H-contexts obviously being one of them:

$$A_H \ \frac{\textit{Wiles knows that Fermat believed Fermat's Last Theorem (FLT).}}{\textit{Something is such that Wiles knows that Fermat believed it.}}$$

Tichý's [1988] hyperintensional logical framework — a TT whose extension applicable to *natural language* (*NL*) is called *Transparent intensional logic* (*TIL*) — is very useful for the investigation of fine-grained hyperintensionality. I will deploy its novel convenient modification, called TT^*, and its extension called *Transparent hyperintensional logic* (*THL*). I adapt Tichý's early results to a hyperintensional setting and offer the rule (EG) (not promoted by Tichý himself or any of his followers) that is uniformly applicable within E-, I- and H-contexts.

The second important objective of my study is

2. To propose an adequate definition of the *substitution operator* (t/x) occurring in (EG), together with the rules of substitution.

Though for many logics that extend FOL substitution is external or 'metalinguistic', in comprehensive systems such as higher-order logics for hyperintensionality, substitution must be treat internally. For the logics aim to provide adequate formalisation of sentences involving H-contexts and "*A substitutes t for x in φ*" is clearly one of them. This is why a precise treatment of (t/x) inside such a logic is needed.

Moreover,

3. The present paper offers a *natural deduction* (*ND*) system, called $\mathsf{ND}_{\mathsf{TT}^*}$, within which (EG) is derived and which is suitable for the control of the (so-called) syntactic validity of arguments.

TT^* has a sufficient model-theoretic specification, but it may be identified just with $\mathsf{ND}_{\mathsf{TT}^*}$, since it is mainly the deduction system which determines the unique consequence relation of the logic in question.[1]

The importance of proposing (EG) as a derived rule might be overlooked by those who confuse (EG) with the rule of $\exists$-*introduction*,

$$(\exists\text{-I})\ \frac{\varphi_{[t]}}{\exists x.\varphi_{[x]}}$$

[1] In contrast, Tichý's [1982] *ND*, bound by the limits of his simple TT, determines a different consequence relation. Another consequence relation is also determined by Tichý's [1988] late logical framework which lacks a deduction system.

where $\varphi_{[t]}$ indicates a formula that contains term t. The rule (∃-I) may be primitive in various *ND*, but must be primitive in *ND* for partial logics (such as the *ND* proposed below) alongside the introduction rule for ∀, since ∀ and ∃ are not interdefinable in partial logics. To illustrate, the formula $\neg\exists x BK(x)$, where BK is the predicate "*be (identical to) the KF*", is currently true, while $\forall x \neg BK(x)$ is false since not every individual is decidable as (not) being BK (to work properly, the law for the exchange of quantifiers must be adjusted using the strong truth predicate [see, e.g., Raclavský, 2014, 2018]).

It is important to note that in works on *ND* [e.g. Prawitz, 2006] the above rule (EG) is often called (∃-I), while substitution is either neglected, or moved from the rule formulated within a language to metalanguage and so even the above rule (∃-I) is called (EG) (both cases occur in [Prawitz, 2006]). The present paper cannot follow this practice for two reasons:

a. Substitution is in its focus while it is made explicit and therefore 'intra-logical'; i.e. the substitution operator is a genuine constant of the logical language $\mathcal{L}$ and is managed by appropriate introduction and elimination rules.
b. (EG) is distinguished from (∃-I) for that reason that, though both concern non-emptiness of a certain set, (∃-I) captures it in a more rudimentary way and so (EG) is derivable using (∃-I), but not *vice versa*.

Structure of the paper. In Section 2, I provide the core part of the logic TT^* — which is published here for the first time — which is adequate for the formalisation of the above arguments and managing their inferential properties. I also define the operation of substitution there and offer an *ND* system for TT^*. The logic TT^* is originally based on Tichý's [1988] logic, from which I provide a number of departures. In Section 3, I derive the rule (EG) within $\mathsf{ND}_{\mathsf{TT}^*}$. Section 4 shows TT^*'s application to *NL* via its extension THL. I demonstrate the (so-called) syntactic validity of (the formalisations of) arguments A_E–A_H and explain the impossibility of constructing such proofs (cf. Section 2.3 for more) for their *de dicto* readings.[2] The appendix contains the list of $\mathsf{ND}_{\mathsf{TT}^*}$'s main rules. Section 5 provides a brief conclusion.

[2] Only the sole rule (EG), the substitution function and the (analysed) example A_H appeared in [Raclavský, 2020]; the present paper offers a number of further ideas.

2. The logic for fine-grained hyperintensionality

As mentioned in the introductory section above, the formalisation of arguments A_E and A_I presents no problem for intensional logics, but A_H's formalisation requires a logic suitable for analysis of *belief sentences*, i.e. sentences that report *belief attitudes* (also known as *propositional attitudes*). Church's translational argument showed that the objects of belief attitudes cannot be sentences/formulas, but language-independent propositions. According to PWS used in common intensional (incl. modal) logics, *propositions* are sets of *possible worlds*.

However, Cresswell [1975], Hintikka [1975] and others showed that such a proposal meets the *paradox of hyperintensional contexts* (cf. the quotation in Section 1) and the *paradox of logical omniscience*. To avoid the problems, *hyperintensions* enabling discrimination between logically equivalent, but non-identical expressions have been argued for. Recently, many have followed [Lewis, 1970] in emphasizing that hyperintensions should be *fine-grained*: meanings modelled by hyperintensions are to be as fine-grained as linguistic expressions are.[3]

A logic suitable for capturing and handling fine-grained hyperintensional meanings was developed by Tichý during 1976–1988; see his [1988; 2004]. Similar neo-Fregean algorithmic proposals have been recently defended by Moschovakis [2006], Muskens [2005] and, of course, Tichý's followers, e.g. Duží and Jespersen [2015] and Raclavský [2020].

Tichý [1986] called his hyperintensions *constructions*, alluding to geometry where (say) one intersection can be constructed by infinitely many congruent ways. Similarly, the truth value $\mathtt{T}$ (True) can be constructed by the application of $\neg$ (negation) to $\mathtt{F}$ (False), or much less effectively by $\forall x \forall y \forall z \forall n((x^n + y^n = z^n) \rightarrow (n < 3))$ (i.e. FLT), or trivially and directly by T. Constructions are thus not necessarily effective, acyclic algorithmic computations of an object.

In the next two subsections, I propose the logic TT^* which is suitable for handling constructions and objects v-constructed by them. It is a substantial modification of Tichý's [1988] logic. TT^*'s extension and application to natural language occurs in Section 2.2.

[3] For a lively debate on these topics and further references, see e.g. [Berto and Nolan, 2021; Raclavský, 2020].

2.1. The logic TT^*: its language and semantics

Since constructions are best representable by λ-terms, TT^*'s language is:

$$\mathcal{L}_{\mathsf{TT}^*} \qquad C ::= X \,|\, x \,|\, C_0(\bar{C}_m) \,|\, \lambda \tilde{x}_m.C_0 \,|\, \ulcorner C_0 \urcorner \,|\, [\![C_0]\!]_\tau$$

where X are *constants* (constants of NL are written in bold, $\mathbf{X}$), x are *variables*, $C_0(\bar{C}_m)$ are *applications*, $\lambda \tilde{x}_m.C_0$ are λ-*abstracts*, $\ulcorner C_0 \urcorner$ are *acquisitions*, $[\![C_0]\!]_\tau$ are *immersions* (τ, a type, is explained below); and

$$\tilde{E}_m \text{ is short for a string of entities } E_1 ... E_m$$
$$\bar{E}_m \text{ is short for } E_1, ..., E_m$$

I will also use auxiliary brackets [,].[4]

$\mathcal{L}_{\mathsf{TT}^*}$'s BNF entails the notion of a *subconstruction* of a construction (see also below).

Each construction constructs dependently on *valuation* v, we say that it v-*constructs* an object; "$v(\mathrm{X}/x)$" denotes v which differs from v in at most that it assigns X to x. Some constructions v-construct nothing at all, which I denote by "_"; they are called v-*improper*. Consequently, the terms expressing such constructions do not denote anything (which I also denote by "_").

Partiality already occurs on the level of functions, since *total* and *partial* multiargument *functions* are treated by Tichý's TT and TT^*. It is important to stress that functions are considered here as *graphs* (not as certain relations or 'computations') which map all (the case of total functions) or some but not all (the case of partial functions) elements of their domain D_F to some elements of their codomain C^{D_F}; (co)domains may consist of functions, so there is a hierarchy of functions. *Partial functions* are thus undefined for some elements of their domains; for example, $\div$ is undefined for couples of the form $\langle n, 0 \rangle$. The missing functional value is also denoted by "_"; these gaps are no objects and so they cannot be typed.

The need for partiality for analysis of our conceptual scheme was stressed e.g. by Feferman [1995], Muskens [1995], Farmer [1990].

The exact behaviour of constructions can be described in terms of Henkin-style *denotational semantics* for $\mathcal{L}_{\mathsf{TT}^*}$ [see Raclavský, 2018, 2020].

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[4] All brackets, comma, dot and λ are syncategorematic expressions. A non-simplified exposition of $\mathcal{L}_{\mathsf{TT}^*}$ requires its pre-terms to be decorated by type terms which are then mostly eliminated if certain conditions are met; only the resulting $\mathcal{L}_{\mathsf{TT}^*}$'s terms are interpreted.

A *frame* $\mathfrak{F}$ is $\{D_\tau\}_{\tau\in\mathcal{T}}$, i.e. a family of sets indexed by τ from the set of types $\mathcal{T}$. It consists of all interpreted TT^*'s types τ as domains D_τ (for each τ). A *model* $\mathfrak{M}$ is a couple $\langle\mathfrak{F},\mathfrak{I}\rangle$, where $\mathfrak{I}$ is the *interpretation function*. $\mathfrak{I}$ maps constants of $\mathcal{L}_{\mathsf{TT}^*}$, to objects of $\mathfrak{F}$. Each *valuation* v w.r.t. $\mathfrak{F}$ consists of infinite *v-sequences* sq^τ (for each type τ) of τ-objects, and so v supplies every variable (of every type) with a *v-value*.

