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Belnap-Dunn Semantics for the Variants of BN4 and E4 which Contain Routley and Meyer's Logic B

Abstract. The logics BN4 and E4 can be considered as the 4-valued logics of the relevant conditional and (relevant) entailment, respectively. The logic BN4 was developed by Brady in 1982 and the logic E4 by Robles and Méndez in 2016. The aim of this paper is to investigate the implicative variants (of both systems) which contain Routley and Meyer's logic B and endow them with a Belnap-Dunn type bivalent semantics.

Keywords: 4-valued logics; many-valued logics; Belnap-Dunn semantics; Routley and Meyer's logic B; relevant logics

1. Introduction

Belnap-Dunn semantics (BD-semantics) was originally introduced by means of the well-known logic B4 developed by Belnap and Dunn to treat inconsistent and incomplete information [cf. 5, 6, 9, 10]. According to this semantics, there is the possibility of assigning T, F, both T and F or neither T nor F to a formula (T represents truth and F represents falsity). The logic B4 is equivalent to Anderson and Belnap's First Degree Entailment (FDE), a logic characterized (determined) by Smiley's matrix MSm4 [cf. 1, pp. 161–162], which is in its turn a simplification of Belnap's 8-elements matrix M_0 [cf. 4], a matrix of considerable importance in the development of relevant logics [cf. 21, pp. 176–179].

Brady defined in 1982 the system BN4 [cf. 7], a logic built upon the matrix MBN4 which is also a modification of the matrix MSm4 referred to above. This system is closely related to both Belnap and Dunn's logic B4 and Routley and Meyer's logic B. On the one hand, BN4 can be

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considered a strengthening of B4 obtained by implicatively expanding the latter. On the other hand, the logic BN4 was first developed by taking as the starting point the axiomatization of B, as Brady himself stated. Therefore, BN4 can also be seen as a 4-valued extension of Routley and Meyer's logic B. As a matter of fact, even though it is tempting to read BN4 as B(oth) and N(either) 4-valued logic, the label was chosen by Brady because "the system contains the basic system B of Routley *et al.* 1982, Chapter 4, and has a characteristic 4-valued matrix set, one of the values being 'n', representing neither truth nor falsity" [cf. 7, p. 32, note 1]. BN4 may be seen as an interesting 4-valued extension of B since Meyer *et al.* maintain that "BN4 is the correct logic for the 4-valued situation where the extra values are to be interpreted in the both and neither senses" [14, p. 25] and according to Slaney, BN4 has the truth-functional implication most naturally associated with the logic FDE referred to above (and equivalent to B4; cf. [22, p. 289]). Moreover, the system BN4 is a central non-classical logic not only due to its relation with the family of relevant logics but also because of its position among other important many-valued logics. In particular, the "strong implication" of the bilattice logic $GLB_{\supset}$ [cf. 2, 3] of considerable importance in artificial intelligence is actually the conditional of BN4 [cf. 13, Appendix].

The logic E4 is built upon a modification of Brady's matrix MBN4 and was developed by Robles and Méndez as a companion to the system BN4 worthy of consideration [cf. 18]. They believe that E4 is related to BN4 in a similar way to which Anderson and Belnap's logic of entailment E is related to the system R [cf. 1 about the logics E and R]. In this sense, the system E4 could be understood as the 4-valued entailment counterpart to the 4-valued relevant conditional presented in BN4.

In the conclusions of [18], Robles and Méndez suggest that E4 might not be the only alternative to BN4 and set out six different tables for $f_{\to}$ which could be of interest. I will investigate the logics built upon these tables and prove that these matrices are the only implicative variants of MBN4 and ME4 that verify Routley and Meyer's logic B (cf. §3; they also verify FDE since B is a proper extension of FDE). In that sense, we will talk about the class of all implicative expansions of B4 verifying B while maintaining the conditional structure of MBN4 or ME4. The aim of this paper is to display the systems built upon these six alternative tables for $f_{\to}$ and endow them with a Belnap-Dunn semantics in order to investigate whether any of these tables can advantageously substitute those of MBN4 and ME4. The results also contribute to pursue further investigation on implicative expansions of FDE[1], which is well-known to be a core non-classical system among many-valued and relevant logics [cf. 15 and references therein].

The structure of the paper is now explained. In Section 2, Belnap and Dunn's matrix MB4 is defined while in Section 3, the six implicative variants of MBN4 and ME4 are displayed and the fact that they are the only ones of their kind verifying Routley and Meyer's logic B is proved. In Section 4, a basic sublogic (labeled "b4") contained in all the systems developed upon the previous matrices (including BN4 and E4) is presented and some of its properties are proved. In Section 5, all the logics characterized by those matrices (let them be called L*ti*-logics) are axiomatized in a Hilbert-style way as extensions of b4. In Section 6, each L*ti*-logic is endowed with a Belnap-Dunn semantics and the soundness theorems are proved. In Section 7, I display the extension and primeness lemmas which will be useful in the completeness theorem proved later, in Section 8. Finally, in Section 9, some properties of L*ti*-logics are explained: L*ti*-logics have the quasi-relevance property [1, p. 417, Proposition 9.4 of this paper] and their characteristic matrices are natural implicative expansions[2] of MB4 (i.e., FDE; cf. Corollary 9.12) among other related properties.

2. The matrix MB4

In this section, the matrix characteristic of Belnap and Dunn's logic B4, MB4[3], will be presented. First of all, some basic notions will be defined.

DEFINITION 2.1 (Languages). The propositional language consists of a denumerable set of propositional variables ($\mathcal{P}$) p_0, p_1, $\ldots$, p_n, $\ldots$ and some or all of the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs ($\mathcal{F}$) are defined in the customary way. A, B (possibly with subscripts 0, 1, $\ldots$, n), etc. are metalinguistic variables.

[1] In [15], some of the most well-studied implicative expansions of the logic FDE are briefly explained, including BN4. In the conclusion of the paper, Omori and Wansing mention some interesting topics for further research, including new expansions of FDE and axiomatizations of relevant logics à la American plan.

[2] I follow Tomova's notion of natural implication as it has been redefined by Robles and Méndez (cf. [23, p. 175] and [19, p. 2, note 2]).

[3] By this label, I will refer to the matrix presented in [5, 6, 10], which is characteristic of the logic FDE.

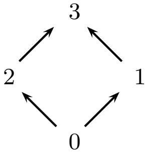
Definition 2.2 (Logics). A logic L is a structure $\langle \mathcal{L}, \vdash_L \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_L$ is a (proof-theoretical) consequence relation defined on $\mathcal{L}$ by a set of axioms and a set of rules of derivation. The notions of *proof* and *theorem* are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_L A$ means that A is derivable from the set of wffs Γ in L; and $\vdash_L A$ means that A is a theorem of L).

Definition 2.3 (Extensions and expansions of a propositional logic L). On the one hand, let $\mathcal{L}$ and $\mathcal{L}'$ be two propositional languages. $\mathcal{L}'$ is a strengthening of $\mathcal{L}$ if the set of wffs of $\mathcal{L}$ is a proper subset of the set of wffs of $\mathcal{L}'$. On the other hand, let L and L' be two logics built upon the propositional languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Furthermore, suppose that every axiom of L is a theorem of L' and all primitive rules of inference of L are provable in L'. Then, L' is an extension of L if $\mathcal{L}$ and $\mathcal{L}'$ are the same propositional language; and L' is an expansion of L if $\mathcal{L}'$ is an strengthening of $\mathcal{L}$. An extension L' of L is a proper extension if L is not an extension of L'.

Definition 2.4 (Logical matrix). A (logical) matrix is a structure $\langle \mathcal{V}, \mathcal{D}, F \rangle$, where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); (3) F is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in F$ such that $f_c\colon \mathcal{V}^n \to \mathcal{V}$.

Definition 2.5 (M-interpretations, M-consequence, M-validity). Let M be a matrix for $\mathcal{L}$. An M-interpretation I is a function from $\mathcal{F}$ to $\mathcal{V}$ according to the functions in F. Then, for any set of wffs Γ and wff A, $\Gamma \vDash_M A$ (A is a consequence of Γ according to M) iff $I(A) \in \mathcal{D}$ whenever $I(\Gamma) \in \mathcal{D}$ for all M-interpretations I ($I(\Gamma) = \inf I(B) \mid B \in \Gamma$, so $I(\Gamma) \in \mathcal{D}$ iff $I(B) \in \mathcal{D}$ for each $B \in \Gamma$). In particular, $\vDash_M A$ (A is M-valid; A is valid in the matrix M) iff $I(A) \in \mathcal{D}$ for all M-interpretations I.

