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LINEAR QUADRATIC GAME OF EXPLOITATION OF COMMON RENEWABLE RESOURCES WITH INHERENT CONSTRAINTS

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Abstract. In this paper, we analyse a linear quadratic multistage game of extraction of a common renewable resource — a fishery — by many players with inherent state dependent constraints for exploitation and an infinite time horizon. To the best of our knowledge, such games have never been studied. We analyse the social optimum and Nash equilibrium for the feedback information structure and compare the results obtained in both cases. For the Nash equilibria, we obtain a value function that is contrary to intuitions from standard linear quadratic games. In our game, we face a situation in which the social optimum results in sustainability, while the Nash equilibrium leads to the depletion of the fishery in a finite time for realistic levels of the initial biomass of fish. Therefore, we also study an introduction of a tax in order to enforce socially optimal behaviour of the players. Besides, this game constitutes a counterexample to simplifications of techniques often used in computation of Nash equilibria and/or optimal control problems.

1. Introduction

Exploitation of common or interdependent renewable resources without an exclusive owner is a very important problem of contemporaneity.

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The most obvious examples of such problems are marine or deep lake fisheries. Those fisheries may be either open access (e.g. high seas fisheries) or formally divided (in the case of marine fisheries, e.g. Exclusive Economic Zones (EEZ) with well-defined owners and borders of each EEZ and exclusive rights of fishing in them). Even in the latter case, exclusiveness is only apparent, since fish does not observe official borders and they can freely migrate between various EEZ.

From the mathematical point of view, the only tool to deal with the whole spectrum of phenomena arising in this kind of problems, in which we have at least two independent decision makers governing the interdependent resources, are dynamic games, since both dynamic optimization methods and static games encompass only fractions of aspects of those problems.

Finding a Nash equilibrium, the basic concept of noncooperative game theory, in dynamic games, requires solving a set of dynamic optimization problems coupled by finding a fixed point of the resulting best response correspondence in some functional space of profiles of strategies. Due to this coupling, the problem becomes much more complicated than analogous dynamic optimization problems. Since the seminal book of Isaacs [18], the number of types of games that have been solved is very restricted. The results that can be regarded as close to complete concern zero sum differential games and linear quadratic differential games.

However, in this paper, we show that even existing results and methods for linear quadratic dynamic games are not sufficient if we try to model real life phenomena like exploitation of common renewable resources.

The aim of this paper is two-fold: applied and theoretical.

The applied part concerns analysis of the problem of extraction of a common renewable resource — a fishery.

We want to address the issue of "the tragedy of the commons", the phenomenon first indicated and named by Hardin [16]. In the context of common fisheries, it may even lead to extinction of whole species at Nash equilibria, while socially optimal management results in sustainability.

Game theoretic models of renewable resource extraction, starting from Levhari and Mirman's seminal paper on Fish Wars [22], have a vast literature (see, e.g. surveys of Long [23], [24] and monographs series, e.g. Carraro and Filar [6]).

Here, we present a discrete time, infinite horizon dynamic game model of extraction of common renewable resource — marine or deep lake fishery — with many players — countries or firms — which sell their catch at a common market. The problem belongs to a class of linear-quadratic dynamic games.

As we have mentioned above, Nash equilibrium problems in dynamic games are very compound, compared both to analogous dynamic optimization problems and standard static Nash equilibrium problems. The class of tractable dynamic games, i.e. games for which Nash equilibria can be calculated, is very small (see e.g. Jørgensen, Zaccour [20]). This limited class further shrinks if we consider state-dependent strategies of players: feedback or closed loop.

Linear quadratic (LQ) dynamic games seem to be the best researched class of dynamic games. Formulae for Nash equilibria for this class of games in their simplest form are now a standard textbook material in dynamic games (see e.g. Haurie, Krawczyk and Zaccour [17], Başar, Olsder [3] or Dockner et al. [9], Başar et al. [2] and there are in-depth monographs on this subject (like Engwerda [11]).