The following definition of the *evaluation function* utilises the expression "$[\![C]\!]^{\mathfrak{M},v}$" which reads 'on valuation v w.r.t. $\mathfrak{F}$, "C" *denotes* in $\mathfrak{M}$ …'. Alternatively, $[\![C]\!]^{\mathfrak{M},v}$ could be read as 'construction C *v-constructs* (in $\mathfrak{M}$) …', and so the definition can be understood as a *specification of modes of constructions* (how they v-construct). Let X be an object or a construction, $C, \bar{C}_m$ constructions, c a variable for constructions, and $\mathrm{C}, \bar{\mathrm{C}}_m$ objects v-constructed by $C, \bar{C}_m$, respectively:

If the term/ construction in $[\![\cdot]\!]^{\mathfrak{M},v}$ *is:*		*it denotes in* $\mathfrak{M}, v/v$*-constructs:*
$[\![x_i^\tau]\!]^{\mathfrak{M},v}$	=	the ith-member X_i of sq^τ, where sq^τ belongs to v
$[\![C]\!]^{\mathfrak{M},v}$	=	$\mathfrak{I}(C)$ where C is a constant
$[\![C(\tilde{C}_m)]\!]^{\mathfrak{M},v}$	=	$\begin{cases}\mathrm{C}(\tilde{\mathrm{C}}_m) & \text{if } [\![C]\!]^{\mathfrak{M},v}=\mathrm{C}\in\langle\bar{\tau}_m\rangle\mapsto\tau, [\![C_1]\!]^{\mathfrak{M},v}=\mathrm{C}_1\in\tau_1, \\ & \ldots, [\![C_m]\!]^{\mathfrak{M},v}=\mathrm{C}_m\in\tau_m \text{ and } \exists x(x=\mathrm{C}(\tilde{\mathrm{C}}_m)) \\ \_ & \text{otherwise}\end{cases}$
$[\![\lambda\tilde{x}_m.C]\!]^{\mathfrak{M},v}$	=	the function $f\in\langle\bar{t}_m\rangle\mapsto\tau$ that takes each $[\![C]\!]^{\mathfrak{M},v^{(\prime)}}\in\tau$ at the respective argument $\langle[\![x_1]\!]^{\mathfrak{M},v^{(\prime)}},\ldots,[\![x_m]\!]^{\mathfrak{M},v^{(\prime)}}\rangle$ where v' is like v except for what it assigns to $\bar{x}_m$ occurring in $\lambda\tilde{x}_m.C$ and for each $1\leqslant i\leqslant m$, $[\![x_i]\!]^{\mathfrak{M},v^{(\prime)}}\in\tau_i$
$[\![\ulcorner C\urcorner]\!]^{\mathfrak{M},v}$	=	C
$[\![\,[\![C]\!]_\tau\,]\!]^{\mathfrak{M},v}$	=	$\begin{cases}X & \text{if } X \text{ is the only } x/\tau \text{ such that } \exists c(x=[\![c]\!]^{\mathfrak{M},v}\wedge c=[\![C]\!]^{\mathfrak{M},v}) \\ \_ & \text{otherwise}\end{cases}$

Note that $\ulcorner C\urcorner$ v-constructs C as such, not the object v-constructed by C; $[\![C]\!]_\tau$ v-constructs the object (if any) of type τ v-constructed by the construction (if any) v-constructed by C. Tichý [1988] used ${}^0X$ as our constant X if X was an object or as our $\ulcorner C\urcorner$ if X was a construction. $[\![C]\!]_\tau$ [proposed in Kuchyňka and Raclavský, 2021] is a rather restricted version of Tichý's ${}^2X$ whose type is elusive and produces contradictions if Tichý's rather restrictive definition of free variables is not abided by.

The above specification of constructions entails a definition of freedom of variables [for details and discussion, see Raclavský, 2020]. Variables can be *bound in C* by λ-operator (any free occurrence of x_i in D is

λ-bound in $C := \lambda\tilde{x}_m.D$), or by $\ulcorner\cdot\urcorner$ (any free occurrence of x in D is $\ulcorner\cdot\urcorner$-bound in $C := \ulcorner D\urcorner$). A rigorous definition would state (with a reference to the notion of order, see below) that if an nth-order construction C contains (an occurrence of) x_i as its nth-order subconstruction, x_i is *free in* C, provided x_i does not occur in the 'body' D of C's nth-order subconstruction $\lambda\tilde{x}_m.D$, where x_i is one of $\bar{x}_m$; moreover, (an occurrence of) x_i is *free in* C, if $C := [\![D]\!]_\tau$ or $C := [\![\ulcorner D\urcorner]\!]_\tau$ and (an occurrence of) x_i is free in D. A variable x is called *not free in* C iff (an occurrence of) x is either bound in C, or x has no occurrence in C at all.

Terms/constructions are typed via *type statements* of the form C/τ; the particular typing rules are stated below. An object belonging to *type* τ of *order* n, i.e. of type τ^n ("n" is usually suppressed), is called an *nth-order* τ*-object*. As indicated above, D_{τ^n} is the set of (all) τ^n-objects.

The definitions of the notions of *type* and *order* are simplified versions of those in [Raclavský, 2020], where one can find a detailed discussion. The essential idea of typing is to restrict ranges of variables etc. to domains D_{τ^n} and so guarantee that no function or construction with a circular formation infects the system with a paradox (e.g. Russell's paradox, Russell-Myhill paradox, or even the Liar paradox).

Let our particular *type base* be $B = \{\iota, o, \omega, \rho\}$, where[5]

Type	*consisting of*	*variable(s) ranging over the type*
ι	*individuals*	x
o	*truth values*	o
ω	*possible worlds*	$w, w', \ldots$
ρ	*real numbers*	$t, t', \ldots, n, n', \ldots$

τ^1 *Types of order* 1: (i) each B's member τ^B is a 1*st-order type*; and (ii) if $\bar{\tau}^1_m$ and τ^1_0 are 1st-order types, $\langle\bar{\tau}^1_m\rangle \mapsto \tau^1_0$ is also a 1*st-order type*; $D_{\langle\bar{\tau}^1_m\rangle\mapsto\tau^1_0}$ consists of all total and partial functions from the Cartesian product of $\bar{D}_{\tau^1_m}$ to $D_{\tau^1_0}$.

Now let $*^n$ be the type of all constructions whose (not necessarily all) subconstructions v-construct objects of nth-order types; D_{*^n} consists of all nth-order constructions.

τ^{n+1} *Types of order* $(n+1)$: (i) $*^n$ and every type of order n is an $(n+1)$*st-order type*; (ii) if $\bar{\tau}^n_m$ and τ^n_0 are $(n+1)$st-order types, then $\langle\bar{\tau}^n_m\rangle \mapsto \tau^n_0$ is also an $(n+1)$*st-order type*; $D_{\langle\bar{\tau}^n_m\rangle\mapsto\tau^n_0}$ consists

[5] Below, x is sometimes used as ranging over an unspecified type τ, which is clear from context surrounding it.

of all total and partial functions from the Cartesian product of $\bar{D}_{\tau_m^n}$ to $D_{\tau_0^n}$.

Below, we will often drop superscripts nearby τs (except for $*^n$s); we will also utilise auxiliary brackets (,).

Examples. o is the type of truth values (True and False) and so $D_o = \{\mathtt{T}, \mathtt{F}\}$. Objects of type $\tau \mapsto o$ represent *sets*; *quantifiers* are of type $(\tau \mapsto o) \mapsto o$, they are 'predicates' applicable to sets (for typing rules see below). Given our B, we may identify *intensions* with objects of type $\rho \mapsto (\omega \mapsto \tau)$; see Section 2.2.

Higher-order types τ^{n+1}, for $0 \leqslant n$, include mainly types such as the type $*^n$ of nth-order constructions. Each nth-order construction — which is an $(n{+}1)$st-order object — v-constructs an nth-order object (if any). 1st-order constructions (i.e. members of $*^1$) v-construct base objects or functions over B (note that the 'bottom order' is 1, not 0). 1st-order constructions are v-constructed by 2nd-order constructions; for example, the 2nd-order construction (a variable) c^1 $v(C^1/c^1)$-constructs a member of $*^1$, namely C^1. Etc.

Here are the *typing rules*:

i. X/τ^n
ii. x/τ^n
iii. $C_0(\bar{C}_m)/\tau_0^n$, where $C_1/\tau_1^n; ...; C_m/\tau_m^n; C_0/\langle\bar{\tau}_m^n\rangle \mapsto \tau_0^n$
iv. $\lambda\tilde{x}_m.C_0/\langle\bar{\tau}_m^n\rangle \mapsto \tau_0^n$, where $x_1/\tau_1^n; ...; x_m^n/\tau_m^n; C_0/\tau_0^n$
v. $\ulcorner C \urcorner/*^n$, where C/τ^n
vi. $[\![C]\!]_{\tau^n}/\tau^n$, where C/τ^n

Note that both $\ulcorner X \urcorner$ and $[\![C]\!]_\tau$ are of higher order (by one) than X or C: they are 'metalogical' devices.

2.2. The logic TT^* extended for the formalisation of natural language

This section provides an extension of TT^* that is suitable for formalisation of A_E–A_H. Following Montague [1974] and others [see e.g. Blackburn et al., 2001; Fitting, 2015; Muskens, 2007; Williamson, 2013] I propose a *higher-order multimodal logic* that is usable for the formalisation of *natural language*. Its constants extend $\mathcal{L}_{\mathsf{TT}^*}$; they often represent higher-order functions (some of them are definable using other primitive constants). For the present paper, the following short introduction should suffice; for details, see [Raclavský, 2020].

It is important to emphasise that the extended logic TT^* (and its predecessors such as Tichý's [1988] TT) simultaneously treats extensions, intensions and also hyperintensions which determine some extensions/intensions/(other) hyperintensions. Tichý [e.g. 1988] also repeatedly and correctly distinguished his TT (with the type base $B = \{o, \iota\}$) from its particular instance TIL, whose $B = \{o, \iota, \omega, \rho\}$ enables the accommodation of intensions as functions from possible worlds (members of ω) which are needed for an adequate formalisation of NL. Similarly, TT^* is a base framework, whose particular instance THL with $B = \{o, \iota, \omega, \rho\}$ is apt for NL's formalisation. Since both TIL and THL are rather characterised by doctrines stating how NL-expressions are to be analysed (as discussed in detail in [Raclavský, 2020]), the analyses can be translated from Tichý's TT to TT^*, and *vice versa*.

Our choice of TT^* is substantiated by the following main facts,

i. TT^*'s $\mathcal{L}_{\mathsf{TT}^*}$ (syntax, semantics — models) is well specified, unlike the languages of Tichý's TT and TIL.
ii. TT^* has a self-sustaining deduction system for all its terms/constructions (see below Section 2.3 and Appendix), unlike Tichý's TT in [1982; 1986], and mainly TT of Tichý [1988] which have no rules for ${}^0X, {}^1X, {}^2X$.

Recently, Kuchyňka (yet unpublished) proved the completeness of (a natural deduction system for) TT^* w.r.t. general models, so TT^* as a system is completed — unlike any version of Tichý's TT, incl. TIL. Crucial reasons for our choice of THL are offered after stating the following notions and ideas.

Tichý [1988, 2004] explicated *intensions* as functions from possible worlds to chronologies of objects. THL — whose main ideas were stated by Kuchyňka (unpublished) and extensively elaborated by Raclavský [e.g. 2018, 2020] — treats many of them equivalently as functions from objects to propositions. *Propositions* are explicated as (total) functions from time instants to (partial) functions from possible worlds to truth values. The type of propositions is thus $\rho \mapsto (\omega \mapsto o)$, which will be abbreviated in notation to π.

Then,

Intension	*its type*
propositions	π
properties of τ-objects	$\tau \mapsto \pi$
relations-in-intension	$\langle \bar{\tau}_m \rangle \mapsto \pi$
τ-*offices* ('concepts')	$\rho \mapsto (\omega \mapsto \tau)$

Consequently, if an object X *instantiates* a certain property F in a particular time instant T and possible world W, the corresponding proposition 'X is F' is true in T and W, i.e. its value is T.

The move from Tichý's intensions of type $\omega \mapsto (\rho \mapsto \tau)$ to Kuchyňka's intensions of type $\omega \mapsto (\rho \mapsto \tau)$ is essential for THL. *Inter alia*, it yields a natural explanation of verification: to verify the sentence "*X is a platypus*" in a time instant T (which serves as an argument for the proposition denoted by the sentence), one needs to take X and check whether it satisfies a certain condition, namely 'be a platypus'. For that purpose, it is enough to check whether X satisfies the easily observable conditions following from it, e.g. 'breast feed one's offspring' and 'be duck-billed'. On Tichý's approach [cf., e.g., 1978], however, one must check the whole universe for the extension of the property of type $\omega \mapsto (\tau \mapsto (o \mapsto \iota))$, and then check whether it involves X, which hardly happens in reality.

Before stating the crucial argument in favour of THL, here are the basic constructions we mention in this paper. They all are of the type $*^1$, except for $\mathscr{K}^n$, $\mathscr{B}^n$ and c^n which are of the type $*^{n+1}$.

