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# A CLASS OF DE GIORGI TYPE AND LOCAL BOUNDEDNESS

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Abstract. Under appropriate assumptions on the $N(\Omega)$-function, the De Giorgi process is presented in the framework of Musielak–Orlicz–Sobolev spaces. As the applications, the local boundedness property of the minimizers for a class of the energy functionals in Musielak–Orlicz–Sobolev spaces is proved; and furthermore, the local boundedness of the weak solutions for a class of fully nonlinear elliptic equations is provided.

## 1. Introduction

In the study of nonlinear differential equations, it is well known that more general functional spaces can handle differential equations with more complex nonlinearities. If one would like to study a general form of differential equations, it is crucial to find a proper functional space in which the solutions may exist. The Musielak–Orlicz–Sobolev (or Musielak–Sobolev) space is such a generalization of the Sobolev space that the classical Sobolev spaces, variable exponent Sobolev spaces and Orlicz–Sobolev spaces can be interpreted as its special cases.

The properties and applications of Orlicz–Sobolev spaces and variable exponent Sobolev spaces have been studied extensively in recent years, see for example

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[3], [5], [6], [11], [16], [17]. In [3], the trace on the inner lower dimensional hyperplanes of Olicz–Sobolev spaces was considered. In [11], Fan investigated the trace embedding of the variable exponent Sobolev spaces. In a recent paper [5], the authors considered the $W^{1,p(\,\cdot\,)}$-regularity for elliptic equations with measurable coefficients in nonsmooth domains. To our best knowledge, however, the properties of Musielak–Sobolev spaces have been little studied. In [4], Benkirane and Sidi presented an embedding theorem in Musielak–Sobolev spaces. In [12], [13], Fan established some properties of these functional spaces, including an embedding theorem, and a compact embedding theorem in a bounded domain. And in [14], Fan and Guan studied the uniform convexity of Musielak–Sobolev spaces and presented some applications. As an application of the embedding results in [13], the authors in [20] showed the existence of solutions to a class of fully nonlinear elliptic equation via variational methods. Very recently the authors of [19] studied the trace regularities of Musielak–Sobolev spaces, including both the trace on the inner lower dimensional hyperplanes and the trace on the boundary; also they established some compact trace embedding results.

Our aim here is to study the local boundedness of the minimizers for functionals defined on Musielak–Sobolev spaces and the local boundedness of weak solutions for the associated fully nonlinear elliptic equations. Recall first some important classical regularity results for the minimizers of integral functionals within the Sobolev, variable exponent, Orlicz and Musielak–Sobolev framework settings in the literature. In an early work [1], Acerbi and Fusco proved that for any $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$-local minimizer $u$ with $1 < p < 2$ of the integral functional $\int f(x,v(x),Dv(x))\,dx$, its gradient $Du$ is locally $\lambda$-Hölder continuous for some $\lambda > 0$ when $f$ fulfills some uniformly $p$-exponent increasing conditions. Some regularity results in the variable exponent spaces framework can be found in the work of Diening, Hästö, Roudenko [7]. They also performed an important study [7] of Triebel–Lizorkin spaces with variable smoothness and integrability, including a trace theorem in the variable indices case. In [2], Adamowicz and Toivanen showed the continuity of quasiminimizers of the energy functionals $\int f(x,u,\nabla u)\,dx$ when $f$ satisfies some uniformly $p(\,\cdot\,)$-exponent monotonicity assumptions. In [8], [9], the authors proved a series of regularity results in Orlicz spaces. More precisely, they proved in [8] the $C^{1,\alpha}$-regularity for local minimizers of functionals with $\varphi$-growth including the decay estimate, where $\varphi$ is a convex $C^1$-function independent of the parameter $x \in \Omega \subset \mathbb{R}^N$; in [9], a local Lipschitz result for the local minimizers of asymptotically convex variational integrals was established. For regularity results in the Musielak–Sobolev setting, the authors of [18] showed that *Harnack's inequality* still holds for quasi-minimizers in the Musielak–Sobolev spaces without any polynomial growth or coercivity conditions, which yields the local Hölder continuity of quasi-minimizers. Comparing with our current study, it is interesting to notice that we propose a different monotonicity assumption for the function $\Phi$ from that in [18]. Also we propose a more general uniform monotonicity condition on the function $N(\Omega)$. Moreover, with the regularity results of the key Lemma 3.6, we can prove *not only* the local boundedness property of the minimizers for a more general class of energy functionals (see Section 4), *but also* the local boundedness of a kind of weak solutions for a class of fully nonlinear elliptic equations (see Section 5).

In our study, the strong nonlinear nature of the problems brings fundamental technical complexity in computations. In order to make computations and exposition more transparent, we explore and develop deep connections between the function $N(\Omega)$, its Young's complementary function, and its Sobolev's conjugate function, see Lemma 3.1. The key lemma of the paper is Lemma 3.4, which is an important fully nonlinear iteration lemma in the De Giorgi process. The lemma is a much more general case for the iteration process. It is our belief that the lemmas established here supply a crucial set of tools in the future studies of regularity properties for critical points of integral functional and the associated weak solutions of fully nonlinear equations in the Musielak–Orlicz–Sobolev spaces framework.

Our results have wide applicability. We claim that not only variable exponent Sobolev spaces satisfy the conditions of Theorems 4.1 and 5.2 (see Example 6.1 in Section 6), but also some more complex cases do (see Example 6.2 in Section 6).

The paper is organized as follows. In Section 2, for the readers' convenience we recall some definitions and properties about Musielak–Orlicz–Sobolev spaces. In Section 3, we establish some crucial lemmas in order to prove main results of this paper. In Section 4, we prove the local boundedness of the minimizers of a class of the energy functionals in Musielak–Orlicz–Sobolev spaces. In Section 5, the local boundedness of the weak solution to a class of fully nonlinear elliptic equations is provided.

## 2. The Musielak–Orlicz–Sobolev spaces

In this section, we list some definitions and propositions related to Musielak–Orlicz–Sobolev spaces. Firstly, we give the definition of an *$N$-function* and *generalized $N$-function.*

Definition 2.1. A function $A\colon \mathbb{R} \to [0,+\infty)$ is called an $N$-function, denoted by $A \in N$, if $A$ is even and convex, $A(0) = 0$, $0 < A(t) \in C^0$ for $t \neq 0$, and the following conditions hold:

$$\lim_{t\to 0+} \frac{A(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{A(t)}{t} = +\infty.$$

Let $\Omega$ be a smooth domain in $\mathbb{R}^n$. A function $A\colon \Omega\times\mathbb{R}\to[0,+\infty)$ is called a generalized $N$-function, denoted by $A\in N(\Omega)$, if for each $t\in[0,+\infty)$, the function $A(\,\cdot\,,t)$ is measurable, and for almost every $x\in\Omega$, we have $A(x,\,\cdot\,)\in N$.

Let $A\in N(\Omega)$, the Musielak–Orlicz space $L^A(\Omega)$ is defined by

$$L^A(\Omega):=\bigg\{u : u \text{ is a measurable real function, and } \exists\,\lambda>0 \\ \text{such that } \int_\Omega A\bigg(x,\frac{|u(x)|}{\lambda}\bigg)\,dx<+\infty\bigg\}$$

with the (Luxemburg) norm

$$\|u\|_{L^A(\Omega)}=\|u\|_A:=\inf\bigg\{\lambda>0:\int_\Omega A\bigg(x,\frac{|u(x)|}{\lambda}\bigg)\,dx\le 1\bigg\}.$$

The Musielak–Sobolev space $W^{1,A}(\Omega)$ is defined by

$$W^{1,A}(\Omega):=\{u\in L^A(\Omega):|\nabla u|\in L^A(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,A}(\Omega)}=\|u\|_{1,A}:=\|u\|_A+\|\nabla u\|_A,$$

where $\|\nabla u\|_A:=\|\,|\nabla u|\,\|_A$.

$A$ is called locally integrable if $A(\,\cdot\,,t_0)\in L^1_{\mathrm{loc}}(\Omega)$ for every $t_0>0$.

Definition 2.2. We say that $a(x,t)$ is the Musielak derivative of $A(x,t)\in N(\Omega)$ at $t$ if for $x\in\Omega$ and $t\ge 0$, $a(x,t)$ is the right-hand derivative of $A(x,\,\cdot\,)$ at $t$; and for $x\in\Omega$ and $t\le 0$, $a(x,t):=-a(x,-t)$.

Define $\widetilde{A}\colon\Omega\times\mathbb{R}\to[0,+\infty)$ by

$$\widetilde{A}(x,s)=\sup_{t\in\mathbb{R}}(st-A(x,t))\quad\text{for } x\in\Omega \text{ and } s\in\mathbb{R}.$$

$\widetilde{A}$ is called the *complementary function* to $A$ in the *sense of Young*. It is well known that if $A\in N(\Omega)$, then $\widetilde{A}\in N(\Omega)$ and $A$ is also the complementary function to $\widetilde{A}$.

For $x\in\Omega$ and $s\ge 0$, we denote by $a_+^{-1}(x,s)$ the right-hand derivative of $A'(x,\,\cdot\,)$ at $s$, at the same time define $a_+^{-1}(x,s)=-a_+^{-1}(x,-s)$ for $x\in\Omega$ and $s\le 0$. Then for $x\in\Omega$ and $s\ge 0$, we have

$$a_+^{-1}(x,s)=\sup\{t\ge 0:a(x,t)\le s\}=\inf\{t>0:a(x,t)>s\}.$$

Proposition 2.3 (see [12], [21]). *Let $A\in N(\Omega)$. Then the following assertions hold*:

(a) $A(x,t)\le a(x,t)t\le A(x,2t)$ *for $x\in\Omega$ and $t\in\mathbb{R}$*;

(b) *$A$ and $\widetilde{A}$ satisfy the Young inequality*

$$st\le A(x,t)+\widetilde{A}(x,s)\quad \textit{for } x\in\Omega \textit{ and } s,t\in\mathbb{R},$$

*and the equality holds if $s=a(x,t)$ or $t=a_+^{-1}(x,s)$.*

Let $A, B \in N(\Omega)$. We say that $A$ is weaker than $B$, denoted by $A \preccurlyeq B$, if there exist positive constants $K_1$, $K_2$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that

$$A(x,t) \le K_1 B(x, K_2 t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in [0,+\infty). \tag{2.1}$$

Proposition 2.4 (see [12], [21]). *Let $A, B \in N(\Omega)$ and $A \preccurlyeq B$. Then $\widetilde{B} \preccurlyeq \widetilde{A}$, $L^B(\Omega) \hookrightarrow L^A(\Omega)$ and $L^{\widetilde{A}}(\Omega) \hookrightarrow L^{\widetilde{B}}(\Omega)$.*

Definition 2.5. We say that a function $A\colon \Omega \times [0,+\infty) \to [0,+\infty)$ satisfies the $\Delta_2(\Omega)$ condition, denoted by $A \in \Delta_2(\Omega)$, if there exist a positive constant $K > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$A(x,2t) \le KA(x,t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in [0,+\infty).$$

If $A(x,t) = A(t)$ is an $N$-function and $h(x) \equiv 0$ in $\Omega$ in Definition 2.5, then $A \in \Delta_2(\Omega)$ if and only if $A$ satisfies the well-known $\Delta_2$ condition defined in [3] and [10].

