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Fractal Analysis of Financial Time Series Using Fractal Dimension and Pointwise Hölder Exponents[**]

A b s t r a c t. This paper presents a fractal analysis application to the verification of assumptions of Fractal Market Hypothesis and the presence of fractal properties in financial time series. In this research, the box-counting dimension and pointwise Hölder exponents are used. Achieved results lead to interesting observations related to nonrandomness of price series and occurrence of relationships binding fractal properties and variability measures with the presence of trends and influence of the economic situation on financial instruments' prices.

K e y w o r d s: fractal analysis, fractal dimension, box-counting dimension, pointwise Hölder exponents, Hurst exponent.

J E L Classification: G14, G15, G17.

Introduction

Ever since Mandelbrot had published his works on the application of R/S analysis to long-memory dependencies in time series (Mandelbrot, Wallis, 1969; Mandelbrot, 1972) and since Peters had presented his Fractal Market Hypothesis (Peters, 1991) as an alternative to commonly acknowledged Efficient Market Hypothesis, this approach is being explored with regard to financial time series. Mulligan examined the use of Lo's modified rescaled range analysis on foreign exchange markets (Mulligan, 2000), proving the

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presence of long memory dependencies and fractal structure of analysed price series. Another published research assumed Hurst exponent estimation using geometrical interpretation (Granero, Segovia, Pérez, 2008) applied to stock market indices or the search for periodic and nonperiodic components in S&P 500 time series (Bohdalová, Greguš, 2010). Another group of studies focused on the analysis of variation of Hurst exponent over time, showing the possible impact of capital flow and trading volume on the decrease of Hurst exponent values (Cajueiro, Tabak, 2004), the influence that the end of Bretton Woods System had on efficiency of US stock markets (Alvarez-Ramirez, Alvarez, Rodriguez, Fernandez-Anaya, 2008), or the relationship between local Hurst exponent and stock market crashes with example of the Warsaw Stock Exchange Index (Grech, Pamuła, 2008). Due to certain limitations of classical R/S analysis approach and Hurst exponent itself, some of the authors explored Hölderian pointwise regularity of some major stock market indices (Bianchi, Pantanella, 2010) and usage of multifractal spectra analysis in order to discover patterns of change in price series before the 1987 market crash and other significant market drawdowns (Los, Yalamova, 2004).

In the presented paper, the initial assumption is that the markets are not efficient, but are fractal in their nature. Despite the fact that Efficient Market Hypothesis (EMH) has been commonly accepted as a default theory explaining the fundamentals of financial markets' behavior, plenty of criticism and doubts have been adressed towards it. The critical remarks are mainly related to too strong assumptions underlying this hypothesis, vastly mangling the real world behavior of the markets. In terms of real market behavior, a chaos theory based Fractal Market Hypothesis (FMH) seems to be much more appriopriate. It assumes – on the contrary to EMH, which uses linear differential equation – that the market is a nonlinear dynamic system, which allows to suppose that „real feedback systems involve long-term correlations and trends, because memories of long-past events can still affect the decisions made in the present" (Peters, 1997, p. 6). Actions of market participants usually generate nonlinear behavior of financial processes. Functions describing investors' attitute towards risk, their expectations towards stock market returns or financial instruments pricing formulae are nonlinear as well (Osińska, 2006, p. 118). The most characteristic property of FMH is acknowledgement that stock market returns time series have fractal (self-similar) structure. FMH also allows chaotic behavior of the market during particular periods and under certain conditions (Peters, 1994, pp. 46–48).

FMH was described by Peters based on the results of long studies conducted by Hurst in the first half of the 20th century (Peters, 1991). The main conclusion was that most of natural systems do not follow the random walk model, but are subject to fractional Brownian motion. Such theory is in line with Peters' assumption about the markets, which says that globally (in the long-term) the market is deterministic, while locally (in the short-term), due to randomly occuring information and emotional reactions of market participants, the market is random (Peters, 1997, pp. 45, 64). Empirical confirmation of this hypothesis is presented in the section 4 of this paper.

1. Fractal Dimension

One of the most substantial characteristics of geometric object is its dimension. However, an attempt to analyse this matter is nontrivial due to the fact that so far the scientists provided many different definitions of dimension, namely: the topological dimension, the Hausdorff dimension, the fractional dimension, the box-counting dimension, the self-similarity dimension etc. The reasoning behind using particular types of dimension depends on certain conditions and while sometimes using different types of dimensions can lead to similar results, it might as well show varying results for the same object (Peitgen, Jürgens, Saupe, 2002, p. 274).