Construction	*type of the v-constructed object*
A (Wiles), **B** (Fermat),	ι
K (the King of France), u	$\rho \mapsto (\omega \mapsto \iota)$
7, **9**	ρ
NP (the number of planets), u^ρ	$\rho \mapsto (\omega \mapsto \rho)$
$>, =^\rho$	$\langle \rho, \rho \rangle \mapsto o$
c^n	$*^n$
H (be bald)	$\iota \mapsto \pi$
$\mathscr{K}^n$ (know), $\mathscr{B}^n$ (believe)	$\langle \iota, *^n \rangle \mapsto \pi$
$\Box$, **Was**	$\pi \mapsto \pi$
$\exists^\tau, \forall^\tau$	$(\tau \mapsto o) \mapsto o$
FLT, T	o

Some constructions are represented here by their abbreviations,[6] some constructions are definable in terms of other constructions.[7]

[6] For example, "**FLT**" is short for "$\forall^\rho \lambda x.\forall^\rho \lambda y.\forall^\rho \lambda z.\forall^\rho \lambda n.((x^n + y^n) =^\rho z^n) \to (n < 3)$" (in which some pairs of round brackets are suppressed), where $x, y, z, n/\rho; +/\langle \rho, \rho \rangle \mapsto \rho$, and "**K**" is short for "$\lambda t.\lambda w.\imath^\iota(\lambda x[[[\mathbf{Ki}(x, \mathbf{Fr})](t)](w)])$", where **Fr** (France), $x/\iota; \imath^\iota/(o \mapsto \iota) \mapsto \iota$ (singularisation operator); **Ki** (be a king of) $/\langle \iota, \iota \rangle \mapsto \pi$.

[7] Example. Let $p/\pi; \wedge, =^o /\langle o, o \rangle \mapsto o$; the notation $\Gamma\{C_1 \Leftrightarrow_x C_2\}$ says that the sequent (see below) $\Gamma, C_1{:}x \Rightarrow C_2{:}x$ is derivable whenever the sequent $\Gamma, C_2{:}x \Rightarrow$

The formalisations of A_E's, A_I's and A_H's premises are thus the following constructions of truth values. Notation:

$$C_{tw} \text{ abbreviates } [C(t)](w), \text{ provided } C/\rho \mapsto (\omega \mapsto \tau)$$

Premiss of	*reading*	*construction*
A_E		$[\mathbf{H}(\mathbf{K}_{tw})]_{tw}$
A_I	*de dicto*	$[\Box(\lambda t'.\lambda w'[\mathbf{NP}_{t'w'} > \mathbf{7}])]_{tw}$
A_I	*de re*	$[\Box(\lambda t'.\lambda w'[\mathbf{NP}_{tw} > \mathbf{7}])]_{tw}$
A_H	*de dicto*	$[\mathscr{K}^2(\mathbf{A}, \ulcorner[\mathbf{Was}(\mathscr{B}^1(\mathbf{B}, \ulcorner\mathbf{FLT}\urcorner))]_{tw}\urcorner)]_{tw}$
A_H	*de re*	$[\mathscr{K}^2\,(\mathbf{A}, Sub^2\,(\ulcorner\mathbf{FLT}\urcorner, \ulcorner c^1\urcorner, \ulcorner[\mathbf{Was}(\mathscr{B}^1(\mathbf{B}, c^1))]_{tw}\urcorner))]_{tw}$

In [Raclavský, 2020], a number of arguments in favour of THL were offered. The crucial argument is as follows. In consequence of THL's type for intensions, its explicata of meanings are simpler and shorter than those proposed by TIL, while they are often straightforwardly intertranslatable and so they have the same explanatory power. Cf. e.g. the following two lists of explicata:

THL's proposal	*TIL's proposal*
$\mathbf{7} > \mathbf{9}$	$\lambda w.\lambda t[\mathbf{7} > \mathbf{9}]$
$[\mathbf{H}(\mathbf{K}_{tw})]_{tw}$	$\lambda w.\lambda t[\mathbf{H}_{wt}(\mathbf{K}_{wt})]$
$[\mathbf{Was}(\mathbf{H}(\mathbf{K}_{tw}))]_{tw}$	$\lambda w.\lambda t[\mathbf{Was}_t(\mathbf{Once}_w(\lambda w'.\lambda t'[\mathbf{H}_{w't'}(\mathbf{K}_{w't'})], \lambda t''[t'' = t])]$

The excessive use of λs in TIL is evident. But it is only partly caused by different types for intensions: Tichý's early idea that a sentence stands for a proposition in any context plays a role here. Its purpose was twofold: (a) to offer propositions as objects of belief attitudes and thus implement PWS; (b) to dismiss Montague's [1974] contextualism, according to which a sentence stands for a truth value in an extensional context, but for a proposition in an intensional context in case of e.g. belief attitudes. In THL, both Montague's and Tichý's proposals are replaced by proposing constructions of truth values as the entities the sentences stand for, regardless of the contexts of their use. The third example shows the biggest price of TIL's types of intensions: its cumbersome and questionable analysis of tenses (for an explanation of the details of TIL's analysis, see e.g. [Tichý, 1980]).

Following [Tichý, 1988], *belief attitudes* are modelled as attitudes towards constructions. The object of a belief attitude is a construction

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$C_1{:}x$ is derivable: $\Gamma\{[[\Box(p)](t)](w) \Leftrightarrow_o \forall^\rho\lambda t.\forall^\omega\lambda w.[p(t)](w)\}$; $\Gamma\{[[\mathbf{Was}(p)](t)](w) \Leftrightarrow_o \exists^\rho\lambda t'([p(t')](w) \wedge (t > t'))\}$.

C (not the entity v-constructed by C) that is usually delivered by the construction of the form $\ulcorner C \urcorner$. The superscript "^{n}" in e.g. "$\mathscr{B}^n$" indicates that the operator's second relatum is an nth-order construction. For an extensive study of belief attitudes within THL (and for further references), see [Raclavský, 2020].

Note that the *de re* reading of A_H, which I denote by "A_H^{re}", must contain substitution (see Section 2.4) of **FLT** for c^1 in the respective open construction, otherwise the belief would not be modelled as a belief *de re*. Note also that the presence of free variables t and w associated with the symbol denoting an intension I of type $\rho \mapsto (\omega \mapsto \tau)$ indicates an application of I to the time instant T and then the possible world W of evaluation; similarly for intensions of other types. On the other hand, t' and w' are λ-bound in all constructions in which they occur; in fact, I used collisionless renaming of variables t to t' and w to w' to make their λ-binding more visible. Below, A_H's formalisation will lack the past tense operator such as $[\mathscr{B}^n_{\mathbf{A}} \ulcorner C \urcorner]_{tw}$ and will be contracted as $[\mathscr{B}^n(\mathbf{A}, \ulcorner C \urcorner)]_{tw}$.

Finally, conclusions of A_E–A_H are each of the form (let C/o; round brackets will often be suppressed) $\exists^\tau(\lambda x.C)$.

2.3. Natural deduction for TT^*

A *natural deduction* system *in sequent style* for TT^* exists and is denoted by $\mathsf{ND}_{\mathsf{TT}^*}$. It is based on [Tichý, 1982, 1986] and [Raclavský, 2014, 2020; Raclavský et al., 2015; Raclavský and Pezlar, 2019], where one finds Kuchyňka's rules for constructions of the forms $\ulcorner C \urcorner$ and $[\![C]\!]_\tau$ which are missing in Tichý's writings.[8]

The rules of $\mathsf{ND}_{\mathsf{TT}^*}$ are made from sequents while sequents are made from congruence statements called *matches* $\mathcal{M}$ (matches, sequents and rules are in fact constructions, see Raclavský [2014, 2020]). Matches are of the form $C{:}x$ or $C{:}X$ or $C{:}\ulcorner C_0 \urcorner$; all the three types of matches are (schematically) represented in notation by $C{:}\mathbf{x}$. Empty matches of the form $C{:}\_$ are also allowed. A valuation v (w.r.t. $\mathfrak{F}$) *satisfies* $\mathcal{M}$ (in $\mathfrak{M}$) iff both entities flanking ":" are v-*congruent* ($\cong$), i.e. they v-construct (in $\mathfrak{M}$) the same object C, or they are both v-improper.

A *sequent* $\mathcal{S}$ is of the form $\Gamma \Rightarrow \mathcal{M}$, where Γ is a finite set of matches and $\Rightarrow$ is a symbol of *(syntactic) consequence*. Γ forms $\mathcal{S}$'s *antecedent*

[8] For techniques of *ND* [see e.g. Indrzejczak, 2010; Negri et al., 2001; Quieroz et al., 2011].

and $\mathcal{M}$ forms its *succedent*. $\mathcal{S}$ is *valid* (in $\mathfrak{M}$) iff every valuation that satisfies (in $\mathfrak{M}$) all members of Γ also satisfies (in $\mathfrak{M}$) $\mathcal{M}$.

A *rule* $\mathcal{R}$ is of the form $\bar{\mathcal{S}}_m \vdash \mathcal{S}$. $\mathcal{R}$ is *valid* (in $\mathfrak{M}$) iff every v that satisfies (in $\mathfrak{M}$) $\bar{\mathcal{S}}_m$ also satisfies (in $\mathfrak{M}$) $\mathcal{S}$. Instead of the horizontal notation with "$\vdash$" I also use the tree notation with "______". I write "Γ, Δ" (where Δ is another set of matches) instead of "$\Gamma \cup \Delta$" and "$\Gamma, \mathcal{M}$" instead of "$\Gamma \cup \{\mathcal{M}\}$"; I use the empty space above "______" (or on the left-hand side of "$\vdash$") to indicate the empty Γ.

A *derivation* $\mathcal{D}$ is a finite *sequence* S *of sequents*; each member of S is either a member of a *set of sequents* H, or the result of an application of a rule from a certain *set of rules* R to some preceding members of S or members of H. $\mathcal{D}$ is also called in brief a *proof* of $\mathcal{D}$'s last sequent; (numbered) members of S are called *steps*. A derived rule revealed by a derivation of $\mathcal{S}$ from S using the rules from R (and $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules) will be denoted by $S \vdash_R \mathcal{S}$.

Here are some important rules utilised in my derivations of (EG) below; for the list of $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules, see Appendix. Let $A, Y, C, \mathbf{y}/\tau; F/\tau \mapsto o; D_1, \mathbf{x}_1/\tau_1, ..., D_m, \mathbf{x}_1/\tau_m; \mathsf{T}$ (True) $/o$; $\exists^\tau/ \ (\tau \mapsto o) \mapsto o$. *Conditions* of the rules should include that the variables occurring within the rules are pairwise distinct and are not free in $\Gamma, \mathcal{M}$ and other constructions occurring in the rule (but, of course, x typically occurs freely in C of $C_{(D/x)}$; $C_{(D/x)}$ and $C_{(\bar{D}_m/\bar{x}_m)}$ are treated in Section 2.4). Let us assume no mishandling of orders.

$$\text{(AX)}\ \frac{}{\Gamma, \mathcal{M} \Rightarrow \mathcal{M}} \qquad \text{(WR)}\ \frac{\Gamma \Rightarrow \mathcal{M}}{\Gamma, \mathcal{M} \Rightarrow \mathcal{M}} \qquad (\exists\text{-I})\ \frac{\Gamma \Rightarrow F(A){:}\mathsf{T}}{\Gamma \Rightarrow \exists^\tau(F){:}\mathsf{T}}$$

$$(\beta\text{-EXP})\ \frac{\Gamma \Rightarrow D_1{:}\mathbf{x}_1;\ \ldots;\ \Gamma \Rightarrow D_m{:}\mathbf{x}_m;\ \Gamma \Rightarrow C_{(\bar{D}_m/\bar{x}_m)}{:}\mathbf{y}}{\Gamma \Rightarrow [\lambda \tilde{x}_m.C](\bar{D}_m){:}\mathbf{y}}$$

(AX) is typically used for an introduction of an assumption into a piece of inference. (WR) is the weakening rule expressing the monotonicity of $\Rightarrow$. (β-EXP) complements the fundamental rule of λ-calculus, the rule of β-contraction; β-contraction captures an application of a function to an argument. ($\exists$-I) says that if it is true that some A is an F, then it is true that F is non-empty.