Proposition 2.6 (see [12]). *Let $A \in N(\Omega)$ satisfy $\Delta_2(\Omega)$. Then the following assertions hold*:

(a) $L^A(\Omega) = \{u : u$ *is a measurable function, and* $\int_\Omega A(x,|u(x)|)\,dx < +\infty\}$;
(b) $\int_\Omega A(x,|u|)\,dx < 1$ (*resp.* $= 1$; $> 1$) *if and only if* $\|u\|_A < 1$ (*resp.* $= 1$; $> 1$), *where* $u \in L^A(\Omega)$;
(c) $\int_\Omega A(x,|u_n|)\,dx \to 0$ (*resp.* 1; $+\infty$) *if and only if* $\|u_n\|_A \to 0$ (*resp.* 1; $+\infty$), *where* $\{u_n\} \subset L^A(\Omega)$;
(d) *if* $u_n \to u$ *in* $L^A(\Omega)$ *then* $\int_\Omega |A(x,|u_n|)\,dx - A(x,|u|)|\,dx \to 0$ *as* $n \to \infty$;
(e) *if* $A'$ *also satisfies* $\Delta_2$, *then*

$$\left| \int_\Omega u(x)v(x)\,dx \right| \le 2\|u\|_A \|v\|_{\widetilde{A}} \quad \text{for all } u \in L^A(\Omega),\ v \in L^{\widetilde{A}}(\Omega);$$

(f) $a(\,\cdot\,, |u(\,\cdot\,)|) \in L^{\widetilde{A}}(\Omega)$ *for every* $u \in L^A(\Omega)$.

The following assumptions will be used:

($C_1$) $\inf\limits_{x\in\Omega} A(x,1) = c_1 > 0$.
($C_2$) For every $t_0 > 0$, there exists $c = c(t_0) > 0$ such that

$$\frac{A(x,t)}{t} \ge c \quad \text{and} \quad \frac{\widetilde{A}(x,t)}{t} \ge c \quad \text{for all } t \ge t_0,\ x \in \Omega.$$

Remark 2.7. Obviously, ($C_2$) $\Rightarrow$ ($C_1$). If $A(x,t) = A(t)$ is an $N$-function, then ($C_1$) and ($C_2$) hold automatically, and $A$ is automatically locally integrable.

Proposition 2.8 (see [12]). *If $A \in N(\Omega)$ satisfies ($C_1$), then $L^A(\Omega) \hookrightarrow L^1(\Omega)$ and $W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega)$.*

Proposition 2.9 (see [12]). *Let $A \in N(\Omega)$, both $A$ and $\widetilde{A}$ be locally integrable and satisfy $\Delta_2(\Omega)$ and $(\mathrm{C}_2)$. Then the space $L^A(\Omega)$ is reflexive, and the mapping $J\colon L^{\widetilde{A}}(\Omega) \to (L^A(\Omega))^*$ defined by*

$$\langle J(v), w\rangle = \int_\Omega v(x)w(x)\,dx \quad \text{for all } v \in L^{\widetilde{A}}(\Omega),\ w \in L^A(\Omega)$$

*is a linear isomorphism, and $\|J(v)\|_{(L^A(\Omega))^*} \le 2\|v\|_{L^{\widetilde{A}}(\Omega)}$.*

Let $A \in N(\Omega)$ be locally integrable. We will denote

$$W_0^{1,A}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,A}(\Omega)}}, \qquad \mathcal{D}_0^{1,A}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_{L^A(\Omega)}}.$$

In the case that $\|\nabla u\|_A$ is an equivalent norm in $W_0^{1,A}(\Omega)$, $W_0^{1,A}(\Omega) = \mathcal{D}_0^{1,A}(\Omega)$.

Proposition 2.10 (see [12]). *Let $A \in N(\Omega)$ be locally integrable and satisfy $(\mathrm{C}_1)$. Then*

(a) *the spaces $W^{1,A}(\Omega), W_0^{1,A}(\Omega)$ and $\mathcal{D}_0^{1,A}(\Omega)$ are separable Banach spaces, and*

$$W_0^{1,A}(\Omega) \hookrightarrow W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega),$$
$$\mathcal{D}_0^{1,A}(\Omega) \hookrightarrow \mathcal{D}_0^{1,1}(\Omega) = W_0^{1,1}(\Omega);$$

(b) *the spaces $W^{1,A}(\Omega), W_0^{1,A}(\Omega)$ and $\mathcal{D}_0^{1,A}(\Omega)$ are reflexive provided $L^A(\Omega)$ is reflexive.*

Proposition 2.11 (see [12]). *Let $A, B \in N(\Omega)$ and $A$ be locally integrable. If there is a compact imbedding $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^B(\Omega)$ and $A \preccurlyeq B$, then there holds the following Poincaré inequality*

$$\|u\|_A \le c\|\nabla u\|_A \quad \text{for all } u \in W_0^{1,A}(\Omega),$$

*which implies that $\|\nabla\cdot\|_A$ is an equivalent norm in $W_0^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega) = \mathcal{D}_0^{1,A}(\Omega)$.*

The following assumptions will be used:

($\mathrm{P}_1$) $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with the cone property, and $A \in N(\Omega)$;

($\mathrm{P}_2$) $A\colon \overline{\Omega} \times \mathbb{R} \to [0,+\infty)$ is continuous and $A(x,t) \in (0,+\infty)$ for $x \in \overline{\Omega}$ and $t \in (0,+\infty)$.

Let $A$ satisfy ($\mathrm{P}_1$) and ($\mathrm{P}_2$). Denote by $A^{-1}(x,\cdot\,)$ the inverse function of $A(x,\cdot\,)$. We always assume that the following condition holds:

($\mathrm{P}_3$) $A \in N(\Omega)$ and

$$\int_0^1 \frac{A^{-1}(x,t)}{t^{(n+1)/n}}\,dt < +\infty \quad \text{for all } x \in \overline{\Omega}. \tag{2.2}$$

Under assumptions (P$_1$)–(P$_3$), for each $x \in \overline{\Omega}$, the function

$$A(x,\,\cdot\,)\colon [0,+\infty) \to [0,+\infty)$$

is a strictly increasing homeomorphism.

Define a function $A_*^{-1}\colon \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ by

$$A_*^{-1}(x,s) = \int_0^s \frac{A^{-1}(x,\tau)}{\tau^{(n+1)/n}}\,d\tau \quad \text{for } x \in \overline{\Omega} \text{ and } s \in [0,+\infty).$$

Then, under assumption (P$_3$), $A_*^{-1}$ is well defined, and $A_*^{-1}(x,\,\cdot\,)$ is strictly increasing, $A_*^{-1}(x,\,\cdot\,) \in C^1((0,+\infty))$, and the function $A_*^{-1}(x,\,\cdot\,)$ is concave for each $x \in \overline{\Omega}$. Set

$$T(x) = \lim_{s\to+\infty} A_*^{-1}(x,s) \quad \text{for all } x \in \overline{\Omega}. \tag{2.3}$$

Then $0 < T(x) \le +\infty$. Define an even function $A_*\colon \overline{\Omega} \times \mathbb{R} \to [0,+\infty)$ by

$$A_*(x,t) = \begin{cases} s & \text{if } x \in \overline{\Omega},\ |t| \in [0,T(x)) \text{ and } A_*^{-1}(x,s) = |t|, \\ +\infty & \text{for } x \in \overline{\Omega} \text{ and } |t| \ge T(x). \end{cases}$$

Then, if $A \in N(\Omega)$ and $T(x) = +\infty$ for any $x \in \overline{\Omega}$, it is well known that $A_* \in N(\Omega)$ (see [3]). $A_*$ is called the Sobolev conjugate function of $A$ (see [3] for the case of Orlicz functions).

Let $X$ be a metric space and $f\colon X \to (-\infty,+\infty]$ be an extended real-valued function. For $x \in X$ with $f(x) \in \mathbb{R}$, the continuity of $f$ at $x$ is well defined. For $x \in X$ with $f(x) = +\infty$, we say that $f$ is continuous at $x$ if, given any $M > 0$, there exists a neighbourhood $U$ of $x$ such that $f(y) > M$ for all $y \in U$. We say that $f\colon X \to (-\infty,+\infty]$ is continuous on $X$ if $f$ is continuous at every $x \in X$. Define $\mathrm{Dom}(f) = \{x \in X : f(x) \in \mathbb{R}\}$ and denote by $C^{1-0}(X)$ the set of all locally Lipschitz continuous real-valued functions defined on $X$.

The following assumptions will also be used:

(P$_4$) $T\colon \overline{\Omega} \to [0,+\infty]$ is continuous on $\overline{\Omega}$ and $T \in C^{1-0}(\mathrm{Dom}(T))$;

(P$_5$) $A_* \in C^{1-0}(\mathrm{Dom}(A_*))$ and there exist three positive constants $\delta_0$, $C_0$ and $t_0$ with $\delta_0 < 1/n$, $0 < t_0 < \min\limits_{x\in\overline{\Omega}} T(x)$ such that

$$|\nabla_x A_*(x,t)| \le C_0 (A_*(x,t))^{1+\delta_0}, \quad j = 1,\dots,n,$$

for $x \in \Omega$ and $|t| \in [t_0, T(x))$ provided $\nabla_x A_*(x,t)$ exists.

Let $A, B \in N(\Omega)$. We say that $A \ll B$ if, for any $k > 0$,

$$\lim_{t\to+\infty} \frac{A(x,kt)}{B(x,t)} = 0 \quad \text{uniformly for } x \in \Omega.$$

Remark 2.12. Suppose that $A, B \in N(\Omega)$, then $A \ll B \Rightarrow A \preccurlyeq B$.

Next we give two embedding theorems for Musielak–Sobolev spaces recently established by Fan in [13].

Theorem 2.13 (see [13], [20]). *Let* $(\mathrm{P}_1)$–$(\mathrm{P}_5)$ *hold. Then*:

(a) *There is a continuous imbedding* $W^{1,A}(\Omega) \hookrightarrow L^{A_*}(\Omega)$.
(b) *Suppose that* $B \in N(\Omega)$, $B\colon \overline{\Omega} \times [0,+\infty) \to [0,+\infty)$ *is continuous, and* $B(x,t) \in (0,+\infty)$ *for* $x \in \Omega$ *and* $t \in (0,+\infty)$. *If* $B \ll A_*$, *then there is a compact imbedding* $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^B(\Omega)$.