In brief, a dimension describes the way in which a geometric object (or time series) fills the space. A common characteristic of all fractal objects is presence of self-similarity, which means that there is a relation between the reduction coefficient (a scale of similarity) and the amount of reduced fragments similar to the original object. While analyzing zigzag-shaped financial instrument time series chart in terms of dimension, it is easy to conclude that its dimension falls into range $(1, 2)$. As zigzag is not a straight line, it has dimension which is definitely distinct from 1, but it is not two-dimensional as it does not fill the entire plane.

There are several kinds of fractal dimensions. One of them is the box-counting dimension, which – in view of its use as a research tool – is worth presenting. Below description of the box-counting dimension was written based on: Peitgen, Jürgens, Saupe (2002), Mastalerz-Kodzis (2003), Kudrewicz (2007) and Borys (2011).

Let F be a certain geometric object embedded in n-dimensional Euclidean space R^n, covered with a set of small cubes (boxes called hypercubes) which sides are equal to ε (e.g., for $n=1$ it will be segments, for $n=2$ squares, for $n=3$ cubes). Theoretically these cubes can be discretionary oriented with respect to the axes of a coordinate system, but they can be as well spheres or other convex solids with a diameter ε. Let $N(F,\varepsilon)$ be a minimum amount of cubes that can completely cover the entire object F. Having satisfied this, below relations are true:

- $N(F,\varepsilon) \sim 1/\varepsilon$ – for F being a segment of a smooth line (amount of cubes is approximately inversely proportional to the length of cube side),
- $N(F,\varepsilon) \sim (1/\varepsilon)^2$ – for F being a piece of smooth plane,
- $N(F,\varepsilon) \sim (1/\varepsilon)^3$ – for F being an area contained in R^3.

With respect to the above, a generalization can be made that there are such geometric objects, for which – assuming small values of ε (which is a scale of similarity, so that the smaller the values of ε, the better the approximation) – below commensurateness is approximately met:

$$N(F,\varepsilon) \sim (1/\varepsilon)^D, \tag{1}$$

whereas – while in majority of cases there is only an approximation of commensurateness between the two sides of the equation – in case of strictly self-similar objects an equality sign can be placed so that:

$$N(F,\varepsilon) = (1/\varepsilon)^D, \tag{2}$$

where D (not necessarily being an integer number) can be treated as a dimension of the object F. Then the limit:

$$D_f(F) = \lim_{\varepsilon \to 0} \frac{\log N(F,\varepsilon)}{\log(1/\varepsilon)}, \tag{3}$$

if it exists, is called a fractal dimension (in this case, a box-counting dimension)[1].

It is easy to conclude from (3), which is applicable to exact fractals, that a dimension of such objects is defined by:

$$D_f(F) = \frac{\log N(F,\varepsilon)}{\log(1/\varepsilon)}. \tag{4}$$

A possibility to apply the box-counting dimension led to creation of algorithms capable of estimating the dimension of time series. In this method, the analyzed structure is placed on a regular grid with a size of ε, followed by

[1] If a given unit gets reduced ε times, the measured line will approximately contain ε times more units than in the previous iteration step (approximation is in this case a result of a possible presence of small curves disturbing the existing commensurateness). Given commensurateness is exact only in the limit, because after reducing the length of the sunit it is possible to achieve a infinitely small unit only in the limit (Tempczyk, 1995, p. 135).

counting all the cubes (boxes), which contain the fragments of analyzed structure. The amount of fragments is obviously dependent on the value of ε. In the next stage it is necessary to repeat the calculations for smaller values of ε and plot the obtained values on the logarithmic chart (marking $\log N(F,\varepsilon)$ with respect to $\log(1/\varepsilon)$). The slope of the straight line fitted to the points marked on the plot determines the box-counting dimension of the time series (Daros, 2010, p. 13).

2. Measures of Variability of the Graph of the Function

One of the most popular and useful measures of variability (irregularity) are Hurst exponent and Hölder exponents. The Hurst exponent is a numerical characteristic of the entire price series, whereas Hölder exponents can be used to analyze the complexity of the function and the trajectory of some stochastic processes in the vicinity of any point of the graph of the function (Mastalerz-Kodzis, 2003, p. 37).

The most important turning point in the subject of long-term dependency analysis of time series was without any doubt the creation of rescaled range analysis $(R/S)_n$ method (Hurst, 1951). A starting point of his analysis was Einstein's work on Brownian motion (Einstein, 1908), which presented an equation for the distance R that a particle travels in time $T,$ which is defined by $R = c\sqrt{T},$ where c is a nonnegative constant. This equation was applicable when the series of increments of the distance travelled by the particle in time was a random walk, characterised by the independency of normally distributed random variables (Weron, Weron, 1998, p. 323). However, during almost fourty years of research, Hurst has reached a conclusion that the majority of natural phenomena is not subject to Gaussian random walk, but rather to processes with „long memory", later called by the name of fractional Brownian motion, which is a combination of a trend and noise (Peters, 1997, p. 64; Mastalerz-Kodzis, 2003, pp. 37–38).