The general form of an *argument* A, which corresponds to a *reading* of an argument formulated in NL, is

$$A := \ C_1, ..., C_m/C$$

where $\bar{C}_m, C/o$, while $\bar{C}_m$ and C are the constructions expressed by NL-sentences $\bar{S}_m$ and S. $\bar{C}_m$ and C are called A's *premises* and A's *conclusion*; the familiar terms will also be used for $\bar{S}_m$ and S. As indicated by the notation $C_{(i)}/o$, each construction $C_{(i)}$ v-constructs an o-object, e.g. T or F, or is v-improper, while it is supposed to v-construct an o-object; we may briefly say that those constructions are o-constructions. For example, $[\mathbf{H}(\mathbf{K}_{tw})]_{tw}$ is an o-construction.

Now an argument A is *semantically valid* iff for every valuation v, if all A's premises $\bar{C}_m$ v-construct T, A's conclusion C also v-constructs T. Given the above definition of a valid sequent, it is clear that to each semantically valid argument A, there corresponds a sequent

$$\mathcal{S} := \ \Gamma, C_1{:}\mathsf{T}, \ldots, C_m{:}\mathsf{T} \Rightarrow C{:}\mathsf{T}$$

containing A's premises $\bar{C}_m$ and A's conclusion C. Philosophically, one follows here the idea of Frege, Tichý and others [see, e.g., Pezlar, 2014], according to which an inference is a 2D-affair: an inferential step contains a statement with all its assumptions, which are listed in the sequent's antecedent: thus, a genuine inference takes you from one 'logical truth' to another 'logical truth'.

There naturally arises a question whether the sequent $\mathcal{S}$ is indeed a 'logical truth', i.e. a (semantically) valid sequent. One can verify it (a) *semantically* by calculating whether $\mathcal{S}$'s succedent $C{:}\mathsf{T}$ is satisfied by all those vs that satisfy all matches $C_i{:}\mathsf{T}$ of $\mathcal{S}$'s antecedent. Or, one can verify it (b) *syntactically* by deriving $\mathcal{S}$ from (i) the set of $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules or from (ii) the set of $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules and the set R consisting of the rules derived from $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules.

Consequently, an argument A can be shown *valid* either (a) *semantically* by demonstrating the sequent $\mathcal{S}$ that corresponds to A is valid; or, (b) *syntactically* by demonstrating $\mathcal{S}$ that corresponds to A is derivable either from $\mathsf{ND}_{\mathsf{TT}^*}$'s primitive rules alone, which is denoted by $\vdash \mathcal{S}$, or from the set R of $\mathsf{ND}_{\mathsf{TT}^*}$'s derived rules (and, of course, also from its primitive rules), which is denoted by $\vdash_R \mathcal{S}$ (if appropriate, R occurs on the right-hand side of "_____"). In the latter case we also say that an argument is *justified by* (a possibly empty) R of $\mathsf{ND}_{\mathsf{TT}^*}$, or that it is *syntactically valid* w.r.t. R.[9]

[9] Instead of the term "syntactic validity", the term "proof-theoretic validity" (contrasting with the term "model-theoretic validity") seems to be appropriate, but it has already been used in proof-theoretic semantics with a meaning different from the one we need in this paper [cf. e.g. Schroeder-Heister, 2006].

Finally, note that an *invalid argument* A cannot be syntactically valid because the sequent $\mathcal{S}$ corresponding to A is not derivable from $\mathsf{ND}_{\mathsf{TT}^*}$. It means that $\mathcal{S}$ is not syntactically valid, which also means that A is not semantically valid.

2.4. Explicit substitution in TT^*

For a sufficient specification of (EG) and some other derivation rules, the operator of substitution must be defined. This is indeed important because the proper character of (EG) changes depending on the logic in which it is embedded and this is mainly due to the nature of the substitution function Sub^n (usually abbreviated in notation to "*Sub*") encoded by the notation (t/x).

In TT^*, I adopt Tichý's [1988] *explicit substitution* which is presented by constructions of the form $[\![Sub(\ulcorner D \urcorner, \ulcorner x \urcorner, \ulcorner C \urcorner)]\!]_\tau$ which is usually abbreviated in notation to $C_{(D/x)}$.

Let "$C_{(\bar{D}_m/\bar{x}_m)}$" be short for "$C_{(D_1/x_1)(D_2/x_2)\cdots(D_m/x_m)}$".[10] The function *Sub*, which is v-constructed by *Sub*, is a function of type $\langle *^n, *^n, *^n \rangle \mapsto *^n$; "*Sub*" is a new constant of $\mathcal{L}_{\mathsf{TT}^*}$. *Sub* maps triples of constructions such as $\langle D, x, C \rangle$ to constructions $C_{(D/x)}$.

The following definition of *Sub* displays C and $C_{(D/x)}$ that preserve the *Compensation Principle* (*CP*): $C_{(D/x)}$ is $v(\mathrm{D}/x)$-congruent with C (where D is v-constructed by D). The principle was stated by Tichý [1988]; for its full proof, see [Raclavský, 2020].[11]

DEFINITION 2.1 (The substitution function *Sub*). Let $C, D, B, \bar{B}_m, x, y$ be nth-order constructions, for $1 \leqslant n$. In II.v and II.vi below, B is of order $n-1$. Let $FV(C)$ stand for the set of all free variables that are nth-order subconstructions of C.

I. If x is not free in C, $C_{(D/x)}$ is identical with C.

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[10] The notion of explicit substitution was popularised by Abadi et al. [1991].

[11] My present definition of *Sub* is based on my and Kuchyňka's definitions in [Raclavský, 2020; Raclavský et al., 2015]. Its original predecessor is Tichý's [1988] definition, which was formulated in an English metalanguage and was based on Tichý's [1982] early definition. I and Kuchyňka [2015] modified Tichý's 1988 version to meet the definition of free variables from [Raclavský, 2009], where the theorem $^{20}C \cong C$ (roughly, $[\![C]\!]_\tau \cong C$) disclosed that a variable free in C is evidently free in $[\![C]\!]_\tau$. Tichý seems to have overlooked this, since he considered x be not free in ^{20}C and thus not susceptible to substitution. My subsequent changes of *Sub*'s definition were motivated by undesirable consequences resulting from our departure from Tichý's too restrictive proposal.

II. If, on the other hand, x is free in C:

	If C *is ...*	$C_{(D/x)}$ *is ...*	*condition:*
i.	x	D	
ii.	$B(\tilde{B}_m)$	$B_{(D/x)}(\tilde{B}_{m(D/x)})$	
iii.	$\lambda y.B$	$\lambda y.B_{(D/x)}$	$x \in FV(B)$ and $y \notin FV(D)$
iv.	$\lambda y.B$	$[\lambda z.B_{(z/y)}]_{(D/x)}$	$x \in FV(B)$ and $y \in FV(D)$ and $z \notin FV(B) \cup FV(D)$
v.	$\llbracket B \rrbracket_\tau$	$\llbracket B_{(D/x)} \rrbracket_\tau$	$\llbracket B \rrbracket_\tau$ is not reducible according to $\llbracket \ulcorner A \urcorner \rrbracket_\tau \cong A$
vi.	$\llbracket B \rrbracket_\tau$	$C'_{(D/x)}$	where C' is the most reduced form of $\llbracket B \rrbracket_\tau$ according to $\llbracket \ulcorner A \urcorner \rrbracket_\tau \cong A$

To illustrate the definition, let $1, 2, 3/\rho; =^\rho / \langle \rho, \rho \rangle \mapsto o$; if $D := ((1+2) =^\rho 3)$ and $C := (x =^\rho x)$; $C_{(D/x)}$ is $\llbracket Sub, \ulcorner (1+2) =^\rho 3 \urcorner, \ulcorner x \urcorner, \ulcorner x =^\rho x \urcorner \rrbracket_\rho$, i.e. $(x + x)_{((1+2)=^\rho 3/x)}$, which can be reduced to $((1+2) =^\rho 3) =^\rho ((1+2) =^\rho 3)$. On the other hand, such D cannot be substituted for x in (say) $C := \ulcorner x \urcorner$ in which x is not free, the result of its substitution is simply C, cf. point I. To illustrate points II.v and II.vi, $C := \llbracket \ulcorner \llbracket B \rrbracket_\tau \urcorner \rrbracket_\tau$ is first simplified to $C' := \llbracket B \rrbracket_\tau$, and one substitutes $\llbracket B_{(D/x)} \rrbracket_\tau$.

Point II.iv of the definition prevents an undesirable binding of free variables by utilising *collisionless renaming* of λ-bound variables: if one substitutes (say) $D := y$ for x in $C := \lambda y.x$ (which v-constructs a constant function), the result is $\lambda z.x_{(y/x)}$ (not $\lambda y.x_{(y/x)}$ which is not $v(y/x)$-congruent to C since it v-constructs the identity function).[12]

The CP is used in proofs of a number of derivability lemmas/theorems concerning substitution which we may adapt from [Tichý, 1982, 1986]. The most important one is

$$(Sub\text{-I})\ \frac{\Gamma \Rightarrow D{:}x \qquad \Gamma \Rightarrow C{:}o}{\Gamma \Rightarrow C_{(D/x)}{:}o}$$

$$(Sub\text{-E})\ \frac{\Gamma \Rightarrow D{:}x \qquad \Gamma \Rightarrow C_{(D/x)}{:}o}{\Gamma \Rightarrow C{:}o}$$

where $D, x/\tau; C, o/o; Sub/\langle *^n, *^n, *^n \rangle \mapsto *^n$.

[12] The collisionless of λ-bound variables renaming is known in λ-calculus as α-*conversion.* Here is an adoption of Tichý's [1982] rule: $\Gamma\{\lambda \tilde{x}_m.C \Leftrightarrow \lambda \tilde{y}_m.C_{(\bar{y}_m/\bar{x}_m)}\}$, where $x_1, y_1/\tau_1; ...; x_m, y_m/\tau_m$.

3. The rule (EG) in *ND* for TT^*

Now we are ready to state the rule (EG); we begin with its conditionalised form. I derive both rules from the primitive rules of $\mathsf{ND}_{\mathsf{TT}^*}$ using an auxiliary theorem.

At first glance, the two rules resemble Tichý's [1986] rules from his Lemmas 11.5 and 11.6, which were formulated in the metalanguage for his simple TT, being tailored to TIL and his not-so-convenient prevention of the collision of variables on substitution. Tichý's proofs were formulated in an English metalanguage and were faithful to the intricacies of his system.

My proposal is generally much simpler, though it deals with more sophisticated constructions and an extended substitution function. The first way in which it is simpler is that I have utilised a much more efficient mechanism of collision avoiding substitution. Tichý's early mechanism forced him, together with TIL, to state restrictive conditions for (EG) and other rules, while those conditions relied on his theorems on exposure and hospitability. The second simplification is the choice of THL, which allows a straightforward substitution into the explicata of sentential meanings, which is not available in the case of TIL's explicata (which are usually λ-abstracts). Consequently, many pages of Tichý's auxiliary definitions and theorems could be abandoned without any loss.

Here are the major general advantages of my proposal in comparison with Tichý's:

i. The rules are formulated inside the hyperintensional system TT^*, unlike Tichý's rules that are restricted to his intensional logic.
ii. The rules employ the intra-theoretical operator *Sub* that v-constructs the substitution function *Sub*, which operates even on 'higher-order' ('meta-logical') constructions such as $\ulcorner C \urcorner$ and $[\![C]\!]_\tau$, which Tichý's early system lacks.
iii. The proofs are sequences of TT^*'s constructions as checkable steps that arise by applications of the derivation rules of $\mathsf{ND}_{\mathsf{TT}^*}$ (unlike Tichý's metalinguistic demonstrations). Because of the differences between my approach and Tichý's approach, the proofs are rather different.
iv. The rules are suitable for THL and so they are generally simpler than Tichý's, since THL offers simpler explicata of NL-meanings than TIL.