By Theorem 2.13, Remark 2.12 and Proposition 2.11, we have the following:

Theorem 2.14 (see [13], [20]). *Let* $(\mathrm{P}_1)$–$(\mathrm{P}_5)$ *hold and furthermore*, $A, A_* \in N(\Omega)$. *Then*:

(a) $A \ll A_*$, *and there is a compact imbedding* $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^A(\Omega)$.
(b) *There holds the Poincaré-type inequality*

$$\|u\|_A \leq C\|\nabla u\|_A \quad \text{for } u \in W_0^{1,A}(\Omega),$$

*i.e.* $\|\nabla u\|_A$ *is an equivalent norm on* $W_0^{1,A}(\Omega)$.

## 3. Some lemmas

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and $A \in N(\Omega)$ satisfies the following condition $(\mathscr{A})$, denoted by $A \in \mathscr{A}$.

$(\mathscr{A})$ $A \in N(\Omega)$ satisfies assumptions $(\mathrm{P}_1)$–$(\mathrm{P}_3)$, $(P_5)$ in Section 2 and the following:
$(\widetilde{\mathrm{P}_4})$ $T(x)$ defined in (2.3) satisfies $T(x) = +\infty$ for all $x \in \overline{\Omega}$.

Lemma 3.1. *Suppose that* $A \in N(\Omega)$, *and there exists a strictly increasing differentiable function* $\mathfrak{A}\colon [0,+\infty) \to [0,+\infty)$ *such that*

$$A(x,\alpha t) \geq \mathfrak{A}(\alpha)A(x,t) \quad \text{for all } \alpha \geq 0,\ t \in \mathbb{R},\ x \in \Omega. \tag{3.1}$$

(a) *Then there exists a strictly increasing differentiable function* $\widehat{\mathfrak{A}}\colon [0,+\infty) \to [0,+\infty)$, *defined by*

$$\widehat{\mathfrak{A}}(\sigma) = \begin{cases} \dfrac{1}{\mathfrak{A}(1/\sigma)} & \text{for } \sigma > 0, \\ 0 & \text{for } \sigma = 0, \end{cases} \tag{3.2}$$

*such that*

$$A(x,\beta t) \leq \widehat{\mathfrak{A}}(\beta)A(x,t), \quad \text{for all } \beta > 0,\ t \in \mathbb{R},\ x \in \Omega, \tag{3.3}$$

*and furthermore* $\widehat{\widehat{\mathfrak{A}}} = \mathfrak{A}$.

(b) *If* $\mathfrak{A}$ *satisfies*

$$n\,\mathfrak{A}(\alpha) > \alpha\,\mathfrak{A}'(\alpha), \tag{3.4}$$

*then* $A_* \in N(\Omega)$*, and there exists a strictly increasing differentiable function* $\mathfrak{A}_*\colon [0,+\infty) \to [0,+\infty)$*, defined by*

$$\mathfrak{A}_*^{-1}(\sigma) = \begin{cases} \dfrac{1}{\sigma^{1/n}\,\mathfrak{A}^{-1}(\sigma^{-1})} & \text{for } \sigma > 0, \\ 0 & \text{for } \sigma = 0, \end{cases} \tag{3.5}$$

*such that*

$$A_*(x, \beta t) \le \mathfrak{A}_*(\beta) A_*(x,t) \quad \text{for all } \beta > 0,\ t \in \mathbb{R},\ x \in \Omega. \tag{3.6}$$

(c) *If* $\mathfrak{A}$ *satisfies*

$$\alpha\,\mathfrak{A}'(\alpha) > \mathfrak{A}(\alpha), \tag{3.7}$$

*then* $\widetilde{A} \in N(\Omega)$*, and there exists a strictly increasing differentiable function* $\widetilde{\mathfrak{A}}\colon [0,+\infty) \to [0,+\infty)$*, defined by*

$$\widetilde{\mathfrak{A}}^{-1}(\sigma) = \begin{cases} \sigma(\mathfrak{A}^{-1}(\sigma))^{-1} & \text{for } \sigma > 0, \\ 0 & \text{for } \sigma = 0, \end{cases} \tag{3.8}$$

*such that*

$$\widetilde{A}(x, \beta t) \le \widetilde{\mathfrak{A}}(\beta)\widetilde{A}(x,t) \quad \text{for all } \beta > 0,\ t \in \mathbb{R},\ x \in \Omega. \tag{3.9}$$

Proof. (a) To prove (3.3), we set $t = s/\alpha$ and $\beta = 1/\alpha$ in (3.1). Then we can see that (3.3) holds with

$$\widehat{\mathfrak{A}}(\beta) = \frac{1}{\mathfrak{A}(\beta^{-1})},$$

which is a strictly increasing differentiable function in the variable $\beta$.

(b) It is clear that $A_* \in N(\Omega)$ by [3]. To prove (3.6), we set

$$t = A^{-1}\biggl(x, \frac{s}{\mathfrak{A}(\alpha)}\biggr)$$

in (3.1). Then, for $\alpha > 0$, we have

$$A\biggl(x, \alpha A^{-1}\biggl(x, \frac{s}{\mathfrak{A}(\alpha)}\biggr)\biggr) \ge \mathfrak{A}(\alpha) A\biggl(x, A^{-1}\biggl(x, \frac{s}{\mathfrak{A}(\alpha)}\biggr)\biggr) = s,$$

or equivalently,

$$\alpha A^{-1}\biggl(x, \frac{s}{\mathfrak{A}(\alpha)}\biggr) \ge A^{-1}(x,s),$$

which implies that

$$\frac{\alpha}{(\mathfrak{A}(\alpha))^{1/n}} \, \frac{A^{-1}(x, s/\mathfrak{A}(\alpha))}{(s/\mathfrak{A}(\alpha))^{(n+1)/n}} \, \frac{1}{\mathfrak{A}(\alpha)} \ge \frac{A^{-1}(x,s)}{s^{(n+1)/n}}.$$

Integrating the above inequality with respect to $s$ from 0 to $t$, we have

$$\frac{\alpha}{(\mathfrak{A}(\alpha))^{1/n}} \int_0^{t/\mathfrak{A}(\alpha)} \frac{A^{-1}(x,r)}{r^{(n+1)/n}}\, dr \ge \int_0^t \frac{A^{-1}(x,s)}{s^{(n+1)/n}}\, ds,$$

where $r = s/\mathfrak{A}(\alpha)$. Then the definition of $A_*$ yields that

$$\frac{\alpha}{(\mathfrak{A}(\alpha))^{1/n}} A_*^{-1}\bigg(x, \frac{t}{\mathfrak{A}(\alpha)}\bigg) \ge A_*^{-1}(x,t). \tag{3.10}$$

If we set

$$t = \mathfrak{A}(\alpha) A_*\bigg(x, \frac{(\mathfrak{A}(\alpha))^{1/n}}{\alpha}\, z\bigg),$$

then we conclude that

$$z \ge A_*^{-1}\bigg(x, \mathfrak{A}(\alpha) A_*\bigg(x, \frac{(\mathfrak{A}(\alpha))^{1/n}}{\alpha}\, z\bigg)\bigg),$$

or equivalently,

$$A_*(x,z) \ge \mathfrak{A}(\alpha) A_*\bigg(x, \frac{(\mathfrak{A}(\alpha))^{1/n}}{\alpha}\, z\bigg). \tag{3.11}$$

Set $\beta = (\mathfrak{A}(\alpha))^{1/n}/\alpha$. Then, by basic computation and assumption (3.4), we have

$$\frac{d\beta}{d\alpha} = \frac{(\mathfrak{A}(\alpha))^{1/n-1}}{n\alpha^2}(\alpha\mathfrak{A}'(\alpha) - n\mathfrak{A}(\alpha)) < 0 \quad \text{for all } \alpha > 0,$$

which implies, by the implicit function theorem, that we can interpret $\alpha$ as a function of $\beta$. Denote $\alpha = \alpha(\beta)$. So inequality (3.11) implies that there exists a function $\mathfrak{A}_*(\beta) := (\mathfrak{A}(\alpha(\beta)))^{-1}$ such that

$$A_*(x, \beta z) \le \mathfrak{A}_*(\beta) A_*(x,z) \quad \text{for all } \beta > 0,\ z \in \mathbb{R}.$$

Set $\sigma = 1/\mathfrak{A}(\alpha)$, we can get (3.5). To see that $\mathfrak{A}_*(\beta)$ is strictly increasing with respect to the variable $\beta$, we compute to obtain

$$\frac{d}{d\beta}(\mathfrak{A}_*(\beta)) = \frac{d}{d\alpha}\bigg(\frac{1}{\mathfrak{A}(\alpha)}\bigg)\frac{d\alpha}{d\beta} = \frac{n\alpha^2\, \mathfrak{A}'(\alpha)}{\mathfrak{A}^{1+1/n}(\alpha)(n\,\mathfrak{A}(\alpha) - \alpha\,\mathfrak{A}'(\alpha))} > 0.$$

The above inequality completes the proof of (b).

(c) By the definition of $\widetilde{A}$ and (3.1), we deduce that

$$\begin{aligned}\widetilde{A}(x, \alpha s) &= \sup_{t\in\mathbb{R}}(\alpha s t - A(x,t)) \ge \sup_{t\in\mathbb{R}}\bigg(s\alpha t - \frac{1}{\mathfrak{A}(\alpha)} A(x,\alpha t)\bigg) \\ &= \frac{1}{\mathfrak{A}(\alpha)} \sup_{t\in\mathbb{R}}(\mathfrak{A}(\alpha)st - A(x,t)) = \frac{1}{\mathfrak{A}(\alpha)} \widetilde{A}(x, \mathfrak{A}(\alpha)s).\end{aligned}$$

Setting $s = t/\alpha$ and $\beta = \mathfrak{A}(\alpha)/\alpha$, we obtain that

$$\widetilde{A}(x, \beta t) \le \mathfrak{A}(\alpha)\widetilde{A}(x,t). \tag{3.12}$$

In view of (3.7) and by direct computation, we conclude that

$$\frac{d\alpha}{d\beta} = \frac{\alpha^2}{\alpha\,\mathfrak{A}'(\alpha) - \mathfrak{A}(\alpha)} > 0 \quad \text{for all } \alpha > 0.$$

Then, by the implicit function theorem, we can interpret $\alpha$ as a function of $\beta$. Denote $\alpha = \alpha(\beta)$. So inequality (3.12) implies that there exists a function $\widetilde{\mathfrak{A}}(\beta) := \mathfrak{A}(\alpha(\beta))$ such that

$$\widetilde{A}(x, \beta t) \le \widetilde{\mathfrak{A}}(\beta)\widetilde{A}(x,t).$$

Set $\sigma = 1/\mathfrak{A}(\alpha)$, we can get (3.8). To see that $\widetilde{\mathfrak{A}}(\beta)$ is strictly increasing with respect to the variable $\beta$, we compute to obtain

$$\frac{d}{d\beta}\,(\widetilde{\mathfrak{A}}(\beta)) = \mathfrak{A}'_\alpha(\alpha)\,\frac{d\alpha}{d\beta} = \frac{\alpha^2\,\mathfrak{A}'(\alpha)}{\alpha\,\mathfrak{A}'(\alpha) - \mathfrak{A}(\alpha)} > 0.$$

The above inequality completes the proof of (c). $\square$

Definition 3.2. We say that $\mathfrak{C}\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition $\Delta_{\mathbb{R}^+}$, denoted by $\mathfrak{C} \in \Delta_{\mathbb{R}^+}$, if there exists a constant $M_0 > 0$ such that

$$\mathfrak{C}(\alpha\beta) \le M_0\,\mathfrak{C}(\alpha)\mathfrak{C}(\beta) \quad \text{for all } \alpha, \beta > 0. \tag{3.13}$$

By equations (3.2), (3.5) and (3.8), it is clear that the following remark holds.