Derivation of a formula for rescaled range allowed the comparison of different types of time series. Creation of this dimensionless indicator, which should increase over time, allowed to formulate the following equation being an extension of Brownian motion model proposed by Einstein.

$$(R/S)_n = c \cdot n^H, \tag{5}$$

where (R/S) – rescaled range, n – number of observations, c – positive constant, H – Hurst exponent.

In order to calculate Hurst exponent, one has to calculate the average value of $(R/S)_n$ for different n and then solve the following equation using linear regression:

$$\log E(R/S)_n = H\log(n) + \log(c), \qquad (6)$$

where $E(R/S)_n$ – expected value of rescaled range.

In above equation, the Hurst exponent can be treated as the regression coefficient and estimated using the least squares method (Jajuga, Papla, 1997).

The Hurst exponent is strictly linked to the fractal dimension of time series, therefore the search for the Hurst exponent is in fact a search for the fractal properties of the series. This relation is described by the following equation (Grech, 2012, p. 10):

$$D_f = 2 - H. \qquad (7)$$

This equation has a huge practical importance as it can be used to classify the type of a time series depending on the fractal dimension of a given object.

Following cases can be distinguished based on Hurst exponent values (Peters, 1997, p. 76–77):

- if $0 < H < 0.5$, then $1.5 < D < 2$ (antipersistent time series),
- if $H = 0.5$, then $D = 1.5$ (random walk),
- if $0.5 < H < 1$, then $1 < D < 1.5$ (persistent time series).

First case ($0 \le H < 0.5$) applies to antipersistent (ergodic) time series. Such a series has a mean reversion tendency. If in a given period the value of the series increased, then in the following period it will most probably decrease and vice versa. The closer the value of H to 0, the more ergodic the behavior of the system and the time series graph has more jagged line, which is a result of a frequent trend reversion. In such case the fractal dimension of the series $D_f \to 2$ as the series fills the plane more and more. The lower the H value, the more noise can be observed in the system. Speaking in the probability language, if e.g. $H = 0.2$, then there is 80% probability that in the future the market will change the direction, which will be equal to trend reversion (Stawicki, Janiak, Müller-Frączek, 1997, p. 37). Despite the fact that mean reversion plays a dominant role in economic and financial literature, so far only a few antipersistent time series have been observed.

The second case applies to a situation, when with $n \to \infty$ $H = 0.5$, which corresponds to a random walk (the consecutive elements of the series are independent). The fractal dimension of the series equals $D = 1.5,$ the series itself is unpredictable, the present does not influence the future, and the past did not influence the present. The probability distribution function can be Gaussian, but not necessarily. Both in natural and economic phenomena the H exponent's value usually differs from 0.5, and the natural processes most often have a long-term data dependency.

When $0.5 < H \leq 1$, the time series are persistent, which means that they bolster the trend. It is caused by the presence of long-term data dependency. When $H \to 1$, the trend gets stronger. As Hurst exponent defines the probability of consecutive rises or drops of the prices, with $H \to 1$, there are more consecutive rises or drops and the level of noise becomes smaller. For example, if $H = 0.8$, then there is 80% probability that a given trend will be sustained in the future. The fractal dimension $D \in (1, 1.5)$ as the more persistent the time series, the less it fills the plane and the smoother are the curves created by a given system. Fractal time series is obviously not purely deterministic, it is rather an intermediate form between a completely random time series and a deterministic system. Persistent time series are fractional Brownian motions, which means that their important feature is a biased random walk, and the strength of bias increases when $H \to 1$, so when the Hurst exponent value recedes from 0.5.

Despite $(R/S)_n$ analysis is a very useful tool, it poses a major disadvantage – it does not take into consideration the changes in particular subperiods. For example, if a particular stock is subject to rapid price changes, whereas the prices of other stocks show only minor price changes, it is not possible to detect this periodical changes using Hurst exponent. Therefore, a good measure of variability of the graph of a function over time is a Hölder function[2], which values in particular points are equal to pointwise Hölder exponents (Mastalerz-Kodzis, 2003, p. 115).

In order to further discuss pointwise Hölder exponents, let us first define the Hölder function[3]. Let (X, d_X) and (Y, d_Y) be metric spaces, then function $f : X \to Y$ is called a Hölder function with exponent α, where $\alpha > 0$, if

[2] Although pointwise Hölder exponents are considered to be the best measure of function regularity in the vicinity of a certain point, other measures used include: local box-counting dimension, local Hausdorff dimension, the degree of fractional differentiability.

[3] The definition of Hölder function and pointwise Hölder exponents was written based on: Mastalerz-Kodzis 2003, pp. 49–51; Kuperin, Schastlivtsev, 2008, pp. 4–6.

for each $x, y \in X$ such that $d_X(x,y) < 1$ the function satisfies the following inequality:

$$d_Y[f(x), f(y)] \leq c d_X(x,y)^{\alpha}, \tag{8}$$

where c – a positive constant.