3.1. The Rule of Conditionalised Existential Generalisation

Let us begin with the conditionalised variant of (EG).

THEOREM 3.1 (Conditionalised Existential Generalisation). *Let* $D, d, x/\tau$; $C, \mathsf{T}/o$; $\exists^\tau/(\tau \mapsto o) \mapsto o$; $Sub/\langle *^n, *^n, *^n\rangle \mapsto *^n$. *Then,*

$$\text{(cond-EG)}\ \frac{}{\Gamma, D{:}d, C_{(D/x)}{:}\mathsf{T} \Rightarrow \exists^\tau \lambda x.C{:}\mathsf{T}}$$

is a valid derived rule of $\mathsf{ND_{TT^*}}$.

PROOF. We are going to show that (cond-EG)'s final sequent is derivable from the set of $\mathsf{ND_{TT^*}}$'s primitive rules. The validity of (cond-EG) thus relies on the validity of the primitive rules of $\mathsf{ND_{TT^*}}$.

1. $\Gamma, C_{(D/x)}{:}\mathsf{T} \Rightarrow C_{(D/x)}{:}\mathsf{T}$ — by (AX)
2. $\Gamma, D{:}d \Rightarrow D{:}d$ — by (AX)
3. $\Gamma, D{:}d, C_{(D/x)}{:}\mathsf{T} \Rightarrow C_{(D/x)}{:}\mathsf{T}$ — from 1 by (WR)
4. $\Gamma, D{:}d, C_{(D/x)}{:}\mathsf{T} \Rightarrow D{:}d$ — from 2 by (WR)
5. $\Gamma, D{:}d, C_{(D/x)}{:}\mathsf{T} \Rightarrow [\lambda x.C](D){:}\mathsf{T}$ — from 3 and 4 by (β-EXP)
6. $\Gamma, D{:}d, C_{(D/x)}{:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x.C){:}\mathsf{T}$ — from 5 by ($\exists$-I) $\dashv$

The rule (cond-EG) contains the match $D{:}d$ which says that D is v-proper. $D{:}d$ is apparently convertible to $\exists^\tau(\lambda d(D = d)){:}\mathsf{T}$ which makes the *semantic presupposition* that "D" is a denoting term even more obvious. The final sequent of (cond-EG) thus (roughly) corresponds to arguments such as

$$\frac{\begin{array}{c}\textit{Donna exists.}\\ \textit{Donna is a girl.}\end{array}}{\textit{Someone is a girl.}}\ \text{(cond-EG)}$$

In usual reasoning, however, one often performs existential generalisation without stating that 'existential' condition as an assumption — a proper rule (EG) should therefore lack it.

3.2. The Rule of Instantiation of Exposure

A proper rule (EG) can be obtained if we use the theorem I prove in this section.

First, to provide a generalisation of our formulations of the theorem and also (EG) for cases when one substitutes not D, but e.g. D_{tw} that v-constructs the value of the intension v-constructed by D, let constructions of the forms C, C_t, C_w, C_{tw} be written as $C_{(0,0)}$, $C_{(1,0)}, C_{(0,1)}$,

$C_{(1,1)}$, being thus of the form $C_{(k,l)}$ for $0 \leqslant k, l \leqslant 1$. Let a similar convention apply to types, so we get $\tau_{(l,k)}$. For example, $\tau_{(1,0)}$ is $\omega \mapsto \tau$, which corresponds to C_w, i.e. $C_{(0,1)}$ (note the inverse order of indices).

The following rule is called (exp-INST) due to its surface resemblance to Tichý's [1986] (meta-linguistic) theorem 11.1. The rule is a particular version of the primitive rule ($[\![.]\!]$-INST.i), cf. Appendix. The rule (exp-INST) says that if $C_{(D_{(k,l)}/x)}$ is v-proper and follows from Γ, then $\mathcal{M}$ follows from Γ even without assuming that $D_{(k,l)}$ is v-proper, since this follows from $C_{(D_{(k,l)}/x)}$'s being v-proper.

THEOREM 3.2 (Instantiation of Exposure).
Let $C/o; d, x/\tau; D/\tau_{(l,k)}$; $0 \leqslant k, l \leqslant 1$. Then,

$$\text{(exp-INST)}\ \frac{\Gamma \Rightarrow C_{(D_{(k,l)}/x)}{:}o \qquad \Gamma, D_{(k,l)}{:}d \Rightarrow \mathcal{M}}{\Gamma \Rightarrow \mathcal{M}}$$

is a valid rule of $\mathsf{ND}_{\mathsf{TT}^*}$.

My *correctness proof* of the rule draws on the complexity of constructions, for which the following definition of *construction's rank* $r(C)$ [borrowed from Raclavský, 2020; Raclavský et al., 2015] is indispensable:

i. If $C := X$ (where X is a constant) or $C := x$, then $r(C) = 1$.
ii. If $C := \ulcorner C_0 \urcorner$ or $C := [\![C_0]\!]_\tau$ or $C := \lambda\tilde{x}_m.C_0$, then $r(C) = n + 1$, where n is the rank of C_0.
iii. If $C := F(\bar{X}_m)$, then $r(C) = n + 1$, where n is the greatest rank among the ranks of $F, \bar{X}_m$.

PROOF. First, note carefully that $C{:}x$ reads "the construction C is v-proper". The proof is based on the idea that the match $C_{(D_{(k,l)}/x)}{:}o$ is entailed by Γ if the 'assumption' $D_{(k,l)}{:}d$ is already implicitly involved in Γ. Consequently, the 'assumption' $D_{(k,l)}{:}d$ is dispensable in entailing $\mathcal{M}$ from Γ, which means that the sequent $\Gamma \Rightarrow \mathcal{M}$ is derivable from the sequents $\Gamma \Rightarrow C_{(D_{(k,l)}/x)}{:}o$ and $\Gamma, D_{(k,l)}{:}d \Rightarrow \mathcal{M}$.

There are two principal cases, I and II.

Case I: C does not contain a free occurrence of x. Then, the question whether $C_{(D_{(k,l)}/x)}{:}o$ is independent of the question whether $D_{(k,l)}{:}d$, for $C_{(D_{(k,l)}/x)} = C$ by *Sub*'s definition part I. Thus, the theorem is obviously satisfied by the C in question.

Case II: C contains a free occurrence of x. To see whether such a C also satisfies the theorem, constructions of Case II are divided in two groups in accordance with their rank. The first group presents (a) an induction base, the rest is covered by (b) induction step. Because of the variety of forms of constructions, (a) consists of constructions of ranks $r = 1$ and $r = 2$, while an inspection of (b) must also proceed in more stages than one. A welcome simplification consists in that constructions of the form $C := \lambda x.C_0$ trivially satisfy the theorem, since they never v-construct an object of type o, and we thus need not investigate them below.

$/r(C) = 1/$ If $r(C) = 1$, the only possible case is $C := x$. Since $x_{(D_{(k,l)}/x)}{:}o$, $D_{(k,l)}$ must be a v-proper construction of an o-object, i.e. $D_{(k,l)}{:}d$, and so the theorem is satisfied by the C in question.

$/r(C) = 2/$ If $r(C) = 2$, the only possible cases are $C := \llbracket x \rrbracket_o$ and $C := f(\bar{x}_m)$. Let us consider $C := \llbracket x \rrbracket_o$ first. $\llbracket x \rrbracket_{o(D_{(k,l)}/x)}$ reduces to $\llbracket x_{(D_{(k,l)}/x)} \rrbracket_o$ (by *Sub*'s definition, part II.v). Since $\llbracket x_{(D_{(k,l)}/x)} \rrbracket_o{:}o$, it trivially follows that $D_{(k,l)}{:}d$, and so the theorem is satisfied by the C in question.

Now consider $C := f(\bar{x}_m)$. $[f(\bar{x}_m)]_{(D_{(k,l)}/x)}$ reduces to $f_{(D_{(k,l)}/x)}(\bar{x}_{m(D_{(k,l)}/x)})$ (by *Sub*'s definition, part II.ii). $f_{(D_{(k,l)}/x)}(\bar{x}_{m(D_{(k,l)}/x)}){:}o$ only if $f_{(D_{(k,l)}/x)}$ (even if assuming $x := f$) v-constructs a function F defined for the argument v-constructed by $\bar{x}_{m(D_{(k,l)}/x)}$. From which follows that $D_{(k,l)}{:}d$, and so the theorem is satisfied by the C in question.

To provide an induction step, let us assume that constructions of—and up to—rank n satisfy the theorem.

$/r(C) = n + 1/$ If $r(C) = n + 1$, there are two relevant cases: $C := F(\bar{A}_m)$ and $C := \llbracket C_0 \rrbracket_o$, whose subconstructions are of rank n or lower. Let us consider $C := F(\bar{A}_m)$ first. $[F(\bar{A}_m)]_{(D_{(k,l)}/x)}$ reduces to $F_{(D_{(k,l)}/x)}(\bar{A}_{m(D_{(k,l)}/x)})$ (by *Sub*'s definition, part II.ii). Since $F_{(D_{(k,l)}/x)}(\bar{A}_{m(D_{(k,l)}/x)}){:}o$, $F_{(D_{(k,l)}/x)}$ v-constructs a total m-ary function F and $\bar{A}_{m(D_{(k,l)}/x)}$ v-constructs a type-theoretically appropriate m-ary argument for F. For that, $D_{(k,l)}{:}d$ is an inevitable condition; so the theorem is satisfied by the C in question.

Now consider $C := \llbracket C_0 \rrbracket_o$. $\llbracket C_0 \rrbracket_{o(D_{(k,l)}/x)}$ reduces to $\llbracket C_{0(D_{(k,l)}/x)} \rrbracket_o$ (by *Sub*'s definition, part II.v). Since $\llbracket C_{0(D_{(k,l)}/x)} \rrbracket_o{:}o$, C_0 v-constructs a construction that v-constructs an o-object. Since C_0 is of rank n, it belongs to constructions that already satisfy the theorem. So the theorem is satisfied by the C in question.

$/r(C) = n + 2/$ Finally, we must consider $C := \llbracket \ulcorner C_0 \urcorner \rrbracket_o$. $\llbracket \ulcorner C_0 \urcorner \rrbracket_{o(D_{(k,l)}/x)}$ reduces to $C_{0(D_{(k,l)}/x)}$ (by *Sub*'s definition, part II.vi).

Since $C_{0(D_{(k,l)}/x)}{:}o$ and C_0 belongs to constructions of rank $n-1$ for which the theorem is already satisfied, the theorem is also satisfied by this particular C. ⊣

3.3. The Rule of Existential Generalisation

Finally we are ready to formulate the derived rule (EG).

THEOREM 3.3 (Existential Generalisation). *Let* $x, d/\tau; D/\tau_{(l,k)}; C, \mathsf{T}/o; \exists^\tau/(\tau \mapsto o) \mapsto o; Sub/\langle *^n, *^n, *^n\rangle \mapsto *^n; 0 \leqslant k, l \leqslant 1$. *Then,*

$$(\mathrm{EG})\ \frac{}{\Gamma, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x.C){:}\mathsf{T}}$$

is a valid derived rule of $\mathsf{ND}_{\mathsf{TT}^*}$.

PROOF. Steps 1–6 mimic the above proof of (cond-EG).

1. $\Gamma, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow C_{(D_{(k,l)}/x)}{:}\mathsf{T}$ by (AX)
2. $\Gamma, D_{(k,l)}{:}d \Rightarrow D_{(k,l)}{:}d$ by (AX)
3. $\Gamma, D_{(k,l)}{:}d, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow C_{(D_{(k,l)}/x)}{:}\mathsf{T}$ from 1 by (WR)
4. $\Gamma, D_{(k,l)}{:}d, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow D_{(k,l)}{:}d$ from 2 by (WR)
5. $\Gamma, D_{(k,l)}{:}d, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow [\lambda x.C](D_{(k,l)}){:}\mathsf{T}$ from 3 and 4 by (β-EXP)
6. $\Gamma, D_{(k,l)}{:}d, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x.C){:}\mathsf{T}$ from 5 by (∃-I)
7. $\Gamma, C_{(D_{(k,l)}/x)}{:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x.C){:}\mathsf{T}$ from 1, 6 by (exp-INST) ⊣

Notice that (EG) can be used in a proof of the following derived rule.