However, most of the papers on this subject are continuous time, differential games (e.g. Engwerda [10]), while in the discrete time they concern mainly Stackelberg equilibria (e.g. Abou-Kandil [1] and Chen, Zadrozny [7]).

In the context of discrete time linear quadratic games, Nash equilibria are examined in Jank, Abou-Kandil [19] in open loop strategies only and in Hämäläinen [15], whose analysis contains the feedback case but only with finite time horizon. This class of games was also mentioned by de Zeeuw, van der Ploeg [8] and considered in Kydland [21] with a finite time horizon.

In this paper, the quadratic form of payoff is slightly more general than the canonical form, however games with payoffs extended in this way have already been used, mainly in economic applications.

The main innovation of this paper from a theoretical point of view, is the introduction of state dependent constraints on players' decisions, simple and inherent in the context of natural resources exploitation. This small modification results in behaviour of feedback Nash equilibrium strategies and value functions contrary to expectations of readers familiar with the literature on linear quadratic dynamic games. Note that a constraint "you cannot produce negative amount of goods" has already been considered in continuous time dynamic games (see e.g. Fershtman, Kamien [12] and Wiszniewska-Matyszkiel, Bodnar, Mirota [32]) and nonstandard value function appeared as a consequence. However, in many other papers in which such a constraint is not fulfilled for some initial conditions, it is omitted, which leads to results which are only partially true. Nevertheless, in all those cases, the value function consists of at most two parts described by analytic formulae, which is much simpler than the form which we obtain in this paper for the discrete time as a result of imposing an inherent upper bound.

Some study of linear quadratic dynamic games with discrete time and linear constraints appears in Reddy and Zaccour [25]. However, again, only a finite time horizon and only the open loop set of strategies are examined, which leads to substantially different conclusions and methods of calculations. As it is well known, open loop (with strategies being functions of time only) Nash equilibria, usually not equivalent to feedback Nash equilibria, are not strongly time consistent, therefore, using feedback strategies leads to more realistic results.

Explicitly written state dependent constraints have already appeared in the context of common resource extraction in Levhari and Mirman [22], Fischer and Mirman [13] and Wiszniewska-Matyszkiel [27], [28] with logarithmic current payoff functions.

To the best of the authors' knowledge, state dependent constraints which can be active at equilibrium have never appeared in discrete time linear-quadratic infinite horizon games modelling ecological problems.

This innovation is important, since, if we want to concentrate on applications, especially applications to interdependent renewable resource extraction, we have to deal with problems which, to the best of the authors' knowledge, have not been solved before. Constraints of the form "you can extract non-negative amount of the resource but not more than the share of it which is in your zone", inherent in common or interdependent renewable resource extraction, change the form of the value function and the equilibrium substantially, as we show in this paper.

To complete the applied part of analysis, we study the results of a decomposition of the decision making structure of the same mass of consumers of the resource and we introduce a tax enforcing social optimality in the case of maximal level of this decomposition.

The theoretical contribution of the paper is extending the knowledge about LQ dynamic games in the case when constraints on decisions are imposed.

A crucial side effect of the research presented in this paper is the fact that the problems considered constitute a counterexample to a simplification of the methodology which is regarded as correct and widely used in dynamic games and dynamic optimization problems.

2. Formulation of the problem

We consider a dynamic game of exploitation of one common renewable resource — a fishery — with many players: either n players or a continuum of players (i.e. the set of players, denoted by $\mathbb{I}$, is either $\{1, \ldots, n\}$ or $[0,1]$) with discrete time and infinite horizon.

The renewable resource that we consider is one species of fish uniformly dispersed in a marine or deep lake fishery equally partitioned between the players. What is specific to this kind of resource is that, although not common from legal point of view, they are actually completely interdependent, since fish can easily migrate from one Exclusive Economic Zone to another with lower density of biomass of the species considered and therefore, abundance of food. Whenever we consider the dynamics of fish reproduction under fishing, it is inherent to model it as a discrete time process because of spawning period and resultant closed season, which divide the time into separate consecutive intervals.