Definition 2.6 (Belnap and Dunn's matrix MB4). The propositional language $\mathcal{L}$ consists of the connectives $\wedge$, $\vee$ and $\neg$. Belnap and Dunn's matrix MB4 is the structure $\langle \mathcal{V}, \mathcal{D}, F \rangle$, where (i) $\mathcal{V}$ is $\{0, 1, 2, 3\}$ and it is partially ordered as shown in the following lattice



(ii) $\mathcal{D} = \{2, 3\}$; (iii) $F = \{f_\wedge, f_\vee, f_\neg\}$, where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice join), respectively. Finally, $f_\neg$ is an involution with $f_\neg(0) = 3$, $f_\neg(3) = 0$, $f_\neg(1) = 1$ and $f_\neg(2) = 2$. Tables for $\wedge$, $\vee$ and $\neg$ are displayed below.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

	0	1	2	3
$\neg$	3	1	2	0

The notions of *MB4-interpretation*, *MB4-consequence* and *MB4-validity* are defined according to Definition 2.5.

Remark 2.7 (On the intuitive meaning of the four values). The truth values 0, 1, 2 and 3 can intuitively be interpreted in MB4 as follows. Let T and F represent truth and falsity. Then, $0 = F$, $1 = N$ (either), $2 = B$ (both) and $3 = T$. Or, in terms of subsets of $\{T, F\}$, we have: $0 = \{F\}$, $1 = \emptyset$, $2 = \{T, F\}$ and $3 = \{T\}$.

The following notion will be especially useful throughout this paper.

Definition 2.8 (Logics determined by matrices). Let $\mathcal{L}$ be a propositional language, M a matrix for $\mathcal{L}$ and $\vdash_L$ a (proof theoretical) consequence relation defined on $\mathcal{L}$. Then, the logic L (cf. Definition 2.2) is determined by M iff for every set of wffs Γ and wff A, $\Gamma \vdash_L A$ iff $\Gamma \vDash_M A$. In particular, the logic L (considered as the set of its theorems) is determined by M iff for every wff A, $\vdash_L A$ iff $\vDash_M A$ (cf. Definition 2.5).

3. Implicative variants of MBN4 and ME4 which verify Routley and Meyer's logic B

On the one hand, Brady developed the system BN4 as a 4-valued extension of B [cf. 21, Chapter 4]. On the other, the system E4 was presented by Robles and Méndez [18] as a companion of BN4 worthy of consideration. As stated by them, E4 is the 4-valued entailment counterpart to the 4-valued relevant conditional represented in BN4. However, they also pointed out that E4 may not be the only interesting alternative to BN4. As a matter of fact, they suggested that there would be at least six other interesting alternatives to it. The aim of this section is to present these matrices and prove that they are the only ones (of their kind) verifying Routley and Meyer's basic logic B.

I start by displaying the matrices upon which logics BN4 and E4 are built.

DEFINITION 3.1 (Brady's 4-valued matrix MBN4 and Robles and Méndez's 4-valued matrix ME4). The propositional language consists of the connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$. MBN4 and ME4 are structures $(\mathcal{V}, \mathcal{D}, F)$ where $\mathcal{V}$, $\mathcal{D}$, $f_\wedge$, $f_\vee$ and $f_\neg$ are defined as in MB4 (cf. Definition 2.6) and $f_\rightarrow$ is defined according to the following implicative tables:

BN4 (t1)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	1	2	3
3	0	1	0	3

E4 (t5)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

On the other hand, the notions of *interpretation*, *consequence* and *validity* in MBN4 and ME4 are understood according to Definition 2.5.

As it has been mentioned above, Robles and Méndez asserted that E4 may not be the only interesting alternative to BN4. In this sense, they suggested that the matrices built upon the following six implicational tables are the only implicative variants of MBN4 and ME4 that verify Routley and Meyer's basic logic B:

t2

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	2	3
3	0	0	0	3

t3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	0	2	3
3	0	0	0	3

t4

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	1	2	3
3	0	1	0	3

t6

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	1	2	3
3	0	0	0	3

t7

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	0	2	3
3	0	0	0	3

t8

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	1	2	3
3	0	0	0	3

The label Mti ($1 \leqslant i \leqslant 8$) will be used to refer to the matrix characterized by the implicative table ti.[4] It is worth underlining that the matrices characterized by the expansions of MB4 with those tables, together with MBN4 and ME4, are the only ones which satisfy B while

[4] I will use the labels Mt1 and Mt5 to refer to MBN4 and ME4, respectively. Consequently, labels Mt2-Mt4 will be used to refer to the implicative variants of MBN4 and similarly, labels Mt6-Mt8 will be employed to allude to the variants of ME4.

preserving one of the following general structures (i.e., either BN4 or E4 implicative matrix structure):

(BN4-type)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1		3		3
2			2	3
3				3

(E4-type)

$\rightarrow$	0	1	2	3
0	3	3	3	3
1		2		3
2			2	3
3				3

Now, suppose that the entries recorded in the tables above are fixed and the blank spaces can be filled with any truth-value from $\mathcal{V}$ (cf. Definition 2.6). Next, we will prove that MBN4, ME4 and the matrices characterized by the rest of the implicative tables shown above (i.e., t2–t4 and t6–t8) are the only ones verifying B.

Proposition 3.2 (Expansions of MB4 which verify Routley and Meyer's basic logic B). *Given the general implicative structures shown above, there are only eight implicative expansions of* MB4 (Mt1–Mt8) *that verify Routley and Meyer's basic logic* B — *which can be axiomatized with* (A1)–(A7) *of* b4 (cf. Definition 4.1) *and modus ponens, adjunction, prefixing, suffixing and contraposition in the form $A \rightarrow \neg B \Rightarrow B \rightarrow \neg A$, as rules of inference — while being based on one of those structures. Those expansions of* MB4 *can be defined as follows: each* Mti ($1 \leqslant i \leqslant 8$) *is a structure $\langle \mathcal{V}, \mathcal{D}, F \rangle$, where $\mathcal{V}$, $\mathcal{D}$, $f_\wedge$, $f_\vee$ and $f_\neg$ are defined as in* MB4 (cf. Definition 2.6) *and $f_\rightarrow$ is defined according to the corresponding* ti *above* (t1 is MBN4 and t5 is ME4).

Proof. The fact that the rules and axioms of B are valid in Mti ($1 \leqslant i \leqslant 8$) can be easily proved with the help of a tester [cf. 12]. However, we also want to specify the reasons why those matrices are the only possible ones (having one of those implicative structures) that verify B. The method will consist in constraining the possibilities for blank spaces in such a way that the validity of (firstly) the rules and (then) the axioms of B is preserved. (1) In order to make the rule MP valid, the following possibilities have to be non-designated: $f_\rightarrow(2,0)$, $f_\rightarrow(3,0)$, $f_\rightarrow(2,1)$, $f_\rightarrow(3,1)$. Once we have limited the possibilities of blank spaces according to (1), we take these results into account and keep limiting possibilities until the validity of the rules and axioms of B is preserved. (2) To validate the rule contraposition, the succeeding possibilities need to take a non-designated value: $f_\rightarrow(3,2)$, $f_\rightarrow(1,2)$, $f_\rightarrow(1,0)$. (3) Likewise, the restriction of the values resulting from the following possibilities to

0 is needed to preserve prefixing or suffixing rules: $f_{\to}(2,0)$, $f_{\to}(3,2)$, $f_{\to}(3,0)$. According to the points made so far, our implicative structures will be restricted as shown below:[5]

(BN4-type)

$\to$	0	1	2	3
0	3	3	3	3
1	a_1	3	a_2	3
2	0	a_3	2	3
3	0	a_4	0	3

(E4-type)

$\to$	0	1	2	3
0	3	3	3	3
1	b_1	2	b_2	3
2	0	b_3	2	3
3	0	b_4	0	3

The previous restrictions are still not sufficient for our matrices to validate the prefixing and suffixing rules, let alone every theorem of B. Thus, the restrictions just remarked will be separately analyzed from now on. Let us start with BN4-type implicational table. For the BN4-type, we find that there are two pairs of values which are related to each other. (4) To verify prefixing rule and A3 of B ($[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$) [cf. 21, Chapter 4], a_1 and a_2 should be the same value (0 or 1); and (5) the same goes for the pair a_3 and a_4 in order to verify the suffixing rule and A5 of B ($[(A \to C) \wedge (A \to B)] \to [(A \vee B) \to C]$) [cf. 21, Chapter 4]. The final result of the restrictions just made are tables t1-t4 shown above. Now, let us consider E4-type implicational table. To assure that rules of B are valid in E4-type matrices, two more restrictions are needed: (6) b_1 and b_4 should be the value 0. Therefore, restrictions in (6) together with those in (1)–(3) result in implicational tables t5–t8 displayed on the previous page. □

It would be easy to verify that if any of the restrictions remarked above is not followed, the resulting implicational tables will fail to verify some rules (or theorems) of B. For instance, let us skip restriction (2). Then, let $f_{\to}(3,2) \in \mathcal{D}$ be the case. For any distinct propositional variables p and q, there is an interpretation I such that $I(p) = 3$ and $I(q) = 2$. Therefore, the rule of contraposition will fail since we have $I(p \to q) \in \mathcal{D}$ but $I(\neg q \to \neg p) \notin \mathcal{D}$, giving $f_{\neg}$ and the restrictions previously made to verify MP (1) –in particular, $f_{\to}(2,\ 0) \notin D$.