Remark 3.3. Under assumptions in Lemma 3.1, if $\mathfrak{A}, \mathfrak{A}^{-1} \in \Delta_{\mathbb{R}^+}$, $\widehat{\mathfrak{A}}$, $\mathfrak{A}_*^{-1}$, $\widetilde{\mathfrak{A}}^{-1}$, $\widehat{\widetilde{\mathfrak{A}}}^{-1} \in \Delta_{\mathbb{R}^+}$.

Now we are in a position to prove the following result.

Lemma 3.4. *Let $\{y_h\} \subset \mathbb{R}$ be a sequence satisfying*

$$y_{h+1} \le \frac{1}{\beta}\,\mathfrak{A}_*\biggl(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,c\,2^h\mathfrak{A}^{-1}(c\,\mathfrak{A}_*(2^{h+2})\,y_h)\biggr) \quad \textit{for all } \beta > 0, \tag{3.14}$$

*where $c$ is a positive constant. If $\mathfrak{A}, \mathfrak{A}^{-1}, \mathfrak{A}_* \in \Delta_{\mathbb{R}^+}$, then there exists $y_0^* > 0$ such that, for $y_0 \le y_0^*$, $y_h \to 0$ as $h \to \infty$.*

Proof. Denote $a_h := c2^{h+2}$, $s_h := c\mathfrak{A}_*(a_h)$, $k_h := \mathfrak{A}^{-1}(\widetilde{M}s_h)$, where $\widetilde{M}$ is a constant to be determined later and set $\beta = \beta_h := (k_h a_h)^n$ in (3.14).

Claim. For $y_0$ small enough, we have

$$y_h \le \frac{\mathfrak{A}(k_h/\mathfrak{A}^{-1}(\beta_h))}{s_h}. \tag{3.15}$$

Indeed, we can prove (3.15) by induction on $h$. Suppose (3.15) holds for some $h$. We will prove that (3.15) holds for $h+1$. By (3.14), we have

$$y_{h+1} \le \frac{1}{\beta_h}\,\mathfrak{A}_*\biggl(\frac{\mathfrak{A}^{-1}(\beta_h)}{\beta_h^{1/n}}\,a_h\,\mathfrak{A}^{-1}(s_h y_h)\biggr) \le \frac{\mathfrak{A}_*(1)}{(a_h k_h)^n}. \tag{3.16}$$

At the same time since $\mathfrak{A}, \mathfrak{A}^{-1}, \mathfrak{A}_* \in \Delta_{\mathbb{R}^+}$, there exist constants $M_1, M_2, M_3 > 0$ such that

$$\mathfrak{A}\biggl(\frac{\mathfrak{A}^{-1}(M\alpha)}{\mathfrak{A}^{-1}(\beta)}\biggr) \ge \frac{\alpha}{\beta} \quad \text{for all } M \ge M_1,\ \alpha, \beta > 0, \tag{3.17}$$

$$\frac{\mathfrak{A}^{-1}(\alpha)}{\mathfrak{A}^{-1}(\beta)} \le M_2\, \mathfrak{A}^{-1}\left(\frac{\alpha}{\beta}\right) \quad \text{for all } \alpha, \beta > 0,$$

and

$$\frac{\mathfrak{A}_*(\alpha)}{\mathfrak{A}_*(\beta)} \le M_3\, \mathfrak{A}_*\left(\frac{\alpha}{\beta}\right) \quad \text{for all } \alpha, \beta > 0.$$

Then

$$\begin{aligned}\frac{k_{h+1}}{k_h} &= \frac{\mathfrak{A}^{-1}(c\widetilde{M}\mathfrak{A}_*(a_{h+1}))}{\mathfrak{A}^{-1}(c\widetilde{M}\mathfrak{A}_*(a_h))} \le M_2\mathfrak{A}^{-1}\left(\frac{\mathfrak{A}_*(a_{h+1})}{\mathfrak{A}_*(a_h)}\right) \\ &\le M_2\mathfrak{A}^{-1}\left(M_3\mathfrak{A}_*\left(\frac{a_{h+1}}{a_h}\right)\right) = M_2\mathfrak{A}^{-1}(M_3\mathfrak{A}_*(2)) := 2^{-1}M_4^{1/n}.\end{aligned}$$

By (3.17), we can choose $\widetilde{M} = M_1M_4\mathfrak{A}_*(1)$. Then

$$\begin{aligned}\frac{\mathfrak{A}\left(\dfrac{k_{h+1}}{\mathfrak{A}^{-1}((k_{h+1}a_{h+1})^n)}\right)}{s_{h+1}} &= \frac{\mathfrak{A}\left(\dfrac{\mathfrak{A}^{-1}(\widetilde{M}s_{h+1})}{\mathfrak{A}^{-1}((k_{h+1}a_{h+1})^n)}\right)}{s_{h+1}} \\ &\ge \frac{M_4\mathfrak{A}_*(1)}{(k_{h+1}a_{h+1})^n} \ge \frac{\mathfrak{A}_*(1)}{(k_ha_h)^n}.\end{aligned} \tag{3.18}$$

Together with (3.16) and (3.18), we conclude

$$y_{h+1} \le \frac{\mathfrak{A}\left(\dfrac{k_{h+1}}{\mathfrak{A}^{-1}((k_{h+1}a_{h+1})^n)}\right)}{s_{h+1}},$$

which completes the induction on $h$ of the claim. Furthermore, by the claim we have

$$y_{h+1} \le \frac{\mathfrak{A}_*(1)}{(a_hk_h)^n}.$$

Then for

$$y_0 \le \frac{\mathfrak{A}\left(\dfrac{k_0}{\mathfrak{A}^{-1}(\beta_0)}\right)}{s_0} = \frac{\mathfrak{A}\left(\dfrac{\mathfrak{A}^{-1}(\widetilde{M}c\mathfrak{A}_*(4c))}{\mathfrak{A}^{-1}((4c\mathfrak{A}^{-1}(\widetilde{M}c\mathfrak{A}_*(4c)))^n)}\right)}{c\mathfrak{A}_*(4c)} := y_0^*$$

we conclude $y_h \to 0$ as $h \to \infty$. □

Lemma 3.5. *Let $f(t)$ be a nonnegative bounded function defined on $[r_0, r_1]$, $r_0 \ge 0$. Suppose that for $r_0 \le s < t \le r_1$ we have*

$$f(s) \le [A((t-s)^{-1}) + B] + \theta f(t), \tag{3.19}$$

*where $B$, $\theta$ are nonnegative constants with $0 \le \theta < 1$ and $A\colon [0, +\infty) \to [0, +\infty)$ is a function satisfying*

$$A(\alpha t) \le \mathfrak{A}(\alpha)A(t), \tag{3.20}$$

*where* $\mathfrak{A}\colon [0,+\infty) \to [0,+\infty)$ *is a strictly increasing continuous function with* $\mathfrak{A}(0) = 0$ *and there exists a* $T_0 \in \mathbb{R}$ *with* $T_0 \geq 1$ *such that* $\mathfrak{A}(T_0) = 1$. *Then for all* $r_0 \leq \rho < R \leq r_1$ *we have*

$$f(\rho) \leq C[A((R-\rho)^{-1}) + B], \tag{3.21}$$

*where* $C$ *is a constant depending on* $\theta$ *and* $\mathfrak{A}$.

PROOF. For fixed $\rho$ and $R$, consider the sequence $\{t_i\}_{i=1}^{+\infty}$ defined by

$$t_0 = \rho, \qquad t_{i+1} - t_i = \left(1 - \frac{r}{T_0}\right)\left(\frac{r}{T_0}\right)^i (R-\rho),$$

where $0 < r < 1 \leq T_0$. By iteration from (3.19) and (3.20) we get

$$\begin{aligned} f(t_0) \leq{}& \theta^k f(t_k) + \left[A\left(\left(1 - \frac{r}{T_0}\right)^{-1}(R-\rho)^{-1}\right) + B\right] \sum_{i=0}^{k-1}\left(\theta\,\mathfrak{A}\left(\frac{T_0}{r}\right)\right)^i \\ \leq{}& \theta^k f(t_k) \\ &+ \max\left\{\mathfrak{A}\left(\left(1 - \frac{r}{T_0}\right)^{-1}\right), 1\right\}[A((R-\rho)^{-1}) + B] \sum_{i=0}^{k-1}\left(\theta\,\mathfrak{A}\left(\frac{T_0}{r}\right)\right)^i. \end{aligned}$$

By the definition of $\mathfrak{A}$ there exists a $0 < r < T_0$ such that $\theta\mathfrak{A}(T_0/r) < 1$, then from (3.20) we can get (3.21) by sending $k \to \infty$ with

$$C = \max\left\{\mathfrak{A}\left(\left(1 - \frac{r}{T_0}\right)^{-1}\right), 1\right\}\left(1 - \theta\,\mathfrak{A}\left(\frac{T_0}{r}\right)\right)^{-1}. \qquad \square$$

In the following lemma $A \in N(\Omega) \cap \mathscr{A}$ satisfies the following assumptions:

($A_1$) There exists a strictly increasing differentiable function $\mathfrak{A}\colon [0,+\infty) \to [0,+\infty)$ satisfying

$$n\,\mathfrak{A}(\alpha) > \alpha\,\mathfrak{A}'(\alpha) \tag{3.22}$$

such that

($A_{11}$) $A(x,\alpha t) \geq \mathfrak{A}(\alpha)A(x,t)$ for all $\alpha \geq 0$, $t \in \mathbb{R}$, $x \in \Omega$;

($A_{12}$) $\mathfrak{A}, \mathfrak{A}^{-1}, \mathfrak{A}_* \in \Delta_{\mathbb{R}^+}$.