The *state of the resource*, denoting the *biomass of fish at the beginning of time period considered*, is denoted by $x \geq 0$. At each time moment, player i extracts amount s_i, these s_i, in common, constitute a *static profile* (or a *profile of decisions*) $\mathbf{s}$. Given a state x, the decisions have to fulfil $s_i \in [0, cx]$.

The catch is sold at a common market at a price $p = \mathrm{price}(\mathbf{s})$ dependent on the aggregate catch. The price is defined by a linear function (in economics, called *inverse demand function*) $\mathrm{price}(\mathbf{s}) = A - u^{\mathbf{s}}$, where $u^{\mathbf{s}}$ denotes the *average/aggregate* extraction of $\mathbf{s}$, which is defined below.

Every player has *current payoff* function equal to their revenue minus cost

$$P_i(\mathbf{s}) = \mathrm{price}(\mathbf{s}) \cdot s_i - \mathrm{cost}(s_i),$$

where the *cost* is identical and quadratic for every player and it is equal to

$$\mathrm{cost}(s_i) = f s_i + \frac{1}{2}\, s_i^2.$$

We assume $A > f$. In economic applications, A is substantially greater than f.

Next, we define the *average/aggregate* extraction (throughout the paper, we shall refer to it as to the aggregate extraction). In the n player game,

$$u^{\mathbf{s}} = \sum_{j=1}^{n} \frac{s_i}{n}.$$

We use division by n in order that we could compare games with various n treated as decomposing the same set of individuals into decisive units of decreasing size (e.g. the global population decomposed into continents, countries, regions etc.)

At the maximal level of this decomposition, we consider the limit game with a continuum of players. Games with a continuum of players model situations when the number of players is large enough so that a single player i is insignificant — *negligible*. The set of players $\mathbb{I}$ is then equal to $[0, 1]$, on which the Lebesgue measure λ is considered. Profiles of decisions in this case are only measurable functions $\mathbf{s}\colon \mathbb{I} \to \mathbb{R}_+$ fulfilling the constraints. However, for consistency of notation with n player games, we shall also denote the strategy of player i at a profile $\mathbf{s}$ by s_i.

For the continuum of players case, the aggregate $u^{\mathbf{s}}$ is defined by

$$u^{\mathbf{s}} = \int_{\mathbb{I}} s_i \cdot d\lambda(i).$$

Strategies describe what decision to choose at each stage of the game. In our paper, we are interested in feedback strategies (sometimes also called closed loop or Markovian), i.e. state dependent $S_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for every state x, $S_i(x) \in [0, c \cdot x]$. We denote this set of strategies by $\mathbb{S}$.

A *profile of strategies* is any assignment of strategies to the players, $S \in \mathbb{S}^{\mathbb{I}}$, in the case of finitely many players.

In the continuum of players case, a measurability assumption is additionally required, which can be written as *for every x, $S_i(x)$ is a profile of decisions.* We denote the set of strategy profiles by Ω.

To complete the definition of the game, we introduce the payoffs in the game and the behaviour of the state variable — the biomass.

Player i maximizes his/her payoff in the game, equal to the sum of discounted current payoffs. For a profile S, the *payoff* is

$$\Pi_i(S, x_0) = \sum_{t=0}^{\infty} P_i(S(X(t))) \cdot \beta^t \tag{2.1}$$

for a discount factor $\beta \in (0,1)$, where X denotes the *trajectory of the state variable* (the biomass) resulting from choosing the strategy profile S given the initial state x_0. We assume that $x_0 > 0$.

The trajectory X is defined by

$$X(t+1) = (1+\xi) \cdot X(t) - u^{S(t)}, \tag{2.2}$$

with $X(0) = x_0$, for $\xi > 0$ being the *regeneration rate of the resource*, representing the growth rate of population without human interference.

Using a linear dynamic is, obviously, a simplification of the reality, in which there exists the maximal capacity of the environment. Nevertheless, a linear dynamics describes reasonably well the behaviour of the biomass for its values which are far from saturation. When we consider contemporary marine or deep lake fisheries, including open access high seas fisheries, which are fisheries after many years of intensive overexploitation by many users, the biomass is usually much closer to extinction than saturation. Although changing the dynamics on a part of the set of states in a dynamic optimization problem or a dynamic game usually changes the optima or equilibria by changing the value function, this is not the case in our paper. As we show in Remarks 3.5 and 4.3, changing the dynamics for large biomass of fish in order to reflect saturation does not change the results whatever the initial x_0 is.