4. The basic logic b4 and its properties

Before defining the eight logics determined by each Mti ($1 \leqslant i \leqslant 8$), it will be useful to define the logic b4. b4 is a new system contained

[5] Where $a_i \in \{0,1\}$ ($1 \leqslant i \leqslant 4$) and $b_j \in \{0,1\}$ ($1 \leqslant j \leqslant 4$).

in all the logics we are going to define in this paper and serves a mere instrumental purpose.[6] The label b4 is intended to abbreviate "basic logic included in every variant of BN4 or E4 which contain Routley and Meyer's logic B". In what follows, the basic logic b4 is defined and some of its properties are proved.

Definition 4.1 (b4). The logic b4 is axiomatized with the following axioms and rules of inference.

Axioms:

A1. $A \to A$
A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A7. $\neg\neg A \to A$
A8. $A \to \neg\neg A$
A9. $\neg A \to [A \vee (A \to B)]$
A10. $B \to [\neg B \vee (A \to B)]$
A11. $(A \vee \neg B) \vee (A \to B)$
A12. $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$

Rules of inference:

Adjunction (ADJ): $A,\ B \Rightarrow A \wedge B$
Modus Ponens (MP): $A,\ A \to B \Rightarrow B$
Disjunctive Modus Ponens (dMP): $C \vee A$, $C \vee (A \to B) \Rightarrow C \vee B$
Disjunctive prefixing (dPREF):
$C \vee (A \to B) \Rightarrow C \vee [(D \to A) \to (D \to B)]$
Disjunctive suffixing (dSUF): $C \vee (A \to B) \Rightarrow C \vee [(B \to D) \to (A \to D)]$
Disjunctive Contraposition (dCON): $C \vee (A \to B) \Rightarrow C \vee (\neg B \to \neg A)$
Disjunctive Counterexample (dCTE): $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \to B)$

Remark 4.2 (About b4). b4 is the result of adding the axioms A9–A12 and rules dMP, dPREF, dSUF, dCON and dCTE to Routley and Meyer's

[6] In order to homogenize the proofs developed throughout this paper, I will use a common axiomatic base as wide as possible (i.e., the instrumental system b4 even if that means including some weak rules of inference which are not necessary to axiomatize every expansion of B4 considered (cf. Remark 2 in the Conclusion). Otherwise, we would have to specify different proofs for all the eight logics determined by each M*ti*. (Regarding the need for disjunctive rules in Completeness proofs cf. [8].)

basic logic B. As a matter of fact, b4 can be seen as an extension of dB (i.e., the disjunctive version of Routley and Meyer's logic B).

PROPOSITION 4.3. *The following theorems are derivable in* b4:

T1 $A \leftrightarrow (A \vee A)$
T2 $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
T3 $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$

PROOF. T1–T3 are theorems of FDE, a system included in b4 [see, e.g., 1, p. 158]. □

In the following lines, I will introduce the notion of a Eb4-theory and the types of Eb4-theories of interest in this paper in order to prove some predicable properties of both b4-theories and Eb4-theories. By the label EL, I will refer to an extension (or an expansion, as the case may be) of the logic L (cf. Definition 2.3).

DEFINITION 4.4 (Eb4-theories). Let L be an Eb4-logic. An L-theory $\mathcal{T}$ is a set of wffs closed under adjunction (Adj) and provable L-entailment (L-ent). That is to say, a set of wffs is closed under Adj iff, whenever A, $B \in \mathcal{T}$, then $A \wedge B \in \mathcal{T}$; a set of wffs is closed under L-ent iff, whenever $A \to B$ is a theorem of L and $A \in \mathcal{T}$, then $B \in \mathcal{T}$.

DEFINITION 4.5 (Types of Eb4-theories). Let L be an Eb4-logic and $\mathcal{T}$ be an L-theory. We set (1) $\mathcal{T}$ is prime iff, for wffs A and B, whenever $A \vee B \in \mathcal{T}$, then either $A \in \mathcal{T}$ or $B \in \mathcal{T}$; (2) $\mathcal{T}$ is regular iff $\mathcal{T}$ contains all theorems in L; (3) $\mathcal{T}$ is trivial iff it contains every wff; (4) $\mathcal{T}$ is a-consistent (consistent in an absolute sense) iff $\mathcal{T}$ is not trivial; (5) $\mathcal{T}$ is empty iff it contains no wff.

Immediate by A7 and A8 and the fact that L-theories for Eb4-logics are closed by L-entailment we obtain:

LEMMA 4.6. *For theory $\mathcal{T}$ for an* Eb4-*logic*: $A \in \mathcal{T}$ *iff* $\neg\neg A \in \mathcal{T}$.

LEMMA 4.7. *Let L be an Eb4-logic and $\mathcal{T}$ be a prime L-theory. Then,* (1a) $A \wedge B \in \mathcal{T}$ *iff* $A \in \mathcal{T}$ *and* $B \in \mathcal{T}$; (1b) $\neg(A \wedge B) \in \mathcal{T}$ *iff* $\neg A \in \mathcal{T}$ *or* $\neg B \in \mathcal{T}$; (2a) $A \vee B \in \mathcal{T}$ *iff* $A \in \mathcal{T}$ *or* $B \in \mathcal{T}$; (2b) $\neg(A \vee B) \in \mathcal{T}$ *iff* $\neg A \in \mathcal{T}$ *and* $\neg B \in \mathcal{T}$.

PROOF. (1a) From left to right (⇒): by A2 and the fact that L-theories are closed by L-ent.[7] From right to left (⇐): by the fact that L-theories

[7] This justification, namely, the fact that L-theories are closed by L-ent is constantly used throughout these proofs. Therefore, I will omit it from now on.

are closed by Adj. (1b) ($\Rightarrow$): by T2 ($\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$) and the fact that $\mathcal{T}$ is a prime theory. ($\Leftarrow$): by A4. (2a) ($\Rightarrow$): by the fact that $\mathcal{T}$ is prime. ($\Leftarrow$): just by A4. (2b) ($\Rightarrow$): by T3 ($(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$) and A2. ($\Leftarrow$): by T3 and the fact that $\mathcal{T}$ is closed under Adj. □

DEFINITION 4.8 (Sets of wffs closed under a certain rule). A set of wffs Γ is closed under a rule R iff the conclusion of R belongs to Γ whenever the hypothesis belongs to Γ.

DEFINITION 4.9 (Full regularity). Let L be an Eb4-logic, a L-theory $\mathcal{T}$ is fully regular iff it is a regular L-theory (cf. Definitions 4.4 and 4.5) which is closed under the following rules: MP, dMP, dCON, dPREF, dSUF, dCTE (cf. Definition 4.8).

PROPOSITION 4.10 (Derived rules under which fully regular Eb4-theories are closed). *Let L be an* Eb4*-logic, if $\mathcal{T}$ is a fully regular L-theory, then it is closed under* (1) CON, (2) PREF, (3) SUF, (4) CTE, (5) MT *and* (6) TRAN.

PROOF. Cases (1)–(4): by A4 and T1 ($A \leftrightarrow (A \vee A)$) and the fact that $\mathcal{T}$ is fully regular (i.e., closed under dCON, dPREF, dSUF and dCTE, respectively for each case). Cases (5)-(6): by hypothesis, $\mathcal{T}$ is fully regular (therefore closed under MP) and by the fact that $\mathcal{T}$ is closed under CON and SUF (given what has already been proved in cases (1) and (3)), respectively for each case. □

LEMMA 4.11. *Let L be an* Eb4*-logic and $\mathcal{T}$ be a prime full regular L-theory.*[8] *Then, $A \to B \in \mathcal{T}$ iff either $A \notin \mathcal{T}$ or $B \in \mathcal{T}$ and either $\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$.*

PROOF. (a) ($\Rightarrow$): Suppose (1) $A \to B \in \mathcal{T}$ and, by reductio, (2i) ($A \in \mathcal{T}$ and $B \notin \mathcal{T}$) or (2ii) ($\neg A \notin \mathcal{T}$ and $\neg B \in \mathcal{T}$). If we suppose (2i), given that $\mathcal{T}$ is closed under MP, we get $B \in \mathcal{T}$, which is impossible. Let us suppose then (2ii). We will get $\neg A \in \mathcal{T}$ since $\mathcal{T}$ is closed under MT (cf. Proposition 4.10). But this is also impossible.[9] (b) ($\Leftarrow$): Suppose

[8] We do not actually need our theories to be fully regular in order to prove this lemma, but we do need them to be regular and closed under MP and MT at least. However, being closed under MP and MT also requires being closed under dMP and dCON for some of the logics considered in this paper.

[9] Subcases (2i) and (2ii) could be also proved by means of the corresponding axioms instead of MP and MT rules. However, since only some of the logics we are considering in this paper validate those axioms (cf. Definition 5.1), I will use the rules to generalize the proof as much as possible.