THEOREM 3.4 (η-Expanded Rule of ∃-Introduction). *Let* $x, X/\tau; F/\tau \mapsto o; \mathsf{T}/o; \exists^\tau/(\tau \mapsto o) \mapsto o; Sub/\langle *^n, *^n, *^n\rangle \mapsto *^n; 0 \leqslant k, l \leqslant 1$. *Then,*

$$(\exists\text{-I}^\eta)\ \frac{}{\Gamma, F(X){:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x[F(x)]){:}\mathsf{T}}$$

is a valid derived rule of $\mathsf{ND}_{\mathsf{TT}^*}$.

Proof of ($\exists$-I$^\eta$) that uses the Rules of η-Conversion (derived by Kuchyňka in [Raclavský et al., 2015]; see also [Kuchyňka and Raclavský, 2021; Raclavský, 2020]), which seems to be an obvious choice of a rule in ($\exists$-I$^\eta$)'s derivation, is rather lengthy, which is why it is omitted here. On the other hand, ($\exists$-I$^\eta$) is straightforwardly derivable using (EG).

1. $\Gamma, F_{(X/x)}{:}\mathsf{T} \Rightarrow F_{(X/x)}{:}\mathsf{T}$ (AX)
2. $\Gamma, F(X){:}\mathsf{T} \Rightarrow F_{(X/x)}{:}\mathsf{T}$ from 1 by Definition of *Sub*
3. $\Gamma, F(X){:}\mathsf{T} \Rightarrow \exists^\tau(\lambda x[F(x)]){:}\mathsf{T}$ from 2 by (EG)

($\exists$-I$^\eta$)'s derivability indicates that arguments justifiable by ($\exists$-I$^\eta$) are justifiable by (EG) — but obviously not *vice versa*, since C in (EG) need not be of the form $F(X)$.[13]

4. Application of (EG) and quantifying in

In this section, I first show how to demonstrate the syntactic validity w.r.t. (EG) of those readings of $A_E - A_H$ which are (pre-theoretically) valid (Section 4.1). Then, I explain the mechanism that formally prevents a construction of such proofs for the readings which are not justifiable by (EG), despite it appearing so (Section 4.2).

4.1. Justification of arguments by the rule (EG)

First, it should be noted that my formulation of (EG) does not need variants designed differently for E-, I- and H-contexts, for it is applicable within contexts of these different types uniformly. The uniform usability of rules employing the substitution function *Sub* across various contexts relies on the fact that *Sub* operates on constructions, regardless of the type of objects they v-construct.

To exhibit how (EG) applies within various contexts, one may thus choose any of them. For simplicity reasons, I choose the E-context. Suppose, therefore, we are going to show that

$$A_E \qquad \frac{[\mathbf{H}(\mathbf{K}_{tw})]_{tw}}{\exists^\iota(\lambda x[\mathbf{H}(x)]_{tw})}\,(\text{EG})$$

is a syntactically valid argument. Recall from Section 2.3 that the proof of its syntactic validity w.r.t. $R = \{(\text{EG})\}$ consists in the demonstration

[13] Duží and Jespersen [2015] offered some rules of quantification in attitude contexts, while their rule (R_1) has (if adjusted) the form of ($\exists$-I$^\eta$): $[Att(A, \ulcorner C \urcorner)]_{tw} \vdash \exists^{*^n}(\lambda c^n[Att(A, c^n)]_{tw})$, where $Att(itude)/\langle\iota, *^n\rangle \mapsto \pi; A/\iota; \ulcorner C \urcorner/*^n$. Their other rules are distinct, though their conclusions use $\exists$. The metalinguistic proofs of their rules refer to certain usually unstated, metalinguistic rules. One of them is referred to by "EG" but inspection reveals that it should have the form $\neg Empty(\lambda x.F(x)) \vdash \exists(\lambda x.F(x))$. Since $\neg Empty(F)$ is straightforwardly reducible to (definable by) $\exists(F)$, "EG" is quite dissimilar to our (EG) (which is also classical). Their informal *ND* is a sort of Bochvar three-valued logic, which is significantly inferior to other familiar 3V logics such as e.g. strong Kleene [cf., e.g., Blamey, 1986], and also to our $\mathsf{ND}_{\mathsf{TT}^*}$ which successfully escapes the 3V paradigm.

that the sequent

$$\mathcal{S}_{A_E} := \Gamma, [\mathbf{H}(\mathbf{K}_{tw})]_{tw}{:}\mathsf{T} \Rightarrow \exists^{\iota}(\lambda x[\mathbf{H}(x)]_{tw}){:}\mathsf{T}$$

containing A_E's premiss and conclusion is derivable from the above rules R of $\mathsf{ND}_{\mathsf{TT}^*}$, i.e. that it is justified by (EG) (i.e. $\vdash_R \mathcal{S}_{A_E}$).

Proof demonstrating A_E's syntactic validity w.r.t. $R = \{(\text{EG})\}$:

1. $\Gamma, [[\mathbf{H}(x)]_{tw}]_{(\mathbf{K}_{tw}/x)}{:}\mathsf{T} \Rightarrow [[\mathbf{H}(x)]_{tw}]_{(\mathbf{K}_{tw}/x)}{:}\mathsf{T}$ by (AX)
2. $\Gamma, [\mathbf{H}(\mathbf{K}_{tw})]_{tw}{:}\mathsf{T} \Rightarrow [[\mathbf{H}(x)]_{tw}]_{(\mathbf{K}_{tw}/x)}{:}\mathsf{T}$ from 1 by Definition of *Sub*
3. $\Gamma, [\mathbf{H}(\mathbf{K}_{tw})]_{tw}{:}\mathsf{T} \Rightarrow \exists^{\iota}(\lambda x[\mathbf{H}(x)]_{tw}){:}\mathsf{T}$ from 2 by (EG)

The arguments A_I and A_H are also syntactically valid w.r.t. $R = \{(\text{EG})\}$ in their *de re* readings, see Section 4.2. The respective proofs are exactly similar to A_E's validity proof, so they are left to the reader.[14]

On the other hand, A_I and A_H are not justified by (EG) in their *de dicto* readings. No relevant proofs of their syntactic validity exist — which is discussed in Section 4.2.

4.1.1. ... and partiality

Now let me add a note concerning partiality. Our argument A_E allows two ways of logical elucidation. Above, we assumed that the sentence "*The KF is bald.*" is true, cf. $[\mathbf{H}(\mathbf{K}_{tw})]_{tw}{:}\mathsf{T}$. This assumption implies that the KF exists and is bald, so it is then surely the case that someone is bald. Yet we may change this assumption and assume that the sentence "*The KF is bald.*" lacks a truth value e.g. because there is no KF. So the sentence's *existential import* concerning the king is not fulfilled [cf. e.g. Raclavský, 2011, 2018].

Normally, it is certainly possible to reason with sentences (or constructions) denoting no truth value. For example, no assumption concerning the existence of (say) the KF substantially affects the inferential process, cf. e.g.

The KF is bald.
Everybody who is bald is not young.

The KF is not young.

In the case with A_E, however, it is not possible to admit that the KF does not exist, for (EG)'s conclusion says that a property 'be bald' has an instance.

[14] There are even more trivial examples with E-contexts to begin with, e.g. $[\Box(\lambda t'.\lambda w'[\mathbf{9} > \mathbf{7}])]_{tw} \vdash_{(\text{EG})} \exists^{\rho}\lambda n[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}$.

To be sure, we may start an inference that resembles the above proof of (EG) but manipulates the empty match $[\mathbf{H}(\mathbf{K}_{tw})]_{tw}$:_ and employs ($\beta$-EXP) in its novel version with ":_" [cf. Kuchyňka and Raclavský, 2021]. But we would then need a version of ($\exists$-I) whose final sequent is of the form $\Gamma, F(A)$:_ $\Rightarrow \exists^{\tau}(F)$:$b$, where b is T or F.[15]

However, there is no such rule, since from the fact that there is no A and so $F(A)$ lacks a truth value, one cannot obtain that there is — or is not — at least one F. To sum up, (EG) only validates the ':T'-version of the argument, not the ':_'-one.

4.2. (Im)possibility to quantify in

The present subsection focuses on how substitutivity issues affect existential generalisation targeted (i) at belief sentences (the case of H-contexts) and (ii) at sentences with modal operators (the case of I-contexts). Case (i) is concerned with a bare type-theoretically adequate applicability of *Sub*, while case (ii) involves complications with λ-binding.

Before we proceed further, I would like to introduce a useful distinction. First, let us recall the famous post-Russellian [Russell, 1905] [cf. e.g. Quine, 1956] distinction between (a) *narrow* and (b) *wide occurrences* of descriptions in belief sentences, whose old-fashioned formalisation is as follows:

(a) $Bel_a \exists x(F_x \wedge G_x \wedge \forall y(F_x \rightarrow x = y))$
(b) $\exists x Bel_a(F_x \wedge G_x \wedge \forall y(F_x \rightarrow x = y))$

The distinction was later revitalised as a distinction between a (a) *de dicto* and (b) *de re reading of belief sentences*, and the questionable formalisation of belief sentences in terms of $\exists$ was dropped.

The differences between (a) and (b) readings allow distinct substitutions. In case of (a), the occurrence of *Bel* as an hyperintensional operator is genuine, nothing can be substituted in its scope. In case of (b), however, the term x is not bound in the scope of *Bel*, as it is apparent even from the outdated formalisation in terms of $\exists$. On its *de re reading* thus, an argument of the form

$$t_1 = t_2,\ Bel_a \varphi_{(t_1/x)} \ /\ Bel_a \varphi_{(t_2/x)}$$

[15] The possibility that $\exists^{\tau}(F)$ does not v-construct a truth value is excluded by the nature of $\exists^{\tau}$ because $\exists^{\tau}$ returns `T` or `F` dependently on non-emptiness of the total/partial set to which it is applied.

is valid, though it is not valid on its *de dicto* reading which does not allow for such substitutions — as already discussed in Section 1.

To avoid somewhat disputable margins of the *de dicto–de re* distinction, I will utilise a slightly different distinction [borrowed from Raclavský, 2020]:

> (a) *Genuine H-context* / (b) *apparent H-context* is a H-context such that one (a) cannot / (b) can substitute logically equivalent, non-identical expressions/constructions in the scope of the H-operator inducing the H-context.

The notion of a genuine H-context is thus identical to the notion of H-context considered by Cresswell [1975] and others. On the other hand, the apparent H-context only appears to be such: for example, "B" in "*A believes of B that she is an F.*" is substitutable, though it occurs in the scope of "*believes*".