If we want to emphasize the influence of the initial state on players' payoffs, we write the payoff $\Pi_i(S, x)$ (as a function of x denoting variable initial condition $x \geq 0$). A dynamic game of extraction of a common ecosystem with this resource dynamics was considered in Wiszniewska-Matyszkiel [27] and [31], but with logarithmic current payoff function in which constraining exploitation by the amount of resource was never active at equilibrium in n-players games. Calculation of feedback Nash equilibria and social optima was reasonably easy in those cases.

Change of current payoff, especially introducing constraints that may be active at equilibrium, with possibility of depletion of the resource, makes the problem much more complicated. We recall that the constraints on player's decisions are reflected by constraints for strategies: $S_i(x) \in [0, cx]$. Since we want to be able to model a situation in which depletion is possible, we consider $c = (1 + \xi)$. This corresponds to possibility of catching all fish in the player's region, including last generation of offspring.

The discount factor β from (2.1) measures players' patience. It can be rewritten in the form $\beta = 1/(1 + r)$. In economics, when we consider firms maximizing profits, these r corresponds to the market interest rate — the rate of growth of money in the banking system. From the point of view of economics, especially interesting are equilibria at which the rates of growth of both assets — the resource and the money – are identical. When applied to renewable resources extraction, it is known as "the golden rule". Therefore, we are especially interested in the case when $\beta = 1/(1 + \xi)$, i.e. $r = \xi$, however, in some cases we also mention other values of β.

2.1. Definition of social optimum and Nash equilibrium. First, we are interested in Pareto optimal profiles, to be more specific, profiles maximizing the aggregate payoff. Pareto optimal profiles can be results of decision making by a social planner or just full cooperation of all players. In dynamic games, finding a Pareto optimal profile requires solving dynamic optimization problem over the set of profiles.

DEFINITION 2.1. A profile $\overline{S}$ is a social optimum if it maximizes over $S \in \Omega$ the following values:

(a) $\sum_{i=1}^{n} \Pi_i(S, x_0)$ in n players games;

(b) $\int_{\mathbb{I}} \Pi_i(S, x_0)\, d\lambda(i)$ in the game with a continuum of players.

We are going to compare socially optimal profiles to Nash equilibria — profiles resulting from optimization of every player given strategies of the others.

Notational convention. If $\mathbf{s}_{\sim i}$ is the vector of decisions of the other players, then to write the current payoff of player i at a profile of decisions $\mathbf{s}$, we can use notation $P([s_i, \mathbf{s}_{\sim i}])$ instead of $P_i(\mathbf{s})$ to emphasize the decision of player i. Analogously, for a profile of strategies S_i, we write $\Pi([S_i, S_{\sim i}], x_0)$. For an aggregate $u \in \mathbb{R}_+$, for convenience, we shall also use $P(s_i, u)$ to emphasize both the decision of player i and the aggregate.

DEFINITION 2.2. A profile $\overline{S}$ is a Nash equilibrium

(a) in n players game if for every $i \in \mathbb{I}$ and for every strategy S_i of player i, we have
$$\Pi(\overline{S}, x_0) \geq \Pi([S_i, \overline{S}_{\sim i}], x_0);$$

(b) in the continuum of players game, if the inequality from (a) holds with "every i" replaced by "almost every i".

In other words, a *Nash equilibrium* is a profile of strategies, such that no (almost no for the continuum of players game) player can benefit from unilateral deviation from it.

Both concepts require solving some dynamic optimization problems: one joint dynamic optimization in the case of social optimum and n dynamic optimization problems for optimization of each of the players (given strategies of the others) in the case of Nash equilibrium. In both cases, for the n players game, we use the Bellman Equation for the infinite time horizon (see e.g. Bellman [4], Blackwell [5], Stokey, Lucas and Prescott [26] and Wiszniewska-Matyszkiel [29]) allowing us to determine the *value function* V, representing the *optimal payoff* in the problem considered.