(1) ($A \notin \mathcal{T}$ or $B \in \mathcal{T}$) and ($\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$). Four options should be considered: (1i) $A \notin \mathcal{T}$ & $\neg A \in \mathcal{T}$ or (1ii) $B \in \mathcal{T}$ & $\neg B \notin \mathcal{T}$ or (1iii) $A \notin \mathcal{T}$ & $\neg B \notin \mathcal{T}$ or (1iv) $B \in \mathcal{T}$ & $\neg A \in \mathcal{T}$. In cases (1i) and (1ii), we obtain $A \to B \in \mathcal{T}$ by axioms A9 and A10, respectively, and the fact that $\mathcal{T}$ is prime. Similarly, case (1iii) is solved by A11 and the fact that $\mathcal{T}$ is a prime regular L-theory. Suppose now case (1iv) $B \in \mathcal{T}$ and $\neg A \in \mathcal{T}$. Moreover, let us suppose (2) $A \to B \notin \mathcal{T}$ by reductio. Then, given that $\mathcal{T}$ is regular, we get (3) $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)] \in \mathcal{T}$ by A12. Now, considering (2) and the fact that $\mathcal{T}$ is prime, (4) $(\neg A \wedge B) \to (A \to B) \in \mathcal{T}$. To end, we obtain (5) $A \to B \in \mathcal{T}$, by the hypothesis of the case (1iv) and the closure of $\mathcal{T}$ by MP (since $\mathcal{T}$ is fully regular). However, (5) is impossible given (2). □

5. Extensions of the basic logic

In the present section, eight different extensions of b4 are defined. Six of them are the logics built upon the implicative variants of MBN4 and ME4 (cf. §3). The remaining two are BN4 and E4 themselves. All eight systems will be axiomatized by adding to b4 some axioms of those listed below:

A13. $(A \wedge \neg B) \to [(A \wedge \neg B) \to \neg(A \to B)]$
A14. $A \vee [\neg(A \to B) \to A]$
A15. $\neg B \vee [\neg(A \to B) \to \neg B]$
A16. $[A \wedge (A \to B)] \to B$
A17. $[(A \to B) \wedge \neg B] \to \neg A$
A18. $A \to [B \vee \neg(A \to B)]$
A19. $\neg B \to [\neg A \vee \neg(A \to B)]$
A20. $[\neg(A \to B) \wedge \neg A] \to A$
A21. $\neg(A \to B) \to (A \vee \neg B)$
A22. $[\neg(A \to B) \wedge B] \to \neg B$
A23. $B \to \{[B \wedge \neg(A \to B)] \to A\}$
A24. $(A \to B) \vee \neg(A \to B)$
A25. $(\neg A \vee B) \vee \neg(A \to B)$
A26. $[(A \to B) \wedge (A \wedge \neg B)] \to \neg(A \to B)$
A27. $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)]$
A28. $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$
A29. $\{[\neg(A \to B) \wedge B] \to A\} \vee A$

Definition 5.1 (Extensions of b4). We refer by Lti ($1 \leqslant i \leqslant 8$) to the eight extensions of b4 considered in this paper, namely, the eight logics built upon the matrices characterized by the eight implicative tables displayed in §3. It will be proved that the logic Lti is characterized by the matrix Mti ($1 \leqslant i \leqslant 8$). These logics are axiomatized by adding the following axioms to b4:

Lt1 (BN4):	A13-A15
Lt2:	A16-A22
Lt3:	A13, A14, A17, A18, A21-A23
Lt4:	A15, A16, A19-A21
Lt5 (E4):	A16-A20, A22, A24-A26
Lt6:	A16, A19, A20, A22, A25, A27, A28
Lt7:	A13, A17, A18, A20, A22, A25, A29
Lt8:	A13, A20, A22, A25, A28, A29

Next, an essential proposition about the behavior of negated conditionals of Lti ($1 \leqslant i \leqslant 8$) will be proved.

Lemma 5.2 (Negated conditionals in Lti-logics). *Let L be an* ELti*-logic, where* Lti *refers to one of the extensions of* b4 *displayed in Definition 5.1 and let $\mathcal{T}$ be a prime and fully regular L-theory. We have*:

ELt1-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* $A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$.

ELt2-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($A \in \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($\neg A \notin \mathcal{T}$ & $\neg B \in \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$).

ELt3-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($A \in \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$).

ELt4-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($\neg A \notin \mathcal{T}$ & $\neg B \in \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$).

ELt5-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($A \in \mathcal{T}$ *or* $\neg A \notin \mathcal{T}$) & ($B \notin \mathcal{T}$ *or* $\neg B \in \mathcal{T}$).

ELt6-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($\neg A \notin \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$) *or* ($\neg A \notin \mathcal{T}$ & $\neg B \in \mathcal{T}$).

ELt7-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($A \in \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($\neg A \notin \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$).

ELt8-logics: $\neg(A \to B) \in \mathcal{T}$ *iff* ($\neg A \notin \mathcal{T}$ & $B \notin \mathcal{T}$) *or* ($A \in \mathcal{T}$ & $\neg B \in \mathcal{T}$).

Proof. This proof is similar to that of Lemma 4.11.[10] Therefore, it suffices to prove one case as example and provide a brief outline for the others. Let us prove the case of ELt6-logics.

[10] A couple of axioms from the list above are not used in this proof: A16 and A17 (i.e., modus ponens and modus tollens axioms). However, those axioms could have been used in the proof of Lemma 4.11 (instead of their corresponding rules) for those Lti-logics which validate them, as it was already mentioned in that Lemma.

(a) (⇒): Suppose (1) $\neg(A \to B) \in \mathcal{T}$ and, for reductio, (2) ($\neg A \in \mathcal{T}$ or $B \in \mathcal{T}$) & ($A \notin \mathcal{T}$ or $\neg B \notin \mathcal{T}$) & ($\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$). There are 8 subcases to consider:

(2a) $\neg A \in \mathcal{T} \,\&\, A \notin \mathcal{T} \,\&\, \neg A \in \mathcal{T}$
(2b) $\neg A \in \mathcal{T} \,\&\, \neg B \notin \mathcal{T} \,\&\, \neg A \in \mathcal{T}$
(2c) $\neg A \in \mathcal{T} \,\&\, A \notin \mathcal{T} \,\&\, \neg B \notin \mathcal{T}$
(2d) $\neg A \in \mathcal{T} \,\&\, \neg B \notin \mathcal{T} \,\&\, \neg B \notin \mathcal{T}$
(2e) $B \in \mathcal{T} \,\&\, A \notin \mathcal{T} \,\&\, \neg A \in \mathcal{T}$
(2f) $B \in \mathcal{T} \,\&\, A \notin \mathcal{T} \,\&\, \neg B \notin \mathcal{T}$
(2g) $B \in \mathcal{T} \,\&\, \neg B \notin \mathcal{T} \,\&\, \neg A \in \mathcal{T}$
(2h) $B \in \mathcal{T} \,\&\, \neg B \notin \mathcal{T} \,\&\, \neg B \notin \mathcal{T}$

Let us take subcase (2a): Given (1) $\neg(A \to B) \in \mathcal{T}$ and the fact that $\mathcal{T}$ is closed under adjunction, (3) $\neg(A \to B) \wedge \neg A \in \mathcal{T}$. Then, by A20 ($[\neg(A \to B) \wedge \neg A] \to A$), we get (4) $A \in \mathcal{T}$. Thus, a contradiction arises between subcase (2a) and (4). Let us now take subcase (2b): By A28 ($[(\neg(A \to B) \wedge \neg A) \to \neg B] \vee \neg B$), the fact that $\mathcal{T}$ is a regular and prime theory and subcase (2b) ($\neg B \notin \mathcal{T}$), we get $(\neg(A \to B) \wedge \neg A) \to \neg B \in \mathcal{T}$. Finally, contradicting (2b), we have $\neg B \in \mathcal{T}$ given (1), (2b) ($\neg A \in \mathcal{T}$) and the fact that $\mathcal{T}$ is closed under MP. On the other hand, subcases (2c) and (2e) can be solved by A20 similarly to (2a); and subcase (2d) by A28 like subcase (2b). Now, let us take subcase (2f): given $B \in \mathcal{T}$, the fact that $\mathcal{T}$ is closed under adjunction and (1) $\neg(A \to B) \in \mathcal{T}$, we have (5) $\neg(A \to B) \wedge B \in \mathcal{T}$. Then, by A22 ($[\neg(A \to B) \wedge B] \to \neg B$), (6) $\neg B \in \mathcal{T}$. However, (6) contradicts (2f). The rest of the subcases (i.e., (2g) and (2h)) can also be solved using A22 in a similar way.