Assuming that function $f : D \to R$ and that parameter $\alpha \in (0,1)$, function f is a C^{α} class ($f \in C^{\alpha}$) Hölder function, if such $c > 0$ and $h_0 > 0$ constants exist that for every x and every $h \in (0, h_0)$ the following inequality is satisfied:

$$|f(x+h)| - f(x) \leq ch^{\alpha}. \tag{9}$$

Assuming that x_0 is an arbitrary point from the domain of function f, so that $x_0 \in D \subset R$, function $f : D \to R$ is a $C^{\alpha}_{x_0}$ class ($f \in C^{\alpha}_{x_0}$) Hölder function in x_0, if such $c > 0$ and $\varepsilon > 0$ constant exist that for every $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ the following inequality is satisfied:

$$|f(x) - f(x_0)| \leq c|x - x_0|^{\alpha}. \tag{10}$$

By definition Hölder function is continous in its entire domain and when this assumption is satisfied, the graph of the function has fractal nature (Gabryś, 2005, p. 24). If the Hölder function is not continuous, it is called generalized Hölder function. It is worth noting that thanks to its time-varying values, Hölder function can take different types of random walks in different ranges (Kutner, 2009, p. 36).

After providing a Hölder function definition it is possible to define a pointwise Hölder exponent of function f in x_0. By definition it is a number $\alpha_f(x_0)$ given by the following equation:

$$\alpha_f(x_0) = sup\{\alpha : f \in C^{\alpha}_{x_0}\}. \tag{11}$$

Approximated pointwise Hölder exponent is not a flawless measure, however its major advantage is the ability to accomodate the stationarity of the series. Interpretation of pointwise Hölder exponent is the same as for Hurst exponent with a difference that pointwise Hölder exponent estimates local, not global value. It is also worth mentioning that Hölder functions are not constant as they are time-varying (Mastalerz-Kodzis, 2003, p. 121).

3. Tools and Analysis Methodology

3.1. Data

All monthly and daily price series were downloaded from http://stooq.pl. Table 1 presents the list of stock market indices and forex currency pairs used in this research.

Table 1. Financial instruments chosen for the analysis

Symbol	Available Data Period	Market Type
DJIA	1896.05–2012.12	Mature market
S&P 500	1923.01–2012.12	Mature market
DAX	1959.10–2012.12	Mature market
Nikkei225	1949.05–2012.12	Mature market
Hang Seng	1969.11–2012.12	Mature market
WIG20	1991.04–2012.12	Emerging market
Bovespa	1992.01–2012.12	Emerging market
RTS	1995.09–2012.12	Emerging market
SENSEX30	1979.04–2012.12	Emerging market
SCI	1990.12–2012.12	Emerging market
XU100	1990.01–2012.12	Emerging market
EUR/USD	1980.01–2012.12	Major currency pair
GBP/USD	1971.01–2012.12	Major currency pair
USD/JPY	1971.01–2012.12	Major currency pair
CHF/PLN	1990.01–2012.12	Exotic currency pair
EUR/PLN	1990.01–2012.12	Exotic currency pair
USD/BRL	1995.01–2012.12	Exotic currency pair
USD/RUB	1995.10–2012.12	Exotic currency pair
USD/INR	1973.01–2012.12	Exotic currency pair
USD/CNY	1984.01–2012.12	Exotic currency pair
USD/TRY	1984.01–2012.12	Exotic currency pair

The above selection is reasoned by analysis of both mature and emerging markets and the associated currency pairs in order to check if any relationships interesting from the fractal analysis point of view are present.

3.2. Tools

During the analysis, Microsoft Excel was used to calculate the common logarithms of price values and plot the charts of logarithmic price series and pointwise Hölder exponents. For the purposes of box-counting dimension estimation and pointwise Hölder exponents calculation, FracLab 2.0 was used. As quoted from FracLab homepage (http://fraclab.saclay.inria.fr/): „FracLab is a general purpose signal and image processing toolbox based on fractal and multifractal methods. (…) A large number of procedures allow to compute various fractal quantities associated with 1D or 2D signals, such as dimensions, Hölder exponents or multifractal spectra. (…) FracLab is a free software developed in the Regularity team at Inria Saclay/Ecole Centrale de Paris."

3.3. Analysis Methodology

Of the fractal analysis methods described in sections 1 and 2, in this research the box-counting dimension, Hurst exponent and pointwise Hölder exponents were used for long-term dependency analysis of chosen financial time series.