If $u \in W^{1,A}(\Omega)$ and $B_R = B_R(x) := \{y \in \mathbb{R}^n : |y - x| < R\} \subset \Omega$ is any ball, we denote $\Omega_{k,R} := \{x \in B_R : u(x) > k\}$, where $k$ is a real number.

LEMMA 3.6. *Let* $A \in N(\Omega) \cap \mathscr{A}$ *satisfy* ($A_1$), $B \in N(\Omega)$ *satisfy* $B \preccurlyeq A_*$, *and* $u \in W^{1,A}(\Omega)$ *satisfy for any* $B_R \subset \Omega$, $R \leq R_0$, *any* $\sigma \in (0,1)$ *and any* $k \geq t_s =$ *a given constant* $> 0$,

$$\int_{\Omega_{k,\sigma R}} A(x,|\nabla u|)\,dx \leq c\int_{\Omega_{k,R}} A_*\left(x, \frac{u-k}{(1-\sigma)R}\right)dx + c\int_{\Omega_{k,R}} B(x,k)\,dx. \tag{3.23}$$

*Then* $u$ *is locally bounded above in* $\Omega$.

Proof. Fix $B_R \subset \Omega$ with $R < R_0$, $k \geq t_s$ and set

$$\rho_h = \frac{R}{2} + \frac{R}{2^{h+1}}, \qquad \overline{\rho}_h = \frac{\rho_h + \rho_{h+1}}{2},$$
$$k_h = k\bigg(1 - \frac{1}{2^{h+1}}\bigg), \qquad h = 0, 1, 2, \ldots$$

It is clear that $\rho_h \downarrow R/2$, $k_h \uparrow k$, $\rho_{h+1} < \overline{\rho}_h < \rho_h$. Define

$$J_h = \int_{\Omega_{k_h,\rho_h}} A_*(x, |u(x) - k_h|)\, dx,$$

and fix $\xi(t) \in C^1([0, +\infty))$, with $0 \leq \xi(t) \leq 1$, such that

$$\xi(t) = \begin{cases} 1 & \text{for } t \leq 1/2, \\ 0 & \text{for } t \geq 3/4, \end{cases}$$

and $\xi'(t) \leq c$. Denote

$$\xi_h = \xi\bigg(\frac{2^{h+1}}{R}\bigg(|x| - \frac{R}{2}\bigg)\bigg).$$

Then

$$\xi_h(x) = \begin{cases} 1 & \text{for } x \in B_{\rho_{h+1}}, \\ 0 & \text{for } x \notin B_{\overline{\rho}_h}. \end{cases}$$

Then we have

$$\begin{aligned} J_{h+1} &\leq \int_{\Omega_{k_{h+1},\overline{\rho}_h}} A_*(x, |u(x) - k_{h+1}|\xi_h)\, dx \\ &= \int_{B_R} A_*(x, (u(x) - k_{h+1})^+\xi_h)\, dx \\ &= \int_{B_R} A_*\bigg(x, \frac{(u(x) - k_{h+1})^+\xi_h}{|(u - k_{h+1})^+\xi_h|_{A_*;B_R}}\, |(u - k_{h+1})^+\xi_h|_{A_*;B_R}\bigg)\, dx. \end{aligned} \tag{3.24}$$

By (3.11), we have

$$A_*\bigg(x, \frac{(\mathfrak{A}(\alpha))^{1/n}}{\alpha}\, s\bigg) \leq \frac{A_*(x, s)}{\mathfrak{A}(\alpha)} \quad \text{for all } x \in \Omega,\ s \geq 0,\ \alpha \geq 0,$$

or equivalently,

$$A_*\bigg(x, \frac{\beta^{1/n}}{\mathfrak{A}^{-1}(\beta)} s\bigg) \leq \frac{A_*(x, s)}{\beta} \quad \text{for all } x \in \Omega,\ s \geq 0,\ \beta \geq 0.$$

Then by (3.24), the Sobolev inequality and Lemma 3.1, we can see that

$$
\begin{aligned}
(3.25)\qquad J_{h+1} &\le \mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,|(u-k_{h+1})^+\xi_h|_{A_*;B_R}\bigg)\\
&\quad\cdot\int_{B_R} A_*\bigg(x,\frac{\beta^{1/n}}{\mathfrak{A}^{-1}(\beta)}\,\frac{(u(x)-k_{h+1})^+\xi_h}{|(u-k_{h+1})^+\xi_h|_{A_*;B_R}}\bigg)\,dx\\
&\le \mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,|(u-k_{h+1})^+\xi_h|_{A_*;B_R}\bigg)\\
&\quad\cdot\frac{1}{\beta}\int_{B_R} A_*\bigg(x,\frac{(u(x)-k_{h+1})^+\xi_h}{|(u-k_{h+1})^+\xi_h|_{A_*;B_R}}\bigg)\,dx\\
&= \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,|(u-k_{h+1})^+\xi_h|_{A_*;B_R}\bigg)\\
&\le \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,c|\nabla((u-k_{h+1})^+\xi_h)|_{A;B_R}\bigg)\\
&\le \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,\big(c|\nabla u|_{A;\Omega_{k_{h+1},\overline{\rho}_h}}+c2^h|u-k_{h+1}|_{A;\Omega_{k_{h+1},\overline{\rho}_h}}\big)\bigg)\\
&= \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\bigg[c\mathfrak{A}^{-1}\bigg(\mathfrak{A}(|\nabla u|_{A;\Omega_{k_{h+1},\overline{\rho}_h}})\\
&\quad\cdot\int_{\Omega_{k_{h+1},\overline{\rho}_h}} A\bigg(x,\frac{|\nabla u(x)|}{|\nabla u|_{A;\Omega_{k_{h+1},\overline{\rho}_h}}}\bigg)\,dx\bigg)\\
&\quad+c\,2^h\,\mathfrak{A}^{-1}\bigg(\mathfrak{A}(|u-k_{h+1}|_{A;\Omega_{k_{h+1},\overline{\rho}_h}})\\
&\quad\cdot\int_{\Omega_{k_{h+1},\overline{\rho}_h}} A\bigg(x,\frac{|u(x)-k_{h+1}|}{|u-k_{h+1}|_{A;\Omega_{k_{h+1},\overline{\rho}_h}}}\bigg)\,dx\bigg)\bigg]\bigg)\\
&\le \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\bigg[c\,\mathfrak{A}^{-1}\bigg(\int_{\Omega_{k_{h+1},\overline{\rho}_h}} A(x,|\nabla u(x)|)\,dx\bigg)\\
&\quad+c\,2^h\,\mathfrak{A}^{-1}\bigg(\int_{\Omega_{k_{h+1},\overline{\rho}_h}} A(x,|u(x)-k_{h+1}|)\,dx\bigg)\bigg]\bigg)\\
&:= \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,[c\,\mathfrak{A}^{-1}(I)+c\,2^h\,\mathfrak{A}^{-1}(II)]\bigg),
\end{aligned}
$$

for any $\beta > 0$. By inequality (3.23), we have in the right-hand side of the above inequality that

$$
\begin{aligned}
(3.26)\qquad I &\le c\int_{\Omega_{k_{h+1},\rho_h}} A_*\bigg(x,\frac{2^{h+2}}{R}\,(u(x)-k_{h+1})\bigg)\,dx\\
&\quad+c\int_{\Omega_{k_{h+1},\rho_h}} B(x,k_{h+1})\,dx\\
&\le c\,\mathfrak{A}_*\bigg(\frac{1}{R}\bigg)\,\mathfrak{A}_*(2^{h+2})\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,u(x)-k_{h+1})\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ c\int_{\Omega_{k_{h+1},\rho_h}} B(x,k)\,dx \\
&\le c\,\mathfrak{A}_*(2^{h+2})J_h + c\int_{\Omega_{k_{h+1},\rho_h}} B(x,k)\,dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
J_h &\ge \int_{\Omega_{k_{h+1},\rho_h}} A_*(x,|u(x)-k_h|)\,dx \ge \int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k_{h+1}-k_h)\,dx \\
&\ge \widehat{\mathfrak{A}_*}\bigg(\frac{k_{h+1}-k_h}{k}\bigg)\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k)\,dx = \widehat{\mathfrak{A}_*}\bigg(\frac{1}{2^{h+2}}\bigg)\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k)\,dx.
\end{aligned}
$$

Then, by equation (3.2), we have

$$
\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k)\,dx \le \mathfrak{A}_*(2^{h+2})J_h \tag{3.27}
$$

and since $B \preccurlyeq A_*$, we obtain

$$
\int_{\Omega_{k_{h+1},\rho_h}} B(x,k)\,dx \le c\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k)\,dx \le c\,\mathfrak{A}_*(2^{h+2})J_h. \tag{3.28}
$$

Together with (3.26) and (3.28), we conclude that

$$
I \le c\,\mathfrak{A}_*(2^{h+2})J_h. \tag{3.29}
$$

On the other hand, by Theorem 2.14, we have $A \ll A_*$. Then

$$
\begin{aligned}
II &\le \int_{\Omega_{k_{h+1},\overline{\rho}_h}} A_*(x,|u(x)-k_{h+1}|)\,dx + \int_{\Omega_{k_{h+1},\overline{\rho}_h}} c\,dx \\
&\le J_h + c|\Omega_{k_{h+1},\overline{\rho}_h}|.
\end{aligned} \tag{3.30}
$$

By (3.27), we have

$$
\begin{aligned}
|\Omega_{k_{h+1},\overline{\rho}_h}| \le |\Omega_{k_{h+1},\rho_h}| &\le c\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,t_s)\,dx \\
&\le c\int_{\Omega_{k_{h+1},\rho_h}} A_*(x,k)\,dx \le \mathfrak{A}_*(2^{h+2})J_h,
\end{aligned}
$$

from which, together with (3.30), we conclude that

$$
II \le (1+\mathfrak{A}_*(2^{h+2}))J_h \le c\mathfrak{A}_*(2^{h+2})J_h. \tag{3.31}
$$

Combining (3.29), (3.31) and (3.25), we see that, for all $\beta > 0$,

$$
\begin{aligned}
J_{h+1} &\le \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,\big[c\,\mathfrak{A}^{-1}(c\,\mathfrak{A}_*(2^{h+2})J_h) + c\,2^h\mathfrak{A}^{-1}(c\,\mathfrak{A}_*(2^{h+2})J_h)\big]\bigg) \\
&\le \frac{1}{\beta}\,\mathfrak{A}_*\bigg(\frac{\mathfrak{A}^{-1}(\beta)}{\beta^{1/n}}\,c\,2^h\mathfrak{A}^{-1}(c\,\mathfrak{A}_*(2^{h+2})J_h)\bigg).
\end{aligned}
$$

Since we can choose $k$ big enough so that

$$J_0 = \int_{\Omega_{k/2,\rho_0}} A_*\left(x, \left|u(x) - \frac{k}{2}\right|\right) dx$$

is small enough, by Lemma 3.4, we conclude that $J_h \to 0$ as $h \to \infty$. Hence, $u(x) \le k$ for a almost every $x \in B_{R/2}$, which completes the proof. □

## 4. Local boundedness of the minimizer

In this section we consider the local boundedness of the minimizers for a class of functionals defined in $W^{1,A}(\Omega)$. Since we consider only local properties of minimizers, without loss of generality, we can assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$.