(b) (⇐): Suppose the following three alternatives (1i) $\neg A \notin \mathcal{T} \,\&\, B \notin \mathcal{T}$ or (1ii) $A \in \mathcal{T} \,\&\, \neg B \in \mathcal{T}$ or (1iii) $\neg A \notin \mathcal{T} \,\&\, \neg B \in \mathcal{T}$. We have to derive $\neg(A \to B) \in \mathcal{T}$ from each of them. Let us suppose (1i). We get (2) $\neg A \vee B \notin \mathcal{T}$ since $\mathcal{T}$ is prime. Given A25 ($(\neg A \vee B) \vee \neg(A \to B)$) and the fact that $\mathcal{T}$ is regular, (3) $(\neg A \vee B) \vee \neg(A \to B) \in \mathcal{T}$. And finally, by (2), (3) and the primeness of $\mathcal{T}$, $\neg(A \to B) \in \mathcal{T}$. Now, let us suppose subcase (1ii) and by reductio (4) $\neg(A \to B) \notin \mathcal{T}$. By A27, (5) $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)] \in \mathcal{T}$ ($\mathcal{T}$ is regular). Thus, (6) $(A \wedge \neg B) \to \neg(A \to B) \in \mathcal{T}$, given (4), (5) and the primeness of $\mathcal{T}$. Finally, (7) $\neg(A \to B) \in \mathcal{T}$ due to the fact that $\mathcal{T}$ is closed under MP and the subcase (1ii) $A \wedge \neg B \in \mathcal{T}$. However, (7) contradicts (4). Lastly, let us suppose subcase (1iii). By A19 ($\neg B \to [\neg A \vee \neg(A \to B)]$) and (1iii) $\neg B \in \mathcal{T}$, we get (8) $\neg A \vee \neg(A \to B) \in \mathcal{T}$. Given that $\mathcal{T}$ is prime and (1iii) $\neg A \notin \mathcal{T}$, we have (9) $\neg(A \to B) \in \mathcal{T}$.

I will provide now a brief outline of the proof for the rest of the ELti-logics ($1 \leqslant i \leqslant 8$) considered.

ELt1-logics. For ($\Rightarrow$) we can use A14 and A15. As for ($\Leftarrow$), A13 could be used.

ELt2-logics. ($\Rightarrow$) by A20, A21 and A22; ($\Leftarrow$) by A18, A19 and the fact that $\mathcal{T}$ is closed under CTE.

ELt3-logics. ($\Rightarrow$) by A14 and A21-A23; ($\Leftarrow$) by A13 and A18.

ELt4-logics. ($\Rightarrow$) by A15, A20 and A21; ($\Leftarrow$) by A19 and the fact that $\mathcal{T}$ is closed under CTE.

ELt5-logics. ($\Rightarrow$) by A20 and A22; ($\Leftarrow$) by A18, A19, A24-A26.

ELt7-logics. ($\Rightarrow$) by A20, A22 and A29; ($\Leftarrow$) by A13, A18, A25.

ELt8-logics. ($\Rightarrow$) by A20, A22, A28 and A29; ($\Leftarrow$) by A13 and A25.

□

6. Belnap-Dunn semantics for the Lti-logics

In this section, a Belnap-Dunn semantics for the Lti-logics is developed. I start by defining the notions of Lti-models, Lti-consequence and Lti-validity.

DEFINITION 6.1 (Lti-models). An Lti-model is a structure $\langle \text{K4}, I\rangle$, where (i) $\text{K4} = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$ and (ii) I is an Lti-interpretation from $\mathcal{F}$ to K4 defined according to the following conditions for all $p \in \mathcal{P}$ and A, $B \in \mathcal{F}$: (1) $I(p) \in \text{K4}$; (2a) $T \in I(\neg A)$ iff $F \in I(A)$; (2b) $F \in I(\neg A)$ iff $T \in I(A)$; (3a) $T \in I(A \wedge B)$ iff $T \in I(A)$ & $T \in I(B)$; (3b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (4b) $F \in I(A \vee B)$ iff $F \in I(A)$ & $F \in I(B)$; (5a) $T \in I(A \to B)$ iff ($T \notin I(A)$ or $T \in I(B)$) & ($F \in I(A)$ or $F \notin I(B)$). Clause (5b) for each one of the Lti-models is as follows:

Lt1-models: (5b) $F \in I(A \to B)$ iff $T \in I(A)$ & $F \in I(B)$.

Lt2-models: (5b) $F \in I(A \to B)$ iff ($T \in I(A)$ & $T \notin I(B)$) or ($F \notin I(A)$ & $F \in I(B)$) or ($T \in I(A)$ & $F \in I(B)$).

Lt3-models: (5b) $F \in I(A \to B)$ iff ($T \in I(A)$ & $T \notin I(B)$) or ($T \in I(A)$ & $F \in I(B)$).

Lt4-models: (5b) $F \in I(A \to B)$ iff ($F \notin I(A)$ & $F \in I(B)$) or ($T \in I(A)$ & $F \in I(B)$).

Lt5-models: (5b) $F \in I(A \to B)$ iff ($T \in I(A)$ or $F \notin I(A)$) & ($T \notin I(B)$ or $F \in I(B)$).

Lt6-models: (5b) $F \in I(A \to B)$ iff ($F \notin I(A)$ & $T \notin I(B)$) or ($T \in I(A)$ & $F \in I(B)$) or ($F \notin I(A)$ & $F \in I(B)$).

Lt7-models: (5b) $F \in I(A \to B)$ iff $(T \in I(A)$ & $T \notin I(B))$ or $(F \notin I(A)$ & $T \notin I(B))$ or $(T \in I(A)$ & $F \in I(B))$.

Lt8-models: (5b) $F \in I(A \to B)$ iff $(F \notin I(A)$ & $T \notin I(B))$ or $(T \in I(A)$ & $F \in I(B))$.

DEFINITION 6.2 (Lti-consequence, Lti-validity). Let M be an Lti-model $(1 \leqslant i \leqslant 8)$. For any set of wffs Γ and wff A, $\Gamma \vDash_M A$ (A is a consequence of Γ in the Lti-model M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$ [$T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$]. Then, $\Gamma \vDash_{\mathrm{Lt}i} A$ (A is a consequence of Γ in Lti-semantics) iff $\Gamma \vDash_M A$ for each Lti-model M. In particular, $\vDash_{\mathrm{Lt}i} A$ (A is valid in Lti-semantics) iff $\vDash_M A$ for each Lti-model M (i.e., iff $T \in I(A)$ for each Lti-model M). (By $\vDash_{\mathrm{Lt}i}$ we shall refer to the relation just defined.)

The following proposition proves that consequence relations $\vDash_{\mathrm{Mt}i}$ (cf. Definition 2.5 and Proposition 3.2) and $\vDash_{\mathrm{Lt}i}$ (cf. Definition 6.2) are coextensive. Then, the soundness theorem will follow immediately.

PROPOSITION 6.3 (Coextensiveness of $\vDash_{\mathrm{Mt}i}$ and $\vDash_{\mathrm{Lt}i}$). *For any i $(1 \leqslant i \leqslant 8)$, set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{Mt}i} A$ iff $\Gamma \vDash_{\mathrm{Lt}i} A$. In particular, $\vDash_{\mathrm{Mt}i} A$ iff $\vDash_{\mathrm{Lt}i} A$.*

PROOF. For any i $(1 \leqslant i \leqslant 8)$, let I be an Mti-interpretation. Then, define the Lti-interpretation I' corresponding to I as follows: for each propositional variable p set

1. $I'(p) = \{T\}$ iff $I(p) = 3$;
2. $I'(p) = \{T, F\}$ iff $I(p) = 2$;
3. $I'(p) = \emptyset$ iff $I(p) = 1$;
4. $I'(p) = \{F\}$ iff $I(p) = 0$.

I' interprets complex formulas according to clauses (2a)-(5b) (Definition 6.1). Then, by an easy induction we immediately have, for any wff A: (a) $T \in I'(A)$ iff $I(A) = 3$ or $I(A) = 2$; and (b) $F \in I'(A)$ iff $I(A) = 0$ or $I(A) = 2$. In general, for any set of wffs Γ, we have: (a) $T \in I'(\Gamma)$ iff $I(\Gamma) = 3$ or $I(\Gamma) = 2$; and (b) $F \in I'(\Gamma)$ iff $I(\Gamma) = 0$ or $I(\Gamma) = 2$.

On the other hand, given a Lti-interpretation I', the Mti-interpretation I corresponding to I' can be defined in a similar way with analogous results.

Being the latter stated, the proof follows easily by Proposition 3.2 and Definitions 2.5, 2.6 and 6.2. $\square$

Now, soundness is proved.

Theorem 6.4 (Soundness of Lti w.r.t. $\vDash_{\mathrm{Mt}i}$ and $\vDash_{\mathrm{Lt}i}$). *For any* i $(1 \leqslant i \leqslant 8)$, *any set of wffs* Γ *and any* $A \in \mathcal{F}$, *if* $\Gamma \vdash_{\mathrm{Lt}i} A$, *then* (1) $\Gamma \vDash_{\mathrm{Mt}i} A$ *and* (2) $\Gamma \vDash_{\mathrm{Lt}i} A$.