Table 2. Values of fractal dimension D and Hurst exponent H calculated on the entire data range of monthly and daily price series of chosen financial instruments

Symbol	Period	Monthly data		Daily data	
		D	H	D	H
DJIA	1896.05–2012.12	1.44	0.56	1.43	0.57
S&P 500	1923.01–2012.12	1.40	0.60	1.40	0.60
DAX	1959.10–2012.12	1.50	0.50	1.47	0.53
Nikkei225	1949.05–2012.12	1.36	0.64	1.44	0.56
Hang Seng	1969.11–2012.12	1.47	0.53	1.50	0.50
WIG20	1991.04–2012.12	1.43	0.53	1.48	0.52
Bovespa	1992.01–2012.12	1.22	0.78	1.48	0.52
RTS	1995.09–2012.12	1.47	0.53	1.44	0.56
SENSEX30	1979.04–2012.12	1.37	0.63	1.46	0.54
SCI	1990.12–2012.12	1.50	0.50	1.46	0.54
XU100	1990.01–2012.12	1.31	0.69	1.46	0.54
EUR/USD	1980.01–2012.12	1.49	0.51	1.47	0.53
GBP/USD	1971.01–2012.12	1.56	0.44	1.44	0.56
USD/JPY	1971.01–2012.12	1.49	0.51	1.42	0.58
CHF/PLN	1990.01–2012.12	1.32	0.68	1.44	0.56
EUR/PLN	1990.01–2012.12	1.35	0.65	1.43	0.57
USD/BRL	1995.01–2012.12	1.34	0.66	1.40	0.60
USD/RUB	1995.10–2012.12	1.20	0.80	1.36	0.64
USD/INR	1973.01–2012.12	1.23	0.77	1.32	0.68
USD/CNY	1984.01–2012.12	1.16	0.84	1.36	0.64
USD/TRY	1984.01–2012.12	1.08	0.92	1.39	0.61

4. Fractal Analysis Using Fractal Dimension and Pointwise Hölder Exponents

4.1. Presentation of Results

The research was divided into two parts. In the first part, the dimensions of chosen price series were estimated and equation (7) was used to derive Hurst exponent from the estimated dimension. First, the dimensions of price series were calculated for the entire historical data range (see Table 2).

Table 3. Values of fractal dimension D and Hurst exponent H calculated on the Oct 1995 to Dec 2012 data range of monthly and daily price series of chosen financial instruments

Symbol	Period	Monthly data		Daily data	
		D	H	D	H
DJIA	1995.10–2012.12	1.41	0.59	1.48	0.52
S&P 500	1995.10–2012.12	1.36	0.64	1.45	0.55
DAX	1995.10–2012.12	1.39	0.61	1.44	0.56
Nikkei225	1995.10–2012.12	1.50	0.50	1.48	0.52
Hang Seng	1995.10–2012.12	1.60	0.40	1.51	0.49
WIG20	1995.10–2012.12	1.52	0.48	1.48	0.52
Bovespa	1995.10–2012.12	1.46	0.54	1.49	0.51
RTS	1995.10–2012.12	1.46	0.54	1.44	0.56
SENSEX30	1995.10–2012.12	1.51	0.49	1.47	0.53
SCI	1995.10–2012.12	1.53	0.47	1.44	0.56
XU100	1995.10–2012.12	1.37	0.63	1.45	0.55
EUR/USD	1995.10–2012.12	1.53	0.47	1.49	0.51
GBP/USD	1995.10–2012.12	1.52	0.48	1.52	0.48
USD/JPY	1995.10–2012.12	1.50	0.50	1.49	0.51
CHF/PLN	1995.10–2012.12	1.51	0.49	1.50	0.50
EUR/PLN	1995.10–2012.12	1.57	0.43	1.49	0.51
USD/BRL	1995.10–2012.12	1.36	0.64	1.40	0.60
USD/RUB	1995.10–2012.12	1.20	0.80	1.36	0.64
USD/INR	1995.10–2012.12	1..40	0.60	1.39	0.61
USD/CNY	1995.10–2012.12	1.26	0.74	1.56	0.44
USD/TRY	1995.10–2012.12	1.21	0.79	1.40	0.60

In the second part, pointwise Hölder exponents were calculated for six chosen markets. Figures 1–6 present pointwise Hölder exponents plotted together with common logarithms of time series values. The range of values taken by pointwise Hölder exponents' values is presented on the right of each graph.

4.2. Analysis of Results

The interpretation of the results leads to some interesting observations, especially in the light of fractal analysis and fractal market hypothesis.

The part of research based on fractal dimension provided interesting evidence supporting the fractal nature of financial time series. First of all, majority of the investigated markets (16 out of 21) revealed the presence of long-term trends during entire available history of the price series, with 12 out of 21 markets having significantly nonrandom nature, denoted by $H \geq 0.6$. Only 4 out of 21 markets displayed a random or close to random nature, having $0.5 \leq H \leq 0.51$, and just in a single case the antiperistent time series was detected (GBP/USD with $H = 0.44$), which is in line with Peters' remark about rare occurrence of antipersistent time series in nature and economy. Such results seem to prove Peters' concept of financial markets not being a random walk, but rather a combination of trend and noise – a fractional Brownian motion.