Consider the integral functionals as follows:

$$E(v) = E(v,\Omega) = \int_\Omega f(x, v(x), \nabla v(x))\, dx, \tag{4.1}$$

where $v \in W^{1,A}(\Omega)$ and $f(x,s,z)$ is a Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying

$$A\left(x, \sum_{i=1}^n |z_i|\right) - B(x,s) - c \le f(x,s,z) \le C\left(A\left(x, \sum_{i=1}^n |z_i|\right) + B(x,s) + c\right) \tag{4.2}$$

with $c$ and $C$ being non-negative constants; $A \in N(\Omega) \cap \mathscr{A}$ satisfying $(\mathrm{A}_1)$ (see Section 3); $\mathfrak{A}_*$ induced by $A_*$ satisfying that there exists a $T_0 \in \mathbb{R}$ with $T_0 \ge 1$ such that $\mathfrak{A}_*(T_0) = 1$; $N(\Omega) \ni B \preccurlyeq A_*$ satisfying the following $(\mathrm{B}_1)$–$(\mathrm{B}_2)$:

- $(\mathrm{B}_1)$ There exists a strictly increasing differentiable function $\mathfrak{B}\colon [0,+\infty) \to [0,+\infty)$ such that

$$B(x,\alpha t) \ge \mathfrak{B}(\alpha) B(x,t) \quad \text{for all } \alpha \ge 0,\ t \in \mathbb{R},\ x \in \Omega.$$

- $(\mathrm{B}_2)$ There exists a constant $T_{B,\Omega} > 0$ such that $B(x, T_{B,\Omega}) \ge 1$ for any $x \in \overline{\Omega}$.

We say that $u$ is a local minimizer for the integral functional $E$ if for any $\varphi \in W_0^{1,A}(\Omega)$,

$$E(u; \operatorname{supp}\varphi) \le E(u+\varphi; \operatorname{supp}\varphi). \tag{4.3}$$

Our main result is the following.

Theorem 4.1. *Let $f$ satisfy the growth condition* (4.2). *If $u \in W^{1,A}(\Omega)$ is a local minimizer for the functional* (4.1), *then $u \in L^\infty_{\mathrm{loc}}(\Omega)$.*

The proof of the above theorem is a direct consequence of the following lemma.

Lemma 4.2. *Let $f$ satisfy* (4.2). *If $u \in W^{1,A}(\Omega)$ is a local minimizer for the functional* (4.3), *then for any ball $B_R \subset \Omega$ ($R \leq 1$), any $k \geq T_{B,\Omega}$, and any $0 < \sigma < 1$, we have*

$$\int_{\Omega_{k,\sigma R}} A(x, |\nabla u|)\, dx \leq c \int_{\Omega_{k,R}} A_*\bigg(x, \frac{u-k}{(1-\sigma)R}\bigg)\, dx + c \int_{\Omega_{k,R}} B(x,k)\, dx,$$

*where $c$ is a constant independent of $k, R, \sigma$ and $u$.*

Proof. Fix $k \geq T_{B,\Omega}$, $B_R$ for $R \leq 1$ and $0 < \sigma < 1$. For any $s$ and $t$ satisfying $\sigma R < s < t < R$, let us denote by $\eta$ a $C^1$-function such that $\eta \equiv 1$ on $B_s$, $0 \leq \eta \leq 1$, $\operatorname{supp} \eta \subset B_t$, $|D\eta| \leq 2/(t-s)$. Taking $\varphi = -\eta \max\{u-k, 0\}$, from (4.3), we have

$$\int_{\Omega_{k,t}} f(x, u, \nabla u)\, dx \leq \int_{\Omega_{k,t}} f(x, u+\varphi, \nabla u + \nabla \varphi)\, dx,$$

which implies by Lemma 3.1 (a) that

$$\begin{aligned}
&\int_{\Omega_{k,t}} A(x, |\nabla u|)\, dx - \int_{\Omega_{k,t}} B(x, |u|)\, dx - \int_{\Omega_{k,t}} c\, dx \\
&\quad\leq C \bigg[ \int_{\Omega_{k,t}} A(x, |(1-\eta)\nabla u - (u-k)\nabla \eta|)\, dx \\
&\qquad + \int_{\Omega_{k,t}} B(x, |u - \eta(u-k)|)\, dx + \int_{\Omega_{k,t}} c\, dx \bigg] \\
&\quad\leq C \max\{\widehat{\mathfrak{A}}(2), \widehat{\mathfrak{B}}(2)\} \bigg[ \int_{\Omega_{k,t}} A(x, |\nabla u|(1-\eta))\, dx \\
&\qquad + \int_{\Omega_{k,t}} A(x, |\nabla \eta|(u-k))\, dx \\
&\qquad + \int_{\Omega_{k,t}} B(x, |u|)\, dx + \int_{\Omega_{k,t}} B(x, |\eta(u-k)|)\, dx + \int_{\Omega_{k,t}} c\, dx \bigg].
\end{aligned}$$

By $k \geq T_{B,\Omega}$ and $0 < t - s < R \leq R_0 = 1$, from the above inequality and Lemma 3.1 we have

$$\begin{aligned}
\int_{\Omega_{k,t}} A(x, |\nabla u|)\, dx &\leq C_1 \max\{\widehat{\mathfrak{A}}(2), \widehat{\mathfrak{B}}(2)\} \bigg[ \int_{\Omega_{k,t}} A(x, |\nabla u|(1-\eta))\, dx \\
&\quad + \int_{\Omega_{k,t}} A(x, |\nabla \eta|(u-k))\, dx + \int_{\Omega_{k,t}} B(x, |u|)\, dx \\
&\quad + \int_{\Omega_{k,t}} B(x, |\eta|(u-k))\, dx + \int_{\Omega_{k,t}} 1\, dx \bigg] \\
&\leq C_2 \max\{\widehat{\mathfrak{A}}(2), \widehat{\mathfrak{B}}(2)\}\, \widehat{\mathfrak{A}}(2) \bigg[ \int_{\Omega_{k,t} \setminus \Omega_{k,s}} A(x, |\nabla u|)\, dx \\
&\quad + \int_{\Omega_{k,t}} A\bigg(x, \frac{u-k}{t-s}\bigg)\, dx + \int_{\Omega_{k,t}} B(x, (u-k)+k)\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega_{k,t}} B(x,u-k)\,dx+\int_{\Omega_{k,t}} 1\,dx\biggr]\\
&\le C_3\max\{\widehat{\mathfrak{A}}(2),\widehat{\mathfrak{B}}(2)\}\,\widehat{\mathfrak{A}}(2)\,\widehat{\mathfrak{B}}(2)\biggl[\int_{\Omega_{k,t}\setminus\Omega_{k,s}} A(x,|\nabla u|)\,dx\\
&+\int_{\Omega_{k,t}} A_*\Bigl(x,\frac{u-k}{t-s}\Bigr)\,dx+\int_{\Omega_{k,t}} B(x,u-k)\,dx\\
&+\int_{\Omega_{k,t}} B(x,k)\,dx+\int_{\Omega_{k,t}} 1\,dx\biggr]\\
&\le c\int_{\Omega_{k,t}\setminus\Omega_{k,s}} A(x,|\nabla u|)\,dx\\
&+c\int_{\Omega_{k,t}} A_*\Bigl(x,\frac{u-k}{t-s}\Bigr)\,dx+c\int_{\Omega_{k,t}} B(x,k)\,dx,
\end{aligned}
$$

where $c=c(A,B)$ is a positive constant. Adding $c\int_{\Omega_{k,s}} A(x,|\nabla u|)$ to both sides of the above inequality, we have

$$
\begin{aligned}
\int_{\Omega_{k,s}} A(x,|\nabla u|)\,dx &\le \theta\int_{\Omega_{k,t}} A(x,|\nabla u|)\,dx\\
&\quad+c\int_{\Omega_{k,R}} A_*\Bigl(x,\frac{u-k}{t-s}\Bigr)\,dx+c\int_{\Omega_{k,R}} B(x,k)\,dx,
\end{aligned}
$$

where $\theta=c/(c+1)<1$ is a constant depending on the $N(\Omega)$-functions $A$ and $B$. By Lemma 3.5, we conclude that

$$
\int_{\Omega_{k,\sigma R}} A(x,|\nabla u|)\,dx\le C\biggl[\int_{\Omega_{k,R}} A_*\Bigl(x,\frac{u-k}{(1-\sigma)R}\Bigr)\,dx+c\int_{\Omega_{k,R}} B(x,k)\,dx\biggr],
$$

where $C=C(A,B)$ is a constant. □

Proof of Theorem 4.1. By Lemmas 4.2 and 3.6 (taking $R_0=1$), the local minimizer $u$ is locally bounded above in $\Omega$. And similarly $-u$ is also locally bounded above in $\Omega$. Then the conclusion of Theorem 4.1 follows. □

## 5. Local boundedness of weak solutions to a kind of fully nonlinear elliptic equations

In this section, we consider the local bounded regularity of weak solutions of a kind of fully nonlinear elliptic equation. Since we consider only local properties of the weak solutions, without loss of generality, we can suppose that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$.

Consider the second order fully nonlinear elliptic equation as follows:

$$
\operatorname{div} L(x,u,\nabla u)+F(x,u,\nabla u)=0\quad\text{for all } x\in\Omega, \tag{5.1}
$$

where $L\colon\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, $F\colon\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^1$, $u\colon\Omega\to\mathbb{R}$.

The following assumptions on $A,B\in N(\Omega)$ will be used:

($\mathrm{A}_1^+$) There exists a strictly increasing differentiable function $\mathfrak{A}\colon [0,+\infty) \to [0,+\infty)$ satisfying

$$n\,\mathfrak{A}(\alpha) > \alpha\,\mathfrak{A}'(\alpha) > \mathfrak{A}(\alpha) \tag{5.2}$$

such that

($\mathrm{A}_{11}$) $A(x,\alpha t) \geq \mathfrak{A}(\alpha)A(x,t)$ for all $\alpha \geq 0$, $t \in \mathbb{R}$, $x \in \Omega$;

($\mathrm{A}_{12}$) $\mathfrak{A}, \mathfrak{A}^{-1}, \mathfrak{A}_* \in \Delta_{\mathbb{R}^+}$.