Proof. (1) Given one of the considered Lti-logics, it is easy to check that the rules preserve Mti-validity, whereas the axioms of Lti are assigned either 3 or 2 by any Mti-interpretation I. Therefore, if $\Gamma \vdash_{\mathrm{Lt}i} A$, then $\Gamma \vDash_{\mathrm{Mt}i} A$. As for (2), it is immediate given (1) and Proposition 6.3. Finally, if Γ is the empty set, the proof is similar.[11] □

7. Extension and primeness lemmas

In the present section, we shall introduce the extension lemmas. Throughout this section, we display a couple of definitions and some lemmas which will be crucial points in the completeness theorem proved in the next section. We follow the method developed in "Relevant logics and their rivals I" [cf. 21, Chapter 4] and followed by Brady [cf. 7, pp. 24–25]. We shall omit some of those proofs since they are similar to Brady's.[12]

Definitions and Lemmas throughout this section are developed in general for extensions of the basic logic b4 (Eb4-logics). However, we note that the class of Eb4-logics in Lemmas 7.2, 7.4 and 7.5 has to be restricted to those logics closed under the rules that determine the full regularity of Eb4-theories (cf. Definition 4.9). Otherwise it is possible that the Extension Lemma is not provable for some Eb4-logic.

Firstly, we set a preliminary definition.

Definition 7.1 (Disjunctive Eb4-derivability). Let L be an Eb4-logic, Γ and Θ be non-empty sets of wffs, then Θ is disjunctively derivable from Γ in Eb4 (in symbols, $\Gamma \vdash^{d}_{L} \Theta$) iff $A_1 \wedge \cdots \wedge A_n \vdash_L B_1 \vee \cdots \vee B_n$ for some wffs $A_1, \ldots, A_n \in \Gamma$ and $B_1, \ldots, B_n \in \Theta$.

The following lemma is essential in order to prove the *Extension to maximal sets* lemma (see Lemma 7.4). We get it by induction on the length of formulas (cf. p. 27 in [7] and Lemma 6.2 in [18]).

[11] Cf. [12] in case a tester is needed.

[12] We shall also follow Robles and Méndez's structure and method for the extension lemmas [cf. 18, pp. 845–847].

LEMMA 7.2 (Preliminary lemma to the extension lemma). *Let L be an* Eb4-*logic closed under no other rules than those specified in Definition 4.9. For any wffs A, B_1, ..., B_n, if $\{B_1, \ldots, B_n\} \vdash_L A$, then, for any wff C, $C \vee (B_1 \wedge \cdots \wedge B_n) \vdash_L C \vee A$.*

Now, the process of extending sets of wffs to maximal sets is required.

DEFINITION 7.3 (Maximal sets). Let L be an Eb4-logic, Γ is a L-maximal set of wffs iff $\Gamma \nvdash^d_L \overline{\Gamma}$ (where $\overline{\Gamma}$ is the complement of Γ).

Just like Lemma 9 in [7] and Lemma 6.4 in [18], we get:

LEMMA 7.4 (Extension to maximal sets). *Let L be an* Eb4-*logic closed under no other rules than those specified in Definition 4.9, Γ and Θ sets of wffs such that $\Gamma \nvdash^d_L \Theta$. Then, there are sets of wffs Γ' and Θ' such that $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$, $\Theta' = \overline{\Gamma'}$ and $\Gamma' \nvdash^d_L \Theta'$ (i.e., Γ' is a L-maximal set such that $\Gamma' \nvdash^d_L \Theta'$).*

Finally, the Primeness Lemma can now be proved.

LEMMA 7.5 (Primeness). *Let L be an* Eb4-*logic closed under no other rules than those specified in Definition 4.9. If Γ is a L-maximal set, then it is a fully regular prime L-theory.*

PROOF. The fact that Γ is a L-theory (i.e., a set of wffs closed under adjunction and L-entailment) can be proved as Brady did for the logic BN4 (cf. Lemma 8 in [7]). Now, let us prove that Γ is prime and fully regular. (1) Γ is regular (i.e., Γ contains any theorem of L). It is clear that, if Γ is a L-maximal set ($\Gamma \nvdash^d_L \overline{\Gamma}$), Γ must contain every theorem of L. Otherwise, there would be some wff A such that $\vdash_L A$ and $A \notin \Gamma$. Therefore, we would have $A \in \overline{\Gamma}$ and consequently, $\Gamma \vdash^d_L \overline{\Gamma}$, contradicting the maximality of Γ. (2) Γ is fully regular (cf. Definition 4.9). We have to prove that Γ is closed under the rules of L. For instance, let us prove that Γ is closed under dPREF. By reductio, suppose there are wffs A, B, C, D such that $C \vee (A \to B) \in \Gamma$ and $C \vee [(D \to A) \to (D \to B)] \notin \Gamma$. Then, $C \vee (A \to B) \vdash C \vee (A \to B)$ by A1 and, by dPREF, $C \vee (A \to B) \vdash C \vee [(D \to A) \to (D \to B)]$, contradicting the maximality of Γ. (3) Γ is prime. If there were some wffs A, B such that $A \vee B \in \Gamma$ but $A \notin \Gamma$ and $B \notin \Gamma$, then Γ would not be L-maximal given A1 ($(A \vee B) \to (A \vee B)$). □

8. Completeness of the Lti-logics

We shall prove the completeness of the Lti-logics ($1 \leqslant i \leqslant 8$) (cf. Definition 5.1) w.r.t. both $\vDash_{\mathrm{Mt}i}$ and $\vDash_{\mathrm{Lt}i}$. Completeness w.r.t. $\vDash_{\mathrm{Lt}i}$ is proved by means of a canonical model construction. Then, completeness w.r.t. $\vDash_{\mathrm{Mt}i}$ follows immediately by Proposition 6.3.

We begin by the definition of canonical Lti-models. It will be proved that if A is not derivable from Γ in a Lti-logic, then A does not follow from Γ in some canonical Lti-model. The concept of a canonical Lti-model is based upon the notion of a $\mathcal{T}$-interpretation.

Definition 8.1 ($\mathcal{T}$-interpretations). Let L be an Lti-logic and K4 be the set $\{\{T\}, \{F\}, \{T, F\}, \emptyset\}$ as in Definition 6.1. Let $\mathcal{T}$ be a prime, regular and a-consistent L-theory. Then, the function I from the set $\mathcal{F}$ to K4 is defined as follows: for each propositional variable p, we set: (a) $T \in I(p)$ iff $p \in \mathcal{T}$ and (b) $F \in I(p)$ iff $\neg p \in \mathcal{T}$. Next, I assigns a member of K4 to each formula A according to the corresponding conditions 2–5 in Definition 6.1. Then, it is said that I is a $\mathcal{T}$-interpretation. (As in Definition 6.2, $T \in I(\Gamma)$ iff $\forall A \in \Gamma$ $(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma$ $(F \in I(A))$.)

Definition 8.2 (Canonical Lti-models). Let L be an Lti-logic. A canonical L-model is a structure (K4, $I_{\mathcal{T}}$) where K4 is defined according to Definition 6.1 and $I_{\mathcal{T}}$ is a $\mathcal{T}$-interpretation built upon a prime, regular and a-consistent L-theory $\mathcal{T}$.

Definition 8.3 (The canonical relation $\vDash_{I_{\mathcal{T}}}$). Let L be an Lti-logic and (K4, $I_{\mathcal{T}}$) a canonical L-model. The canonical relation $\vDash_{I_{\mathcal{T}}}$ is defined as follows. For any set of wffs Γ and wff A, $\Gamma \vDash_{I_{\mathcal{T}}} A$ (A is a consequence of Γ in the canonical L-model (K4, $I_{\mathcal{T}}$)) iff $T \in I_{\mathcal{T}}(A)$ whenever $T \in I_{\mathcal{T}}(\Gamma)$. In particular, $\vDash_{I_{\mathcal{T}}} A$ (A is true in the canonical L-model (K4, $I_{\mathcal{T}}$), iff $T \in I_{\mathcal{T}}(A)$.

It is clear that any canonical L-model is an L-model given Definitions 8.2 and 8.3.

Proposition 8.4 (Any canonical L-model is an L-model). *Let L be an Lti-logic and $M = \langle \mathrm{K4}, I_{\mathcal{T}} \rangle$ a canonical L-model. Then M is indeed an L-model.*

Proof. It follows immediately by Definitions 6.1, 8.2 and 8.3. (Notice that each propositional variable –and so, each wff — can be assigned $\{T\}$,

$\{F\}$, $\{T, F\}$ or $\emptyset$ since $\mathcal{T}$ is required to be neither complete nor consistent in the classical sense.) □

The following lemma generalizes conditions (a) and (b) in Definition 8.1 to the set $\mathcal{F}$ of all wffs.