However, unlike finding the social optimum, finding a Nash equilibrium requires solving a set of dynamic optimization problems of players that are coupled by finding a fixed point in the space of strategy profiles, since each player solves a dynamic optimization problem given strategies of the others.

3. Calculation of the social optimum

The first problem that we consider is the *social optimum*. In the social optimum, n countries jointly maximize their payoffs. So, the value function $\overline{V}\colon \mathbb{R}_+ \to \mathbb{R}$ is defined by $\overline{V}(x) = \sup_{S\in\Omega} \sum_{i=1}^{n} \Pi_i(S,x)$.

A sufficient condition for a function V to be the value function and S to be the optimal solution of a dynamic optimization problem consists of the Bellman Equation and terminal condition, together with the fact that $S(x)$ maximizes the right hand side of the Bellman Equation. It is specified in the sequel. The Bellman Equation is also a necessary condition whenever the payoff is well defined.

This version of a sufficient condition is equivalent to that proved in Stokey, Lucas and Prescott [26], Theorem 4.3 (rewritten because of different formulation) and it is an immediate consequence of a weaker sufficient condition — the main result of Wiszniewska-Matyszkiel [29].

$$\overline{V}(x) = \sup_{\mathbf{s}\in[0,cx]^n} \sum_{i=1}^{n} P_i(\mathbf{s}) + \beta \cdot \overline{V}\left((1+\xi)x - u^{\mathbf{s}}\right), \tag{3.1}$$

with the terminal condition

$$\lim_{t\to\infty} \beta^t \overline{V}(X(t)) = 0, \quad \text{for every admissible trajectory } X. \tag{3.2}$$

Under (3.1) and (3.2), $\overline{V}$ is the unique value function of the social optimization problem considered, the sum of optimal payoffs of the players is equal to $\overline{V}(x_0)$, and every optimal profile of strategies fulfils, for every x,

$$S(x) \in \operatorname*{Argmax}_{\mathbf{s}\in[0,cx]^n}. \tag{3.3}$$

Equations (3.1) and (3.3) constitute also a necessary condition whenever the payoffs are well defined (Stokey, Lucas and Prescott [26], Theorems 4.2 and 4.4).

Although in finite horizon problems, the dynamic programming method based on the Bellman Equation determines the optimal solution explicitly by backwards induction, in the infinite horizon case with a terminal condition at infinity, it cannot be done. So, we are going to solve it in a different way.

In LQ games, both for the social optimum and Nash equilibrium, we expect that the value function is quadratic, while the optimal control is linear. Presence of constraints changes the solution.

We start the analysis of the model from solving the social planner's optimization problem for arbitrary number of players.

Lemma 3.1. *If the value function $\overline{V}$ fulfilling the Bellman Equation* (3.1) *for the n players social optimum problem is differentiable, then the optimal solution is symmetric.*

Proof. We are going to use the standard Karush–Kuhn–Tucker first order necessary conditions for equation (3.1), given x. The constraints can be written as

$$s_i \geq 0 \quad \text{and} \quad (1+\xi)x - s_j \geq 0 \quad \text{for } i = 1, \dots, n.$$

Let us define the adjoint variables $\mu = (\mu_1, \dots, \mu_n) \geq 0$ and $\nu = (\nu_1, \dots, \nu_n) \geq 0$ for the constraints $s_i \geq 0$ and $(1+\xi)x - s_i \geq 0$, correspondingly. Consider the Lagrangian

$$\mathcal{L}(x, \mathbf{s}, \mu, \nu) = \sum_{i=1}^{n} (A - u^{\mathbf{s}}) s_i - \left(f s_i + \frac{s_i^2}{2} \right) + \beta V((1+\xi)x - u^{\mathbf{s}}) + \sum_{i=1}^{n} \mu_i s_i + \sum_{i=1}^{n} \nu_i ((1+\xi)x - s_i).$$