($\mathrm{B}_1^+$) There exists a strictly increasing differentiable function $\mathfrak{B}\colon [0,+\infty) \to [0,+\infty)$ satisfying

$$\alpha\,\mathfrak{B}'(\alpha) > \mathfrak{B}(\alpha) \tag{5.3}$$

such that $B(x,\alpha t) \geq \mathfrak{B}(\alpha)B(x,t)$ for all $\alpha \geq 0$, $t \in \mathbb{R}$, $x \in \Omega$.

($\mathrm{B}_2$) There exists a constant $T_{B,\Omega} > 0$ such that $B(x, T_{B,\Omega}) \geq 1$ for any $x \in \overline{\Omega}$.

Suppose equation (5.1) satisfies the following growth conditions:

$$L(x,u,z)z \geq a_0 A(x,|z|) - bB(x,u) - c, \tag{5.4}$$

$$|L(x,u,z)| \leq a_1 \widetilde{A}^{-1}A(x,|z|) + b\widetilde{A}^{-1}B(x,u) + c, \tag{5.5}$$

$$|F(x,u,z)| \leq a_2 \widetilde{B}^{-1}A(x,|z|) + b\widetilde{B}^{-1}B(x,u) + c, \tag{5.6}$$

where $a_0, a_1, a_2, b, c$ are positive constants, $A \in N(\Omega) \cap \mathscr{A}$ satisfies ($\mathrm{A}_1^+$), and $N(\Omega) \ni B \preccurlyeq A_*$ satisfies ($\mathrm{B}_1^+$) and ($\mathrm{B}_2$).

Definition 5.1. $u \in W^{1,A}(\Omega)$ is said to be a weak solution of (5.1) if

$$\int_\Omega L(x,u,\nabla u)\nabla v\,dx - \int_\Omega F(x,u,\nabla u)v\,dx = 0 \tag{5.7}$$

for any $v \in W_0^{1,A}(\Omega)$.

Theorem 5.2. *Let the growth conditions* (5.4)–(5.6) *hold. If $u \in W^{1,A}(\Omega)$ is a weak solution of* (5.1), *then $u \in L^\infty_{\mathrm{loc}}(\Omega)$.*

Lemma 5.3. *Let equations* (5.4)–(5.6) *hold. If $u \in W^{1,A}(\Omega)$ is a weak solution of* (5.1), *then for any ball $B_R \subset \Omega$ ($R \leq 1$), any $k \geq T_{B,\Omega}$, and any $0 < \sigma < 1$, we have*

$$\int_{\Omega_{k,\sigma R}} A(x,|\nabla u|)\,dx \leq c\int_{\Omega_{k,R}} A_*\bigg(x, \frac{u-k}{(1-\sigma)R}\bigg)\,dx + c\int_{\Omega_{k,R}} B(x,k)\,dx,$$

*where $c$ is a constant independent of $k, R, \sigma$ and $u$.*

Proof. Let $u$ be a weak solution of (5.1). For arbitrary balls $\overline{B}_s(\widetilde{x}) \subset B_t(\widetilde{x}) \subset \Omega$, let $\xi$ be a $C^\infty$-function such that

$$0 \leq \xi \leq 1, \qquad \operatorname{supp}\xi \subset B_t, \qquad \xi = 1 \quad \text{on } B_s, \qquad |\nabla\xi| \leq \frac{2}{t-s}.$$

For $k \geq T_{B,\Omega}$ set $v = \mathfrak{A}(\xi)\max\{u-k,0\} \in W_0^{1,A}(\Omega)$. Then by (5.7), we obtain

$$\int_{\Omega_{k,t}} \mathfrak{A}(\xi)L(x,u,\nabla u)\cdot\nabla u\,dx + \int_{\Omega_{k,t}} (u-k)L(x,u,\nabla u)\cdot\nabla\mathfrak{A}(\xi)\,dx - \int_{\Omega_{k,t}} \mathfrak{A}(\xi)(u-k)F(x,u,\nabla u)\,dx = 0. \tag{5.8}$$

From (5.5), (5.6) and (5.8) it follows that

$$\begin{aligned} a_0\int_{\Omega_{k,t}} A(x,|\nabla u|)\mathfrak{A}(\xi)\,dx &\leq b\int_{\Omega_{k,t}} B(x,|u|)\mathfrak{A}(\xi)\,dx + c\int_{\Omega_{k,t}} \mathfrak{A}(\xi)\,dx \\ &+ a_1\int_{\Omega_{k,t}} \widetilde{A}^{-1}A(x,|\nabla u|)|\nabla\mathfrak{A}(\xi)|(u-k)\,dx \\ &+ b\int_{\Omega_{k,t}} \widetilde{A}^{-1}B(x,|u|)|\nabla\mathfrak{A}(\xi)|(u-k)\,dx \\ &+ c\int_{\Omega_{k,t}} |\nabla\mathfrak{A}(\xi)|(u-k)\,dx \\ &+ a_2\int_{\Omega_{k,t}} \widetilde{B}^{-1}A(x,|\nabla u|)\mathfrak{A}(\xi)(u-k)\,dx \\ &+ b\int_{\Omega_{k,t}} \widetilde{B}^{-1}B(x,|u|)\mathfrak{A}(\xi)(u-k)\,dx + c\int_{\Omega_{k,t}} \mathfrak{A}(\xi)(u-k)\,dx. \end{aligned} \tag{5.9}$$

Now let us estimate each term of the right-hand side of (5.9). As $t-s<R\leq 1$ we obtain

$$\begin{aligned} \int_{\Omega_{k,t}} B(x,|u|)\mathfrak{A}(\xi)\,dx &\leq \mathfrak{A}(1)\int_{\Omega_{k,t}} B(x,(u-k)+k)\,dx \\ &\leq \mathfrak{A}(1)\int_{\Omega_{k,t}} B(x,2\max\{u-k,k\})\,dx \\ &\leq \mathfrak{A}(1)\widehat{\mathfrak{B}}(2)\int_{\Omega_{k,t}} B(x,\max\{u-k,k\})\,dx \\ &\leq c\int_{\Omega_{k,t}} B(x,u-k)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx \\ &\leq c\int_{\Omega_{k,t}} A_*(x,u-k)\,dx + c|\Omega_{k,t}| + c\int_{\Omega_{k,t}} B(x,k)\,dx \\ &\leq c\int_{\Omega_{k,t}} A_*\left(x,\frac{u-k}{t-s}\right)dx + c\int_{\Omega_{k,t}} B(x,k)\,dx. \end{aligned} \tag{5.10}$$

Obviously,

$$\int_{\Omega_{k,t}} \mathfrak{A}(\xi)\,dx \leq \mathfrak{A}(1)\int_{\Omega_{k,t}} B(x,k)\,dx. \tag{5.11}$$

By the Young inequality, and taking $\varepsilon > 0$ such that $a_1\widetilde{\mathfrak{A}}(n\varepsilon) = a_0/4$, we deduce from the assumption $n\mathfrak{A}(\alpha) > \alpha\mathfrak{A}'(\alpha) > \mathfrak{A}(\alpha)$, Lemma 3.1 (c) and (3.12) that

$$
\begin{aligned}
(5.12)\quad a_1 \int_{\Omega_{k,t}} & \widetilde{A}^{-1}A(x,|\nabla u|)|\nabla\mathfrak{A}(\xi)|(u-k)\,dx \\
&= a_1 \int_{\Omega_{k,t}} \widetilde{A}^{-1}A(x,|\nabla u|)\mathfrak{A}'(\xi)|\nabla\xi|(u-k)\,dx \\
&\le a_1 \int_{\Omega_{k,t}} \widetilde{A}\big(x,\varepsilon\,\mathfrak{A}'(\xi)\cdot\widetilde{A}^{-1}(A(x,|\nabla u|))\big)\,dx \\
&\quad + a_1 \int_{\Omega_{k,t}} A(x,\varepsilon^{-1}|\nabla\xi|(u-k))\,dx \\
&\le a_1 \int_{\Omega_{k,t}} \widetilde{A}\bigg(x,\varepsilon n\,\frac{\mathfrak{A}(\xi)}{\xi}\cdot\widetilde{A}^{-1}(A(x,|\nabla u|))\bigg)\,dx \\
&\quad + a_1 \int_{\Omega_{k,t}} A(x,\varepsilon^{-1}|\nabla\xi|(u-k))\,dx \\
&\le a_1\,\widetilde{\mathfrak{A}}(n\varepsilon) \int_{\Omega_{k,t}} \widetilde{A}\bigg(x,\frac{\mathfrak{A}(\xi)}{\xi}\cdot\widetilde{A}^{-1}(A(x,|\nabla u|))\bigg)\,dx \\
&\quad + a_1\,\widehat{\mathfrak{A}}\Big(\frac{2}{\varepsilon}\Big) \int_{\Omega_{k,t}} A\Big(x,\frac{u-k}{t-s}\Big)\,dx \\
&\le \frac{a_0}{4} \int_{\Omega_{k,t}} \mathfrak{A}(\xi)A(x,|\nabla u|)\,dx + c\int_{\Omega_{k,t}} A_*\Big(x,\frac{u-k}{t-s}\Big)\,dx + c|\Omega_{k,t}| \\
&\le \frac{a_0}{4} \int_{\Omega_{k,t}} \mathfrak{A}(\xi)A(x,|\nabla u|)\,dx \\
&\quad + c\int_{\Omega_{k,t}} A_*\Big(x,\frac{u-k}{t-s}\Big)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx.
\end{aligned}
$$