Lemma 8.5 ($\mathcal{T}$-interpreting the set of wffs $\mathcal{F}$). *Let L be an* Lti*-logic and I be a $\mathcal{T}$-interpretation defined on the L-theory $\mathcal{T}$. For each wff A, we have : (1) $T \in I(A)$ iff $A \in \mathcal{T}$; (2) $F \in I(A)$ iff $\neg A \in \mathcal{T}$.*

Proof. By induction on the length of A (clauses cited in points (b)-(e) below refer to those in Definition 6.1; H.I. abbreviates "hypothesis of induction"). (a) A is a propositional variable: by conditions (a) and (b) in Definition 8.1. (b) A is of the form $\neg B$: (i) $T \in I(\neg B)$ iff (clause 2a) $F \in I(B)$ iff (H.I.) $\neg B \in \mathcal{T}$. (ii) $F \in I(\neg B)$ iff (clause 2b) $T \in I(B)$ iff (H.I) $B \in \mathcal{T}$ iff (Lemma 4.6) $\neg\neg B \in \mathcal{T}$. (c) A is of the form $B \wedge C$: (i) $T \in I(B \wedge C)$ iff (clause 3a) $T \in I(B)$ and $T \in I(C)$ iff (H.I.) $B \in \mathcal{T}$ and $C \in \mathcal{T}$ iff (Lemma 4.7) $B \wedge C \in \mathcal{T}$; (ii) $F \in I(B \wedge C)$ iff (clause 3b) $F \in I(B)$ or $F \in I(C)$ iff (H.I.) $\neg B \in \mathcal{T}$ or $\neg C \in \mathcal{T}$ iff (Lemma 4.7) $\neg(B \wedge C) \in \mathcal{T}$. (d) A is of the form $B \vee C$: the proof is similar to (c) by using clauses 4a, 4b and Lemma 4.7. (e) A is of the form $B \to C$: (i) $T \in I(B \to C)$ iff (clause 5a) ($T \notin I(B)$ or $T \in I(C)$) and ($F \in I(B)$ or $F \notin I(C)$) iff (H.I.) ($B \notin \mathcal{T}$ or $C \in \mathcal{T}$) and ($\neg B \in \mathcal{T}$ or $\neg C \notin \mathcal{T}$) iff (Lemma 4.11) $B \to C \in \mathcal{T}$; (ii) A is a conditional assigned F by I. We have to consider 8 cases. I will choose two cases as examples. The rest of them are proved similarly. Let us take, for instance, the case of the logic Lt3. We have $F \in I(B \to C)$ iff (clause 5b for Lt3-logics) ($T \in I(B)$ and $T \notin I(C)$) or ($T \in I(B)$ and $F \in I(C)$) iff (H.I.) ($B \in \mathcal{T}$ and $C \notin \mathcal{T}$) or ($B \in \mathcal{T}$ and $\neg C \in \mathcal{T}$) iff (Lemma 5.2) $\neg(B \to C) \in \mathcal{T}$. As a second example, let Lti be now Lt6: $F \in I(B \to C)$ iff (clause 5b for L6-logics) ($F \notin I(B)$ and $T \notin I(C)$) or ($T \in I(B)$ and $F \in I(C)$) or ($F \notin I(B)$ and $F \in I(C)$) iff (H.I.) ($\neg B \notin \mathcal{T}$ and $C \notin \mathcal{T}$) or ($B \in \mathcal{T}$ and $\neg C \in \mathcal{T}$) or ($\neg B \notin \mathcal{T}$ and $\neg C \in \mathcal{T}$) iff (Lemma 5.2) $\neg(B \to C) \in \mathcal{T}$. □

Next, we prove completeness. In order to do so, the notion of "set of consequences in Lti of a given set of wffs Γ" is recalled.

Definition 8.6 (The set CnΓ[Lti]). The set of consequences in Lti of a set of wffs Γ (in symbols CnΓ[Lti]) is defined as follows: CnΓ[Lti] $= \{A \mid \Gamma \vdash_{\mathrm{Lt}i} A\}$.

Remark 8.7 (The set of consequences of Γ in Lti is a fully regular theory)*.* It is obvious that for any Γ, CnΓ[Lti] contains all theorems of Lti and is closed under the rules of Lti. Consequently, it is also closed under Lti-entailment.

Theorem 8.8 (Completeness of Lti-logics)*. For any i ($1 \leqslant i \leqslant 8$), any set of wffs Γ and any wff A, (1) if $\Gamma \vDash_{\mathrm{Lt}i} A$, then $\Gamma \vdash_{\mathrm{Lt}i} A$; (2) if $\Gamma \vDash_{\mathrm{Mt}i} A$, then $\Gamma \vdash_{\mathrm{Lt}i} A$.*

Proof. (1) Suppose $\Gamma \nvdash_{\mathrm{Lt}i} A$ for some set of wffs Γ and wff A. We need to prove $\Gamma \nvDash_{\mathrm{Lt}i} A$. If $\Gamma \nvdash_{\mathrm{Lt}i} A$, then clearly we have $A \notin Cn\Gamma[Lti]$. Thus, $Cn\Gamma[Lti] \nvdash^{d}_{\mathrm{Lt}i} \{A\}$; otherwise $B_1 \wedge \cdots \wedge B_n \vdash_{\mathrm{Lt}i} A$ for some $B_1 \wedge \cdots \wedge B_n \in Cn\Gamma[Lti]$ whence A would be in $Cn\Gamma[Lti]$ after all. By Lemma 7.4, there is a maximal set $\mathcal{T}$ such that $Cn\Gamma[Lti] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Lemma 7.5, $\mathcal{T}$ is a fully regular prime theory. On the other hand, given Lema 8.5, $\mathcal{T}$ generates a $\mathcal{T}$-interpretation $I_{\mathcal{T}}$ such that $T \in I_{\mathcal{T}}(\Gamma)$ (since $T \in I_{\mathcal{T}}(\mathcal{T})$ and $Cn\Gamma[Lti] \subseteq \mathcal{T}$) but $T \notin I_{\mathcal{T}}(A)$. Therefore, $\Gamma \nvDash_{I_{\mathcal{T}}} A$, by Definition 8.3 and Proposition 8.4, consequently, $\Gamma \nvDash_{\mathrm{Lt}i} A$, by Definition 6.2. (2) Completeness w.r.t. $\vDash_{\mathrm{Mt}i}$ is immediate by (1) and Proposition 6.3. □

9. Some facts about Lti-logics

I will investigate if some properties characteristic of relevant logics are predicable of the Lti-logics. In particular, I will prove that no Lti-logic ($1 \leqslant i \leqslant 8$) is a relevant logic since they lack the "variable-sharing property" (vsp). However, they are strongly related to the family of relevant logics since they have another related property called "quasi relevant property" (qrp).

Definition 9.1 (Variable-sharing property –vsp). A logic L has the "variable-sharing property" if for every theorem of L of the form $A \to B$, A and B share at least a propositional variable.

Proposition 9.2. *Every* Lti*-logic* ($1 \leqslant i \leqslant 8$) *lacks the vsp.*

Proof. Let M be the matrix determining the logic L. The proof is immediate since, for any distinct propositional variables p and q, the wff $\neg(p \to p) \to (q \to q)$ is M-valid, this is, the wff is valid in any Lti-logic [cf. 12]. □

DEFINITION 9.3 (Quasi-relevance property –qrp). A logic L has the "quasi-relevance property" if for every theorem of L of the form $A \to B$, either A and B share at least a propositional variable or both $\neg A$ and B are also theorems of L.

PROPOSITION 9.4. *Every* Lti*-logic* ($1 \leqslant i \leqslant 8$) *has the qrp.*

PROOF. Let M be the matrix determining the logic L. By reductio, suppose that there are wffs A and B which have no propositional variable in common and such that $A \to B$ is M-valid but either $\neg A$ or B is not.

(i) Let us suppose that $\neg A$ is not M-valid. Then, there is an M-interpretation I such that $I(\neg A) = 0$ or $I(\neg A) = 1$ (i.e., $I(A) = 3$ or $I(A) = 1$). Now, let I' be exactly as I except that for each propositional variable p in B, $I'(p) = 2$. Then, clearly $I'(B) = 2$ since $\{2\}$ is closed under $\to$, $\wedge$, $\vee$ and $\neg$, and either $I'(A) = 3$ or $I'(A) = 1$, since A and B do no share propositional variables. Consequently, we get either $I'(A \to B) = 0$ or $I'(A \to B) = 1$ (depending on i, $1 \leqslant i \leqslant 8$; cf. Mt$i$ in Section 3), contradicting the M-validity of the wff $A \to B$.

(ii) Let us suppose now that B is not M-valid. Then, there is an M-interpretation I such that $I(B) = 0$ or $I(B) = 1$. Let I' be exactly as I except that for each propositional variable p in A, $I'(p) = 2$. Similarly, as in case (i), we get $I'(A) = 2$ and either $I'(B) = 0$ or $I'(B) = 1$. Then, $I'(A \to B) = 0$ or $I'(A \to B) = 1$, contradicting the M-validity of $A \to B$. □

Next, I will prove that Lti-logics are paraconsistent and paracomplete in the sense of the following definitions.