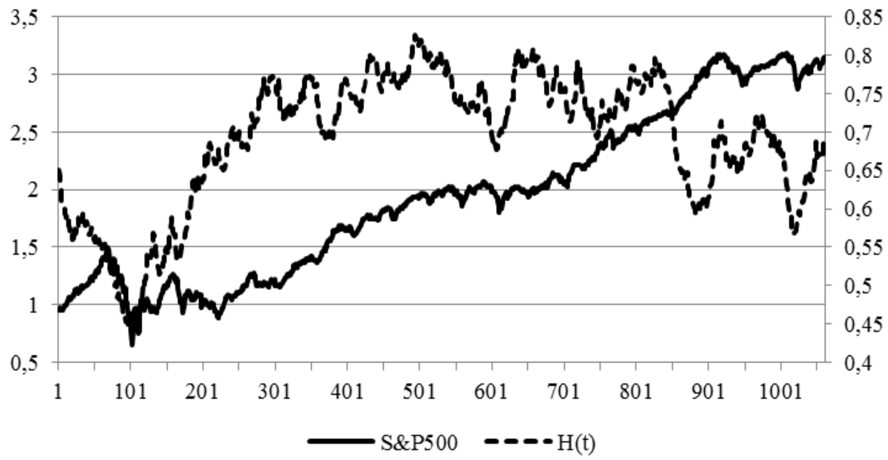

Figure 1. S&P500 price series and pointwise Hölder exponents

Another interesting evidence was gained based on the comparison of the fractional dimension and Hurst exponent values for monthly and daily data. Whereas the research carried out for monthly price series resulted in a wide variety of values ($0.44 \leq H \leq 0.92$ for the entire history of price series), in case of daily data for 16 out of 21 examined markets it turned out that Hurst exponent values are usually considerably lower than respective results for monthly data and fall into a narrow range ($0.5 \leq H \leq 0.6$ for the entire histo-