By the Young inequality, taking $\varepsilon > 0$ such that $a_2\,\widetilde{\mathfrak{B}}(\varepsilon) = a_0/4$ we obtain from the assumption $\alpha\,\mathfrak{B}'(\alpha) > \mathfrak{B}(\alpha)$ and Lemma 3.1 (c) that

$$
\begin{aligned}
(5.13)\quad a_2 \int_{\Omega_{k,t}} & \widetilde{B}^{-1}A(x,|\nabla u|)\mathfrak{A}(\xi)(u-k)\,dx \\
&\le a_2 \int_{\Omega_{k,t}} \mathfrak{A}(\xi)\widetilde{B}\big(x,\varepsilon\widetilde{B}^{-1}A(x,|\nabla u|)\big)\,dx \\
&\quad + a_2 \int_{\Omega_{k,t}} \mathfrak{A}(\xi)B(x,\varepsilon^{-1}(u-k))\,dx \\
&\le a_2\,\widetilde{\mathfrak{B}}(\varepsilon) \int_{\Omega_{k,t}} \mathfrak{A}(\xi)A(x,|\nabla u|)\,dx + a_2\,\mathfrak{A}(1)\widehat{\mathfrak{B}}\Big(\frac{1}{\varepsilon}\Big) \int_{\Omega_{k,t}} B(x,u-k)\,dx \\
&\le \frac{a_0}{4} \int_{\Omega_{k,t}} \mathfrak{A}(\xi)A(x,|\nabla u|)\,dx + c\int_{\Omega_{k,t}} A_*(x,u-k)\,dx + c|\Omega_{k,t}|
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{a_0}{4} \int_{\Omega_{k,t}} \mathfrak{A}(\xi) A(x, |\nabla u|)\, dx \\
&\quad + c \int_{\Omega_{k,t}} A_*\Big(x, \frac{u-k}{t-s}\Big)\, dx + c \int_{\Omega_{k,t}} B(x,k)\, dx.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\Omega_{k,t}} \widetilde{A}^{-1} B(x, |u|) |\nabla \mathfrak{A}(\xi)| (u-k)\, dx \\
&\quad = \int_{\Omega_{k,t}} \widetilde{A}^{-1} B(x, |u|) \mathfrak{A}'(\xi) |\nabla \xi| (u-k)\, dx \\
&\quad \le \int_{\Omega_{k,t}} n\, \frac{\mathfrak{A}(\xi)}{\xi}\, \widetilde{A}^{-1} B(x, |u|) |\nabla \xi| (u-k)\, dx \\
&\quad \le \int_{\Omega_{k,t}} \widetilde{A}\Big(x, n\, \frac{\mathfrak{A}(\xi)}{\xi}\, \widetilde{A}^{-1} B(x, |u|)\Big)\, dx + \int_{\Omega_{k,t}} A(x, |\nabla \xi| (u-k))\, dx \\
&\quad \le \widetilde{\mathfrak{A}}(n) \int_{\Omega_{k,t}} \mathfrak{A}(\xi) \widetilde{A}\big(x, \widetilde{A}^{-1} B(x, |u|)\big)\, dx + \int_{\Omega_{k,t}} A(x, |\nabla \xi| (u-k))\, dx \\
&\quad \le \widetilde{\mathfrak{A}}(n)\, \mathfrak{A}(1) \int_{\Omega_{k,t}} B(x, (u-k)+k)\, dx + \int_{\Omega_{k,t}} A(x, |\nabla \xi| (u-k))\, dx \\
&\quad \le c \int_{\Omega_{k,t}} B(x, u-k)\, dx + c \int_{\Omega_{k,t}} B(x,k)\, dx \\
&\qquad + \mathfrak{A}(2) \int_{\Omega_{k,t}} A\Big(x, \frac{u-k}{t-s}\Big)\, dx \\
&\quad \le c \int_{\Omega_{k,t}} A_*\Big(x, \frac{u-k}{t-s}\Big)\, dx + c \int_{\Omega_{k,t}} B(x,k)\, dx + c|\Omega_{k,t}| \\
&\quad \le c \int_{\Omega_{k,t}} A_*\Big(x, \frac{u-k}{t-s}\Big)\, dx + c \int_{\Omega_{k,t}} B(x,k)\, dx,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
&\int_{\Omega_{k,t}} |\nabla \mathfrak{A}(\xi)| (u-k)\, dx \le \int_{\Omega_{k,t}} n\, \frac{\mathfrak{A}(\xi)}{\xi}\, |\nabla \xi| (u-k)\, dx \\
&\quad \le \mathfrak{A}(2n) \int_{\Omega_{k,t}} A\Big(x, \frac{u-k}{t-s}\Big)\, dx + \int_{\Omega_{k,t}} \widetilde{A}\Big(x, \frac{\mathfrak{A}(\xi)}{\xi}\Big)\, dx \\
&\quad \le c \int_{\Omega_{k,t}} A\Big(x, \frac{u-k}{t-s}\Big)\, dx + \int_{\Omega_{k,t}} \mathfrak{A}(\xi) \widetilde{A}(x,1)\, dx \\
&\quad \le c \int_{\Omega_{k,t}} A_*\Big(x, \frac{u-k}{t-s}\Big)\, dx + c|\Omega_{k,t}| \\
&\quad \le c \int_{\Omega_{k,t}} A_*\Big(x, \frac{u-k}{t-s}\Big)\, dx + c \int_{\Omega_{k,t}} B(x,k)\, dx,
\end{aligned} \tag{5.15}$$

$$\int_{\Omega_{k,t}} \widetilde{B}^{-1} B(x, |u|) \mathfrak{A}(\xi) (u-k)\, dx \tag{5.16}$$

$$
\begin{aligned}
&\le \int_{\Omega_{k,t}} B(x,(u-k)+k)\,dx + \int_{\Omega_{k,t}} B(x,\mathfrak{A}(\xi)(u-k))\,dx \\
&\le c\int_{\Omega_{k,t}} B(x,u-k)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx + c|\Omega_{k,t}| \\
&\le c\int_{\Omega_{k,t}} A_*\bigg(x,\frac{u-k}{t-s}\bigg)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Omega_{k,t}} \mathfrak{A}(\xi)(u-k)\,dx &\le \int_{\Omega_{k,t}} B(x,u-k)\,dx + \int_{\Omega_{k,t}} \widetilde{B}(x,\mathfrak{A}(\xi))\,dx \\
&\le c\int_{\Omega_{k,t}} A_*\bigg(x,\frac{u-k}{t-s}\bigg)\,dx + c|\Omega_{k,t}| \\
&\le c\int_{\Omega_{k,t}} A_*\bigg(x,\frac{u-k}{t-s}\bigg)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx.
\end{aligned} \tag{5.17}
$$

From (5.9)–(5.17), we conclude that

$$
\begin{aligned}
\mathfrak{A}(1)\int_{\Omega_{k,s}} A(x,|\nabla u|)\,dx &\le \int_{\Omega_{k,t}} A(x,|\nabla u|)\mathfrak{A}(\xi)\,dx \\
&\le c\int_{\Omega_{k,t}} A_*\bigg(x,\frac{u-k}{t-s}\bigg)\,dx + c\int_{\Omega_{k,t}} B(x,k)\,dx,
\end{aligned}
$$

which implies the conclusion of the lemma. □

PROOF OF THEOREM 5.2. By Lemmas 5.3 and 3.6 (taking $R_0 = 1$), the local minimizer $u$ is locally bounded above in $\Omega$. And similarly $-u$ is also locally bounded above in $\Omega$. Then the conclusion of Theorem 5.2 follows. □

## 6. Examples

In this section, we give two examples of the function $A$ which satisfy the conditions in our theorems. And we claim that not only variable exponent Sobolev spaces satisfy the conditions in Theorems 4.1 and 5.2 (see Example 6.1), but also some more complex space also satisfies conditions of these theorems (see Example 6.2).

EXAMPLE 6.1. Let $p \in C^{1-0}(\overline{\Omega})$ and $1 < q \le p(x) \le p_+ := \sup_{x\in\overline{\Omega}} p(x) < n$ ($q \in \mathbb{R}$) for $x \in \overline{\Omega}$. Define $A\colon \overline{\Omega}\times\mathbb{R} \to [0,+\infty)$ and $B\colon \overline{\Omega}\times\mathbb{R} \to [0,+\infty)$ by

$$
A(x,t) = |t|^{p(x)}, \quad B(x,t) = |t|^{p^*(x)}, \quad \text{where } p^*(x) = \frac{np(x)}{n-p(x)}.
$$

It is readily checked that $A$ satisfies $(\mathrm{P}_1)$–$(\mathrm{P}_3)$. It is easy to see that $p \in C^{1-0}(\overline{\Omega})$ implies $A \in C^{1-0}(\overline{\Omega})$ and, for $s > 0$,

$$
A_*^{-1}(x,s) = \frac{np(x)}{n-p(x)}\, s^{(n-p(x))/(np(x))}. \tag{6.1}
$$

Then $T(x) = +\infty$, and $(\widetilde{\mathrm{P}_4})$ is satisfied. In addition, for $x \in \Omega$,

$$\nabla_x A(x,t) = |t|^{p(x)} \ln|t| \nabla p(x).$$

Since for any $\varepsilon > 0$, $\ln t/t^{\varepsilon} \to 0$ as $t \to +\infty$, we conclude that there exist constants $\delta_1 < 1/n$, $c_1$ and $t_1$ such that

$$\left|\frac{\partial A(x,t)}{\partial x_j}\right| \leq c_1 A^{1+\delta_1}(x,t),$$

for all $x \in \Omega$ and $t \geq t_1$. Combining $A \in \Delta_2(\Omega)$, from Proposition 3.1 in [13], it is easy to see that condition $(\mathrm{P}_5)$ is satisfied. All growth conditions $(\mathrm{A}_1)$, $(\mathrm{B}_1)$, $(\mathrm{B}_2)$, $(\mathrm{A}_1^+)$, $(\mathrm{B}_1^+)$ and $(\mathrm{B}_2^+)$ are easy to verify. Thus conditions in Theorems 4.1 and 5.2 are verified.

We claim that this example contains and extends in part the conclusion of Theorems 3.1 and 4.1 in [15] and some of its corollaries.

EXAMPLE 6.2. Let $p \in C^{1-0}(\overline{\Omega})$ satisfy $1 < p^- \leq p(x) \leq p_+ := \sup_{x\in\overline{\Omega}} p(x) < n-1$. Define $A\colon \overline{\Omega} \times \mathbb{R} \to [0,+\infty)$ and $B\colon \overline{\Omega} \times \mathbb{R} \to [0,+\infty)$ by

$$A(x,t) = |t|^{p(x)} \log(1+|t|), \qquad B(x,t) = |t|^{p^*(x)-\delta},$$

for some $\delta > 0$, where $p^*(x) = np(x)/(n-p(x))$. It is obvious that $A$ satisfies $(\mathrm{P}_1)$–$(\mathrm{P}_3)$. Pick $\varepsilon > 0$ small enough such that $p^+ + \varepsilon < n$. Then, for $t > 0$ big enough, $A(x,t) \leq ct^{p^+ + \varepsilon}$, which implies that $T(x) = +\infty$ for all $x \in \overline{\Omega}$. Then $(\widetilde{\mathrm{P}_4})$ is satisfied. Since $p \in C^{1-0}(\overline{\Omega})$ and $A \in C^{1-0}(\overline{\Omega} \times \mathbb{R})$, by Proposition 3.1 in [13], $A_* \in C^{1-0}(\overline{\Omega} \times \mathbb{R})$. Combining $A \in \Delta_2(\Omega)$, it is easy to see that condition $(\mathrm{P}_5)$ is satisfied. All growth conditions $(\mathrm{A}_1)$, $(\mathrm{B}_1)$, $(\mathrm{B}_2)$, $(\mathrm{A}_1^+)$, $(\mathrm{B}_1^+)$ and $(\mathrm{B}_2^+)$ are easy to verify. By now conditions in Theorems 4.1 and 5.2 are verified.

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