DEFINITION 9.5 (Paraconsistent logics). Let $\Vdash$ represent a consequence relation (may it be defined either semantically or proof-theoretically). Then, a logic L is paraconsistent if, for any wffs A, B, the rule Ecq (*E contradictione quodlibet*) $A, \neg A \Vdash B$ does not hold in L.

In other words, a logic is paraconsistent if theories built upon L are not necessarily trivial when a contradiction arises.

PROPOSITION 9.6. *Every* Lti*-logic* ($1 \leqslant i \leqslant 8$) *is paraconsistent.*

PROOF. Let M be the matrix determining the logic L and let p and q be distinct propositional variables. There is an M-interpretation I such that $I(p) = 2$ and $I(q) = 0$. Therefore, $\{p, \neg p\} \nvDash_M q$, this is, Ecq does not hold in any Lti-logic. □

DEFINITION 9.7 (Paracomplete logics). A logic L is paracomplete if, for some wff A, the PEM (principle of excluded middle) $A \vee \neg A$ does not hold in L.

In other words, a logic is paracomplete if there are prime, fully regular L-theories that are not complete in the sense that they lack at least both a formula and its negation.

PROPOSITION 9.8. *Every* Lti-*logic* $(1 \leqslant i \leqslant 8)$ *is paracomplete.*

PROOF. Let M be the matrix determining the logic L, then for any propositional variable p there is an M-interpretation I such that $I(p) = 1$. Therefore, $I(p \vee \neg p) = 1$ given Definition 2.6, i.e., PEM does not hold in any Lti-logic. □

Next, I define the natural implicative expansions of Belnap and Dunn's logic B4 and prove that Lti-logics $(1 \leqslant i \leqslant 8)$ are indeed natural implicative expansions of it. First, following Robles and Méndez [19] (who likewise follow Tomova [23]), I introduce the notion of "natural conditionals" and then I will examine the case of 4-valued matrices.

DEFINITION 9.9 (Natural conditionals). Let $\mathcal{L}$ be a propositional language with $\rightarrow$ among its connectives and M be a matrix for $\mathcal{L}$ where the values x and y represent the maximun and the infimum in $\mathcal{V}$. Then, an $f_{\rightarrow}$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:

1. $f_{\rightarrow}$ coincides with the classical conditional when restricted to the subset $\{x, y\}$ of $\mathcal{V}$;
2. $f_{\rightarrow}$ satisfies Modus Ponens, that is, for any a, $b \in \mathcal{V}$, if $a \rightarrow b \in \mathcal{D}$ and $a \in \mathcal{D}$, then $b \in \mathcal{D}$;
3. For any a, $b \in \mathcal{V}$, $a \rightarrow b \in \mathcal{D}$ if $a \leqslant b$.

PROPOSITION 9.10 (Natural conditionals in 4-valued matrices). *Let* $\mathcal{L}$ *be a propositional language and* M *a 4-valued matrix for* $\mathcal{L}$, *where* $\mathcal{V}$ *and* $\mathcal{D}$ *are defined exactly as in B4 (cf. Definition 2.6). Consider the 32.768* $f_{\rightarrow}$-*functions defined in the following general table*:

	$\rightarrow$	0	1	2	3
	0	3	b_1	b_2	3
TI	1	c_1	b_3	c_2	b_4
	2	a_1	a_2	b_5	b_6
	3	0	a_3	c_3	3

where $a_i \in \{0,1\}$ $(1 \leqslant i \leqslant 3)$*,* $b_j \in \mathcal{D}$ $(1 \leqslant j \leqslant 6)$ *and* $c_m \in \mathcal{V}$ $(1 \leqslant m \leqslant 3)$*. The set of functions* (*contained*) *in* TI *is the set of all natural conditionals definable in* M*.*

PROOF. (1) The following cases are needed in order to fulfill clause 1 in Definition 9.9: $f_{\rightarrow}(0,0) = 3$, $f_{\rightarrow}(0,3) = 3$, $f_{\rightarrow}(3,3) = 3$ and $f_{\rightarrow}(3,0) = 0$. (2) Regarding clause 2 in the same definition, a non-designated value (i.e., either 0 or 1) needs to be assigned to the subsequent cases: $f_{\rightarrow}(2,0)$, $f_{\rightarrow}(2,1)$, $f_{\rightarrow}(3,0)$, $f_{\rightarrow}(3,1)$. (3) Finally, we also need $f_{\rightarrow}(0,0) \in \mathcal{D}$, $f_{\rightarrow}(0,1) \in \mathcal{D}$, $f_{\rightarrow}(0,2) \in \mathcal{D}$, $f_{\rightarrow}(0,3) \in \mathcal{D}$, $f_{\rightarrow}(1,1) \in \mathcal{D}$, $f_{\rightarrow}(1,3) \in \mathcal{D}$, $f_{\rightarrow}(2,2) \in \mathcal{D}$, $f_{\rightarrow}(2,3) \in \mathcal{D}$ and $f_{\rightarrow}(3,3) \in \mathcal{D}$ for the last condition in Definition 9.9 to be guaranteed. □

DEFINITION 9.11 (Natural implicative expansions of MB4). Consider the matrix MB4 and let M be the result of adding any $f_{\rightarrow}$-function to it. It is said that M is an implicative expansion of MB4 (cf. Definition 2.3). Then, M is a natural implicative expansion of MB4 if $f_{\rightarrow}$ is any of the $f_{\rightarrow}$-functions (defining one of the conditionals) in TI (see Proposition 9.10).

COROLLARY 9.12. *Each* Mti $(1 \leqslant i \leqslant 8)$ *is a natural implicative expansion of* MB4*.*

PROOF. It is obvious given that each Mti ($1 \leqslant i \leqslant 8$; cf. Section 3) is one of the 32.768 $f_{\rightarrow}$-functions defined in Proposition 9.10 and therefore all of them fulfill the requirements mentioned in Definition 9.9. □

10. Conclusion

By way of conclusion, I shall make five remarks.

1. In the present paper, the class of all implicative expansions of MB4 verifying B while maintaining the conditional structure of MBN4 or ME4 was presented. This class is formed by all the Lti-logics displayed in §5, most of which (with the obvious exception of BN4 and E4) had not been deeply studied before[13] (to the best of my knowledge) and could be seen

[13] It has to be mentioned that Petrukhin and Shangin defined natural deduction systems for binary extensions of FDE from the point of view of correspondence analysis in a recent paper [16]. More specifically, in Section 2 of that paper they refer to the eight matrices discussed in the conclusions of [18], which are the same ones employed to develop the Lti-logics in the present paper. Additionally, it is also

as implicative alternatives to BN4 for situations with incomplete and inconsistent information (in the sense explained in the introduction of the paper). Furthermore, this work also serves as a contribution to the previous research on possible expansions of FDE.[14]

2. Since the main guideline followed to axiomatize Eb4-extensions is to maintain a common base as wide as possible, there is the chance to define all Lti-logics more conspicuously and economically than in Definition 5.1. For instance, disjunctive rules are not mandatory for all Lti-logics to be axiomatized. As a matter of fact, the system E4 (Lt5) can be axiomatized without any disjunctive rule (cf. [17, 18]). On their part, BN4 (Lt1) and Lt8 only need the rule dMP (cf. [7, 17], regarding some alternative axiomatizations of BN4). As for systems Lt2 and Lt6, just the rules dCTE and dCON will be required, respectively. Lastly, Lt4 can be axiomatized with both dCON and dCTE as the only disjunctive rules.

3. Let us now consider the properties proved in §9. On the one hand, all Lti-logics are clearly related to the family of relevant logics since they are natural implicative expansions of Routley and Meyer's system B and they hold the qrp (cf. Proposition 9.4). On the other hand, Lti-logics could also be of interest among many-valued logics since they are also implicative expansions of MB4 (which is the smallest bilattice, according to Ginsberg; cf. [11]). Moreover, all Lti-logics are natural implicative expansions (in the sense of Tomova) of both the logics just mentioned [cf. 23, Corollary 9.12 of this paper].

4. Given the position of Lti-logics in the family of relevant logics, I believe that a ternary relational semantics developed for them could also be a useful tool to compare them to many other different logics of said family. As a matter of fact, the systems BN4 and E4 were already provided with such a semantics [cf. 17]. Furthermore, Brady endowed BN4 with a 2-set-up semantics when he first develop the system in [7]. As for E4, a 2-set-up semantics was developed in [20]. It is worth mentioning that a reduced Routley-Meyer semantics could be more difficult to develop for some Lti-logics due to the apparent ineliminability of disjunctive rules [cf. 8].

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worth mentioning here that J. M. Blanco has been working on the logic built upon Mt2 (an equivalent axiomatization to Lt2) in his doctoral dissertation (written under G. Robles' supervision).

[14] In this case, on implicative expansions of FDE which are also endowed with a Belnap-Dunn semantics, as Omori and Wansing suggested in Section 7 and the conclusion of [15].

5. There are, of course, other conditional structures different from those investigated in this paper that could also be a source of similar interest and another line for further research.[15]

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[15] I am thinking, for example, on logics based on variants of Smiley's matrix MSm4 [cf. 13].

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