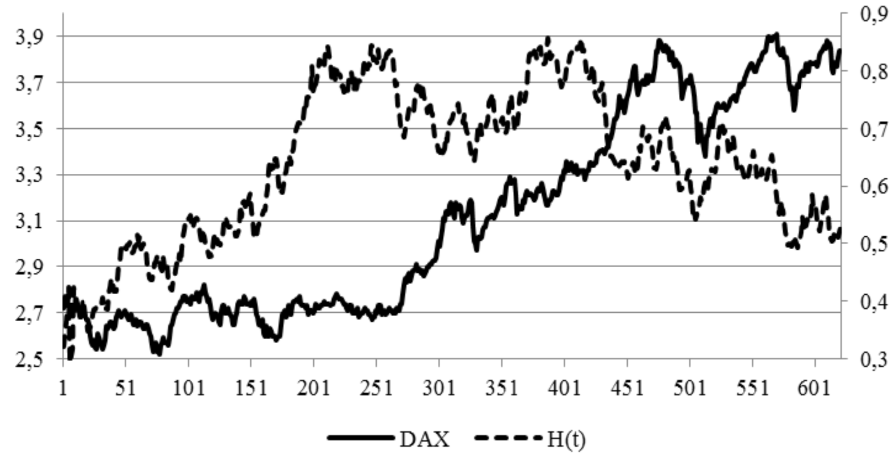

Figure 2. DAX price series and pointwise Hölder exponents

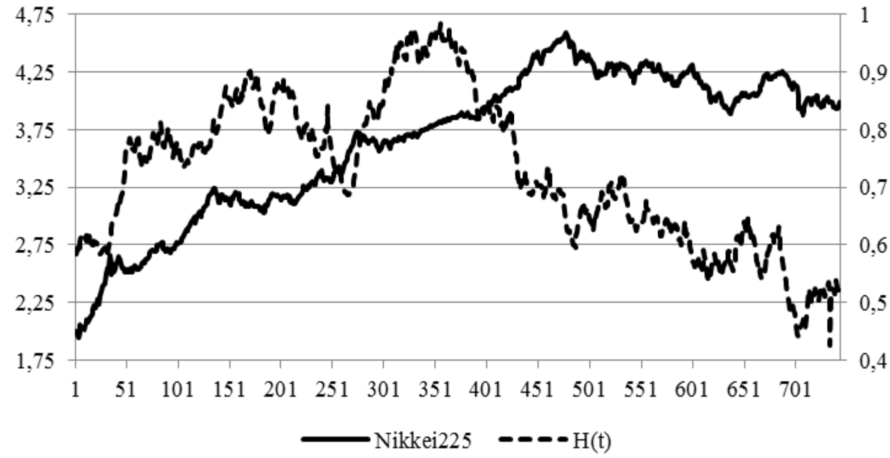

Figure 3. Nikkei225 price series and pointwise Hölder exponents

ry of price series and $0.48 \leq H \leq 0.57$ for October 1995–December 2012 data range). This might mean that another Peters' assumption related to Fractal Market Hypothesis, saying that globally (in the long-term) the market is deterministic, while locally (in the short-term) it is random, can be positively verified. Moreover, it seems that during recent years the growing popularity of electronic trading, larger trading volumes and introduction of automated algorithmic trading and high frequency trading systems, might have increased the efficiency of the markets in the short-term (especially that the biggest increase of efficiency is visible for the world's biggest markets like US and Japan stock market indices or major forex currency pairs). Interestingly, one group of investigated markets is an exception to this rule. All of the exotic currency pairs linked to emerging markets examined in this paper (USD/BRL, USD/CNY, USD/INR, USD/RUB, USD/TRY) show a presence of significant trends on the daily data interval indicated by $H \geq 0.6$, which suggests that the underlying economic phenomena are substantially different for this group of markets. Nevertheless, further research on such hypothesis is out of scope of this paper and might require more studies including not only fractal analysis, but also economic factors, e.g. capital flows.

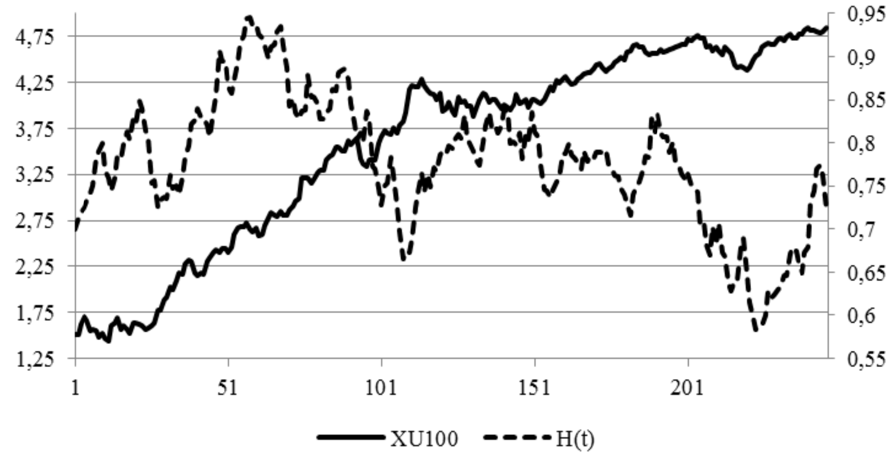

Figure 4. XU100 price series and pointwise Hölder exponents

The observations made during the analysis of results for the entire data range and October 1995–December 2012 data range with respect to market type lead to a conclusion that fractal dimension and Hurst exponent values correspond to the underlying economic situation. For example, the increase of Hurst exponent values for DJIA, S&P500 and DAX during the latter period reflects the dynamic growth and propitious economic conditions in the 1990s and most of 2000s in the economies of United States and Germany. Respectively, the decrease of Hurst exponent values for Nikkei225 and Hang Seng is indicative for the economic slowdown in Japan and South Korea during last two decades after the preceeding very aggresive growth in 1970s and 1980s. Another interesting observations include the growth of efficiency of major forex currency pairs or the randomness of SCI, which probably reflects the huge influence that Chinese government has on the country's economy.