4.2.1. ... and the case of H-contexts

Figure 1 gives two arguments with belief sentences for which it is possible to easily adjust the validity proof above (see Section 4.1) to demonstrate their syntactic validity w.r.t. $R = \{(\mathrm{EG})\}$.[16]

As we observed on step 2 of the above proof (see Section 4.1), the conversion of their premises using definition of *Sub* into their *Sub-forms* (as I will call them) is crucial. Here are the *Sub*-forms relevant to our examples:

Premiss of	*its Sub-form*
A_{H_0}	$[\![Sub^2(\ulcorner\mathbf{FLT}\urcorner, \ulcorner c^1\urcorner, \ulcorner[\mathscr{B}^1_{\mathbf{B}} c^1]_{tw}\urcorner)]\!]_o$
A^{re}_H	$[\![Sub^3(\ulcorner\mathbf{FLT}\urcorner, \ulcorner d^2\urcorner, \ulcorner[\mathscr{K}^2_{\mathbf{A}}\, Sub^2\,(d^2, \ulcorner c^1\urcorner, \ulcorner[\mathscr{B}^1_{tw\mathbf{B}} c^1]_{tw}\urcorner)]_{tw}\urcorner)]\!]_o$

Note that all *Sub*-forms satisfy two principles entailed by the definition of *Sub* (see Section 2.4):

[16] As indicated above, Duží and Jespersen [2015] offered rules which allows justification of the two arguments. The problem is that they did not prove their correctness. They considered the rules as primitive and suggested their correctness proofs that refer to some unstated rules — whose correctness is thus not shown. Moreover, their conception of substitution function is quite unsatisfactory: for example, the result of substitution of D for D' in C (where D and D' need not be a constant and a variable, respectively) is allowed by them to be not $v(\mathrm{D}/D')$-congruent to the initial construction C.

Arguments involving an apparent H-context

A_{H_0}

$$\frac{\textit{Fermat believed FLT.}}{\textit{Some construction is such that Fermat believed it.}}$$

$$\frac{[\mathscr{B}^1_{\mathbf{B}}\ulcorner\mathbf{FLT}\urcorner]_{tw}}{\exists^{*^1}\lambda c^1[\mathscr{B}^1_{\mathbf{B}}\,c^1]_{tw}}\ (\mathrm{EG})$$

A^{re}_H

$$\frac{\textit{Wiles knows FLT is such that Fermat believed it.}}{\textit{Some construction is such that Wiles knows that Fermat believed it.}}$$

$$\frac{[\mathscr{K}^2_{\mathbf{A}}\,Sub^2\,(\ulcorner\mathbf{FLT}\urcorner,\ulcorner c^1\urcorner,\ulcorner[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}\urcorner)]_{tw}}{\exists^{*^2}\lambda d^2[\mathscr{K}^2_{\mathbf{A}}\,Sub^2\,(d^2,\ulcorner c^1\urcorner,\ulcorner[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}\urcorner)]_{tw}}\ (\mathrm{EG})$$

Figure 1.

(i) A premiss and its *Sub*-form are v-congruent and so the conversion of the latter to the former is semantic-value preserving.

(ii) The substitution function *Sub* employed in *Sub*-forms applies to constructions of the same order.

Let us take a closer look at how point (ii) is implemented. Ad *Sub*-A_{H_0}: c^1 and $[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}$ are 2nd-order constructions. Even the 1st-order construction **FLT** is ranked here as a 2nd-order construction, for due to the *Cumulativity Principle for Constructions* [e.g. Raclavský, 2020] which holds in TT^*, every nth-order construction is also an $(n{+}1)$st-order construction. Ad *Sub*-A^{re}_H: d^2 and $[\mathscr{K}^2_{\mathbf{A}}\,Sub^2\,(d^2,\ulcorner c^1\urcorner,\ulcorner[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}\urcorner)]_{tw}$ are 3rd-order constructions; similarly as in the case with *Sub*-A_{H_0}, **FLT** is ranked here as a 3rd-order construction.

On the other hand, *Sub*'s definition does not permit substitution of D for x in C if x is not free in C. In other words, the result of such substitution is C that is unchanged (see part I of the definition). (In the case of genuine H-contexts the impossibility of substitution coincides with the fact that the variable x is of a strictly lower order than the (super)construction C into which one substitutes.) *Sub*'s definition thus prevents situations when one tries to substitute something into a genuine H-context, which is by definition impossible.

To illustrate, here is an example of an argument that could be justified by (EG) if a suitable *Sub*-form of the premiss was obtainable — which, however, it is not. (Assume that the conclusion receives the formalisation displayed below.)

An argument involving a genuine H-context

A_H^{dicto}

Wiles knows that Fermat believed FLT.
Some construction is such that Wiles knows that Fermat believed it.

$$\frac{[\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}\ulcorner\mathbf{FLT}\urcorner]_{tw}\urcorner]_{tw}}{\exists^{*^1}\lambda c^1[\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}\urcorner]_{tw}}\ \text{~~(EG)~~}$$

The alleged *Sub*-form

$$[\![Sub^3(\ulcorner\mathbf{FLT}\urcorner,\ulcorner c^1\urcorner,\ulcorner[\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}\,c^1]_{tw}\urcorner]_{tw}\urcorner)]\!]_o$$

applies Sub^3 to the 3rd-order construction c^1 that is v-constructed by $\ulcorner c^1\urcorner$. But in the 3rd-order construction $C := [\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}\,c^1]_{tw}\urcorner]_{tw}$, the variable c^1 occurs as a 2nd-order construction that is not free in C, for it occurs bound in a construction of the form $\ulcorner C_0\urcorner$. Since this C has no substitutable variable, Sub^3 returns C without any change — by the definition of *Sub*, part I. So the alleged *Sub*-form cannot be v-congruent with A_H^{dicto}'s premiss.

Therefore, the final step of the proof attempting to establish A_H^{dicto}'s syntactic validity would be similar to the proof from Section 4.1, but its last step would be a sequent which does not contain A_H^{dicto}'s premiss, but the reduction (according to *Sub*'s definition) of the alleged *Sub*-form:

$$\Gamma, [\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}\,c^1]_{tw}\urcorner]_{tw}{:}\mathsf{T} \Rightarrow \exists^{*^1}\lambda c^1[\mathscr{K}^2_{\mathbf{A}}\ulcorner[\mathscr{B}^1_{\mathbf{B}}c^1]_{tw}\urcorner]_{tw}{:}\mathsf{T}$$

Thus, not A_H^{dicto}, but another argument is justified by it.

The fact that an argument such as A_H^{dicto} is not justified by (EG) does not necessarily mean that the argument is invalid. But the corresponding sequent is derivable using a rule of attitude logic that says that if it is true that an agent A knows (believes, ...) a construction C, then there is a construction c such that A knows c.

4.2.2. ... and the case of I-contexts

Now let us discuss substitution in I-contexts. Here are two readings of our argument A_I:

Arguments involving modality de dicto/de re	
A_I^{re}	*The NP is necessarily greater than 7.* / *Some number is necessarily greater than 7.* $$\frac{[\Box(\lambda t'.\lambda w'[\mathbf{NP}_{tw} > \mathbf{7}])]_{tw}}{\exists^{\rho}\lambda n[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}}\ \text{(EG)}$$
A_I^{dicto}	*Necessarily, the NP is greater than 7.* / *Some number is such that necessarily, it is greater than 7.* $$\frac{[\Box(\lambda t'.\lambda w'[\mathbf{NP}_{t'w'} > \mathbf{7}])]_{tw}}{\exists^{\rho}\lambda n[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}}\ \text{~~(EG)~~}$$

It is easy to show that the argument A_I^{re} is syntactically valid w.r.t. $R = \{(\text{EG})\}$, i.e. being justified by (EG). The *Sub*-form of A_I^{re}'s premiss is:

Premiss of	*its Sub-form*
A_I^{re}	$[\![Sub^1(\ulcorner\mathbf{NP}_{tw}\urcorner, \ulcorner n\urcorner, \ulcorner[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}\urcorner)]\!]_o$

On the other hand, A_I^{dicto} is not justified by (EG). Its application requires a suitable *Sub*-form, which, however, does not exist. Consider its alleged *Sub*-form:

$$[\![Sub^1(\ulcorner\mathbf{NP}_{t'w'}\urcorner, \ulcorner n\urcorner, \ulcorner[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}\urcorner)]\!]_o$$

When substituting $\mathbf{NP}_{t'w'}$ for n in $[\Box(\lambda t'.\lambda w'.n > \mathbf{7})]_{tw}$, *Sub* prevents λ-binding of $\mathbf{NP}_{t'w'}$'s variables, and so **NP** remains applied to free variables — similarly as in the *de re* case. Thus, the alleged *Sub*-form cannot be v-congruent with $[\Box(\lambda t'\lambda w'[\mathbf{NP}_{t'w'} > \mathbf{7}])]_{tw}$.

As above, the final step of a proof attempting to demonstrate A_I^{dicto}'s syntactic validity (w.r.t. $R = \{(\text{EG})\}$) cannot be the sequent that would contain A_I^{dicto}'s premiss and conclusion, cf.

$$\Gamma, [\Box(\lambda t'.\lambda w'[\mathbf{NP}_{t''w''} > \mathbf{7}])]_{tw}{:}\mathsf{T} \Rightarrow \exists^{\rho}\lambda n[\Box(\lambda t'.\lambda w'[n > \mathbf{7}])]_{tw}{:}\mathsf{T}$$

Similarly as above, though it is not justified by (EG), A_I^{dicto} is semantically valid. Given the meaning of "□" and "*NP*", its premiss is necessarily false and the conclusion is necessarily true. To show its respective syntactic validity, then, the facts expressed in the previous sentence must be rephrased as rules of R that could provide A_I^{dicto}'s justification.

5. Conclusion

In this paper, I have offered the rule (EG) which is applicable within E-, I- and also certain H-contexts. These contexts were managed by an appropriate hyperintensional higher-order logic TT^*, whose fine-grained hyperintensions determine extensions, intensions, and even other hyperintensions.

The logic TT^* enables an explicit substitution, i.e. substitution that occurs within the logic and not within its English metalanguage. The substitution function *Sub* applicable to hyperintensions was defined and the operator *Sub* standing for *Sub* was deployed inside the logic. So even sentences such as "*A believes that B substitutes t for x in* φ" can be subjected to existential generalisation and the respective arguments decided in TT^*.

A natural deduction system for TT^* was offered. Within it, I derived the rule (EG) from the rules such as (∃-I); this crucial result occurred in Section 3.

Let me now emphasise the difference among the rules (EG) and (∃-I) and an η-converted version of (∃-I) (a simplified notation):

$$(\exists\text{-I})\ \frac{F(t)}{\exists(F)} \qquad (\text{EG})\ \frac{\varphi_{(t/x)}}{\exists(\lambda x.\varphi)} \qquad (\exists\text{-I}^\eta)\ \frac{F(t)}{\exists(\lambda x.F(x))}$$

Only (∃-I) is primitive in partial type-theoretic logic, but (EG) and (∃-I$^\eta$) are derivable from (∃-I) and $\mathsf{ND}_{\mathsf{TT}^*}$'s other rules. Unlike (EG), (∃-I) does not contain the substitution operator (t/x). Unlike (EG) and (∃-I$^\eta$), (∃-I) does not contain an extra λx. The common notation of existential generalisation/introduction rule as $\varphi_{[t]} \vdash \exists x.\varphi_{[x]}$ evidently mixes the tree rules together, mingling also the metalinguistic notation "$[t]$" with the object language.

The present work might be understood as an extension of Tichý's work [1982; 1986] which was limited to E- and I-contexts, since his simple TT could not provide an intra-theoretic model of meanings as fine-grained hyperintensional entities. Based on ideas of Tichý's [1988] late TT, my TT^* and its extension THL tailored for the formalisation of NL provide such hyperintensions as internal objects of the logic and treat them within its substitution function and the rules that employ it.

Arguments featuring E-, I- and also certain H-contexts can be decided as justified by (EG) of $\mathsf{ND}_{\mathsf{TT}^*}$ (suitably extended for an analysis for natural language). (EG) relies on *Sub*'s definition; thanks to it, I could explain why some arguments seemingly based on existential generalisation are not justified by (EG).

In particular, (EG) cannot be meaningfully applied to *de dicto* belief sentences, since *Sub*'s definition does not allow a real substitution of something for parts of that-clauses. In case of *de dicto* modalities, *Sub*'s definition does not allow λ-binding such that a term referring to the value of an intension in a time instant and possible world of evaluation would change its reference due to capture of its free variables.