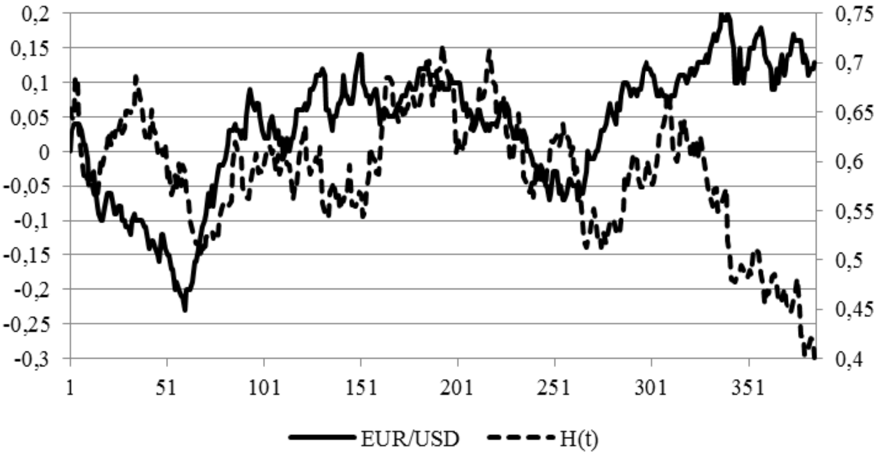

Figure 5. EUR/USD price series and pointwise Hölder exponents

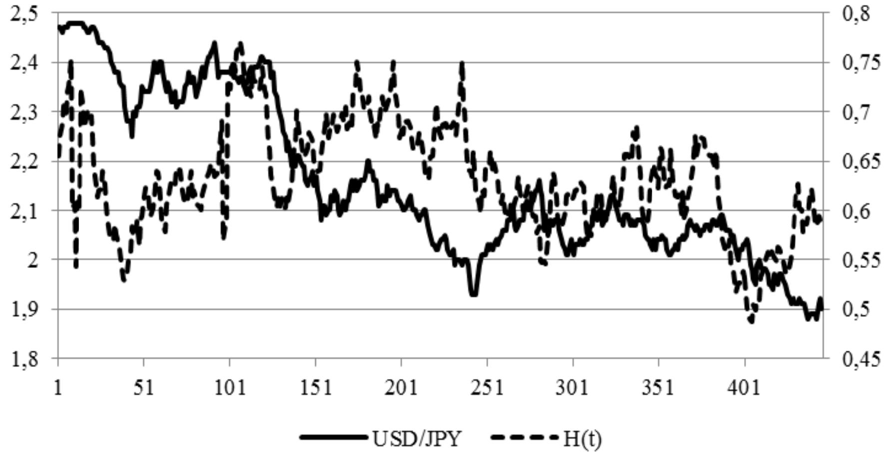

Figure 6. USD/JPY price series and pointwise Hölder exponents

The second part of the research, focused on the application of pointwise Hölder exponents in fractal analysis, provided additional evidence on non-random nature of the markets. For all six examined markets, the values of the pointwise Hölder exponents are significantly higher than 0.5 during most of the entire data range, which proves the presence of long-term dependencies in the investigated time series. Moreover, the graphs expose the relationship between the trend strength and Hölder exponents' values as they tend to considerably decrease and converge to 0.5 during recent years sideways price movements (which is especially visible for Nikkei225, DAX or EUR/USD).

One of the pros of using the box-counting dimension method in this article is that due to its nature, it does not require assumption on the selfsimilarity of the analysed object, which allows to avoid making a possibly false statement even before the beginning of the research. Another considerable advantage of the methods used in this article was their algorithmic accessibility, thus making it affordable from the calculation complexity point of view and available even for the users which do not have a high-end processor cluster at their disposal. Nevertheless, as the estimation method assumes that the Hölder function is constant over each of the intervals, it might introduce some error of method to the results.

Conclusions

The analysis conducted in this research provides solid empirical evidence in favour of Fractal Market Hypothesis, with the presence of long-term dependencies in the financial time series and the confirmation of the global determinism and local randomness of the markets being the most important ones.

An important observation supporting the nonrandomness of the markets is a relationship between fractal properties of the investigated time series and the underlying economic situation. One of the most interesting researches that could be made in the future based on this would be a repetition of the calculations at some fixed intervals of time – after ten, twenty or thirty years. Such approach could possibly verify if the opinions stated by economists a posteriori major changes in economic conditions are in line with the conclusions drawn from the fractal characteristics of analysed markets.

Another study that could be made as a continuation of research conducted in this article could involve inclusion of additional test data, this time not limited to market time series. Inclusion of some regularly published economic indicators like gross domestic product, consumer price index or money supply could potentially reveal some additional relationships and regularities and shed more light on the macroeconomic processes.

Moreover, the analysis of results suggests that fractal analysis can be a valuable tool for the evaluation of market trends, which might be of practical use for institutional and individual investors.

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Fraktalna analiza finansowych szeregów czasowych z wykorzystaniem wymiaru fraktalnego oraz punktowych wykładników Höldera

Z a r y s t r e ś c i. Artykuł przedstawia propozycję zastosowania analizy fraktalnej w celu weryfikacji niektórych założeń hipotezy rynku fraktalnego oraz występowania fraktalnych właściwości w finansowych szeregach czasowych. W celu przeprowadzenia badań wykorzystany został wymiar pudełkowy oraz punktowe wykładniki Höldera. Rezultaty osiągnięte dla badanych rynków pozwoliły dokonać interesujących obserwacji dotyczących nielosowości szeregów cenowych oraz występowania relacji między fraktalnymi właściwościami i miarami zmienności a obecnością trendów i wpływem sytuacji ekonomicznej na ceny instrumentów finansowych.

S ł o w a k l u c z o w e: analiza fraktalna, wymiar fraktalny, wymiar pudełkowy, punktowe wykładniki Höldera, wykładnik Hursta.