Appendix

The main rules of $\mathsf{ND}_{\mathsf{TT}^*}$. Let $A, \mathbf{a}, X, x, Y, \mathbf{y}/\tau; C, o/o; X_1, \mathbf{x}_1/\tau_1, \ldots, X_m, \mathbf{x}_m/\tau_m; \mathsf{F}, \mathbf{g}, F/\langle\bar{\tau}_m\rangle \mapsto \tau$.

(AX) $\vdash \Gamma, \mathcal{M} \Rightarrow \mathcal{M}$
(WR) $\Gamma \Rightarrow \mathcal{M} \vdash \Gamma, \mathcal{M} \Rightarrow \mathcal{M}$
(CUT) $\Gamma \Rightarrow \mathcal{M}_1;\ \Gamma, \mathcal{M}_1 \Rightarrow \mathcal{M}_2 \vdash \Gamma \Rightarrow \mathcal{M}_2$
(TM) $\vdash \Gamma \Rightarrow \mathbf{a}{:}\mathbf{a}$
(EFQ) $\Gamma \Rightarrow \mathcal{M}_1;\ \Gamma \Rightarrow \mathcal{M}_2 \vdash \Gamma \Rightarrow \mathcal{M}$

Condition: matches $\mathcal{M}_1$ and $\mathcal{M}_2$ are *patently incompatible*: they are either of forms $C{:}\mathbf{a}_1$ and $C{:}\mathbf{a}_2$, where $\mathbf{a}_1, \mathbf{a}_2$ are not v-congruent, or $C{:}\mathbf{a}$ and $C{:}\_$. Patently incompatible matches are never satisfied by the same valuation.

(EXH) $\Gamma, A{:}\_ \Rightarrow \mathcal{M};\ \Gamma, A{:}a \Rightarrow \mathcal{M} \vdash \Gamma \Rightarrow \mathcal{M}$
(app-SUB.i) $\Gamma \Rightarrow F(\bar{X}_m){:}\mathbf{y};\ \Gamma \Rightarrow X_1{:}\mathbf{x}_1; \ldots; \Gamma \Rightarrow X_m{:}\mathbf{x}_m \vdash \Gamma \Rightarrow F(\bar{\mathbf{x}}_m){:}\mathbf{y}$
(app-SUB.ii) $\Gamma \Rightarrow F(\bar{\mathbf{x}}_m){:}\mathbf{y};\ \Gamma \Rightarrow X_1{:}\mathbf{x}_1; \ldots; \Gamma \Rightarrow X_m{:}\mathbf{x}_m \vdash \Gamma \Rightarrow F(\bar{X}_m){:}\mathbf{y}$
(EXT) $\Gamma, \mathsf{F}(\bar{x}_m){:}y \Rightarrow \mathbf{g}(\bar{x}_m){:}y;\ \Gamma, \mathbf{g}(\bar{x}_m){:}y \Rightarrow \mathsf{F}(\bar{x}_m){:}y \vdash \Gamma \Rightarrow \mathbf{g}{:}\mathsf{F}$
(app-INST) $\Gamma \Rightarrow F(\bar{X}_m){:}\mathbf{y};\ \Gamma, F{:}f, X_1{:}x_1, \ldots, X_m{:}x_m \Rightarrow \mathcal{M} \vdash \Gamma \Rightarrow \mathcal{M}$
(β-CON) $\Gamma \Rightarrow [\lambda\tilde{x}_m.Y](\bar{X}_m){:}\mathbf{y} \vdash \Gamma \Rightarrow Y_{(\bar{X}_m/\bar{x}_m)}{:}\mathbf{y}$

(β-EXP) $\mathit{\Gamma} \Rightarrow X_1{:}\mathbf{x}_1; \ldots; \mathit{\Gamma} \Rightarrow X_m{:}\mathbf{x}_m; \mathit{\Gamma} \Rightarrow Y_{(\bar{X}_m/\bar{x}_m)}{:}\mathbf{y} \vdash \mathit{\Gamma} \Rightarrow [\lambda\tilde{x}_m.Y](\bar{X}_m){:}\mathbf{y}$

(λ-INST) $\mathit{\Gamma}, \lambda\tilde{x}_m.Y{:}f \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($\ulcorner\cdot\urcorner$-ID) $\mathit{\Gamma} \Rightarrow \ulcorner X\urcorner{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow \ulcorner Y\urcorner{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow X{:}\mathbf{y} \vdash \mathit{\Gamma} \Rightarrow Y{:}\mathbf{y}$

($\ulcorner\cdot\urcorner$-E) $\mathit{\Gamma} \Rightarrow X{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow \ulcorner Y\urcorner{:}\mathbf{x} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

Condition: X is a subconstruction of Y (the type of Y may differ from τ).

($\ulcorner\cdot\urcorner$-INST) $\mathit{\Gamma}, \ulcorner X\urcorner{:}x \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

Condition ($\ulcorner\cdot\urcorner$-I, $\ulcorner\cdot\urcorner$-E, $\ulcorner\cdot\urcorner$-INST): $\mathbf{x}/*^{n-1}; \mathbf{y}, Y, X/\tau^{n-1}$.

($[\![\cdot]\!]$-I) $\mathit{\Gamma} \Rightarrow X{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow \ulcorner Y\urcorner{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow Y{:}\mathbf{y} \vdash \mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y}$

($[\![\cdot]\!]$-E) $\mathit{\Gamma} \Rightarrow X{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow \ulcorner Y\urcorner{:}\mathbf{x}; \mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y} \vdash \mathit{\Gamma} \Rightarrow Y{:}\mathbf{y}$

($[\![\cdot]\!]$-INST.i) $\mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y}; \mathit{\Gamma}, X{:}x \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($[\![\cdot]\!]$-INST.ii) $\mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y}; \mathit{\Gamma} \Rightarrow X{:}\ulcorner Y\urcorner; \mathit{\Gamma}, Y{:}y \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($[\![\cdot]\!]$-SUB.i) $\mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y}; \mathit{\Gamma} \Rightarrow X{:}\mathbf{x} \vdash \mathit{\Gamma} \Rightarrow [\![\mathbf{x}]\!]_\tau{:}\mathbf{y}$

($[\![\cdot]\!]$-SUB.ii) $\mathit{\Gamma} \Rightarrow [\![\mathbf{x}]\!]_\tau{:}\mathbf{y}; \mathit{\Gamma} \Rightarrow X{:}\mathbf{x} \vdash \mathit{\Gamma} \Rightarrow [\![X]\!]_\tau{:}\mathbf{y}$

Condition ($[\![\cdot]\!]$-I, $[\![\cdot]\!]$-E, $[\![\cdot]\!]$-SUB, $[\![\cdot]\!]$-INST): $\mathbf{x}, X/*^{n-1}; \mathbf{y}, Y/\tau^{n-1}$.

(Sub-I) $\mathit{\Gamma} \Rightarrow D{:}x; \mathit{\Gamma} \Rightarrow C{:}o \vdash \mathit{\Gamma} \Rightarrow C_{(D/x)}{:}o$

(Sub-E) $\mathit{\Gamma} \Rightarrow D{:}x; \mathit{\Gamma} \Rightarrow C_{(D/x)}{:}o \vdash \mathit{\Gamma} \Rightarrow C{:}o$

Condition (Sub-I, Sub-E): $\mathit{Sub}/\langle *^n, *^n, *^n\rangle \mapsto *^n$.

We also need rules for logical operators (the list is not comprehensive). Let $\mathsf{T}, \mathsf{F}, o, \mathbf{i}, \mathbf{j}, I, J/o; \neg/o \mapsto o; \rightarrow /\langle o, o\rangle \mapsto o; \forall^\tau, \exists^\tau/(\tau \mapsto o) \mapsto o; =^\tau /\langle \tau, \tau\rangle \mapsto o; \imath^\tau/(\tau \mapsto o) \mapsto \tau$; see also Section 2.2 for more.

($\neg$-I) $\mathit{\Gamma}, \mathbf{j}{:}\mathbf{i} \Rightarrow \mathcal{M}_1; \mathit{\Gamma}, \mathbf{j}{:}\mathbf{i} \Rightarrow \mathcal{M}_2 \vdash \mathit{\Gamma} \Rightarrow \neg\,\mathbf{j}{:}\mathbf{i}$

Condition: matches $\mathcal{M}_1$ and $\mathcal{M}_2$ are *patently incompatible.*

(RA) $\mathit{\Gamma}, \mathbf{i}{:}\mathsf{T} \Rightarrow \mathcal{M}; \mathit{\Gamma}, \mathbf{i}{:}\mathsf{F} \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($\neg$-INST) $\mathit{\Gamma}, \neg\,\mathbf{i}{:}i \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($\rightarrow$-I) $\mathit{\Gamma}, \mathbf{i}{:}\mathsf{T} \Rightarrow \mathbf{j}{:}\mathsf{T} \vdash \mathit{\Gamma} \Rightarrow (\mathbf{i} \rightarrow \mathbf{j}){:}\mathsf{T}$

($\rightarrow$-E) $\mathit{\Gamma} \Rightarrow (I \rightarrow J){:}\mathsf{T}; \mathit{\Gamma} \Rightarrow I{:}\mathsf{T} \vdash \mathit{\Gamma} \Rightarrow J{:}\mathsf{T}$

($\rightarrow$-INST) $\mathit{\Gamma}, (\mathbf{i} \rightarrow \mathbf{j}){:}i \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($\forall$-I) $\mathit{\Gamma} \Rightarrow F(x){:}\mathsf{T} \vdash \mathit{\Gamma} \Rightarrow \forall^\tau(F){:}\mathsf{T}$

($\forall$-E) $\mathit{\Gamma} \Rightarrow \forall^\tau(F){:}\mathsf{T} \vdash \mathit{\Gamma} \Rightarrow F(\mathbf{x}){:}\mathsf{T}$

($\forall$-INST) $\mathit{\Gamma}, \forall^\tau(F){:}i \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

($\exists$-I) $\mathit{\Gamma} \Rightarrow F(A){:}\mathsf{T} \vdash \mathit{\Gamma} \Rightarrow \exists^\tau(F){:}\mathsf{T}$

($\exists$-E) $\mathit{\Gamma} \Rightarrow \exists^\tau(F){:}\mathsf{T}; \mathit{\Gamma}, F(x){:}\mathsf{T} \Rightarrow \mathcal{M} \vdash \mathit{\Gamma} \Rightarrow \mathcal{M}$

(∃-INST)	$\Gamma, \exists^{\tau}(F){:}i \Rightarrow \mathcal{M} \vdash \Gamma \Rightarrow \mathcal{M}$
(=-I)	$\Gamma \Rightarrow X{:}\mathbf{x} \vdash \Gamma \Rightarrow (\mathbf{x} =^{\tau} X){:}\mathsf{T}$
(=-E)	$\Gamma \Rightarrow (\mathbf{x} =^{\tau} X){:}\mathsf{T} \vdash \Gamma \Rightarrow X{:}\mathbf{x}$
(=-INST)	$\Gamma, (\mathbf{x} =^{\tau} \mathbf{y}){:}i \Rightarrow \mathcal{M} \vdash \Gamma \Rightarrow \mathcal{M}$
($\imath$-I)	$\Gamma \Rightarrow F(\mathbf{x}){:}\mathsf{T}; \Gamma, F(\mathbf{x}){:}\mathsf{T} \Rightarrow y{:}\mathbf{x} \vdash \Gamma \Rightarrow \imath^{\tau}(F){:}\mathbf{x}$
($\imath$-E)	$\Gamma \Rightarrow \imath^{\tau}(F){:}\mathbf{x} \vdash \Gamma, F(\mathbf{x}){:}\mathsf{T}$
($\imath$-INST)	$\Gamma \Rightarrow \imath^{\tau}(F){:}\mathrm{x}; \Gamma, F(\mathbf{x}){:}\mathsf{T} \Rightarrow \mathcal{M} \vdash \Gamma \Rightarrow \mathcal{M}$

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