Differentiating the Lagrangian with respect to s_i and substituting

$$\frac{\partial \mathcal{L}(x, \mathbf{s}, \mu, \nu)}{\partial s_i} = 0$$

yields

$$(A - u^{\mathbf{s}}) - \frac{s_i}{n} - (f + s_i) - \left(\frac{\beta}{n}\right) V'((1+\xi)x - u^{\mathbf{s}}) + \mu_i - \nu_i = 0. \tag{3.4}$$

Similarly, for a different j, $\partial \mathcal{L}(x, \mathbf{s}, \mu, \nu)/\partial s_j = 0$ yields

$$(A - u^{\mathbf{s}}) - \frac{s_j}{n} - (f + s_j) - \left(\frac{\beta}{n}\right) V'((1+\xi)x - u^{\mathbf{s}}) + \mu_j - \nu_j = 0. \tag{3.5}$$

There are the following four possibilities.

1. If both $\mu_i, \mu_j = 0$ and $\nu_i, \nu_j = 0$, then by (3.4) and (3.5), the strategies are symmetric, i.e. $s_i = s_j$.

2. Assume two asymmetric boundary points, $s_i = 0$ and $s_j = (1+\xi)x$ with $\mu_i \neq 0$ and $\nu_j \neq 0$. We put $s_i = 0$ into (3.4) and $s_j = (1+\xi)x$ into (3.5) and solve for μ_i and ν_j. We have $\mu_i + \nu_j + (1+1/n)(1+\xi)x = 0$. This is a contradiction, since both $\mu_i, \nu_j \geq 0$.

3. If we consider $s_i = 0$ (so, it cannot be equal to $(1+\xi)x$, which implies $\nu_i = 0$) and $s_j \neq 0$ (which implies $\mu_j = 0$), then, by solving equations (3.4) and (3.5) for μ_i and ν_j, we have, $s_j(1+1/n) + \mu_j + \nu_i = 0$. This is again a contradiction.

4. If we consider $s_i = (1+\xi)x$ and $s_j \neq (1+\xi)x$ (which implies $\mu_i = 0$ and $\nu_j = 0$), then by solving equations (3.4) and (3.5) for μ_i and ν_j, we have, $(1+1/n)((1+\xi)x - s_j) + \mu_j + \nu_i = 0$. This is again a contradiction, since $s_j < (1+\xi)x$.

Therefore, the strategies are symmetric. □

Theorem 3.2. *Consider the* golden rule $\beta = 1/(1+\xi)$.

(a) *For the n players social optimization problem with $n \geq 1$, the value function is*

$$\overline{V}^{\mathrm{SO}}(x) = \begin{cases} \widehat{g} \cdot x + \dfrac{\widehat{h}}{2} \cdot x^2 & \text{if } x \in (0, \widetilde{x}), \\ \widetilde{k} & \text{otherwise}, \end{cases} \tag{3.6}$$

for $\widehat{s} = (A-f)/3$, $\widetilde{x} = \widehat{s}/\xi$, $\widehat{h} = -3\,n\xi\,(1+\xi)$, $\widehat{g} = n(A-f)(1+\xi)$ and $\widetilde{k} = (A-f)^2(1+\xi)n/(6\xi)$.

(b) *For $n \geq 1$ players, the value function per player, $\overline{V}^{\mathrm{SO}}(x)/n$, is independent of the number of players n.*

(c) *The value function for the continuum of players game is same as the value function per player for n players.*

(d) *A profile defined by*

$$\overline{S}_i^{\mathrm{SO}}(x) = \begin{cases} \xi x & \text{for } x \in (0, \widetilde{x}), \\ \widehat{s} & \text{otherwise}, \end{cases} \tag{3.7}$$

is the unique social optimum both for n players and a continuum of players.

The results of Theorem 3.2 are presented in Figures 1 and 2, for constants $A = 1000$, $f = 9$, $\xi = 0.02$ and $\beta = 1/(1+\xi)$. The choice of any constants with properties assumed in the formulation does not change the character of the graph.

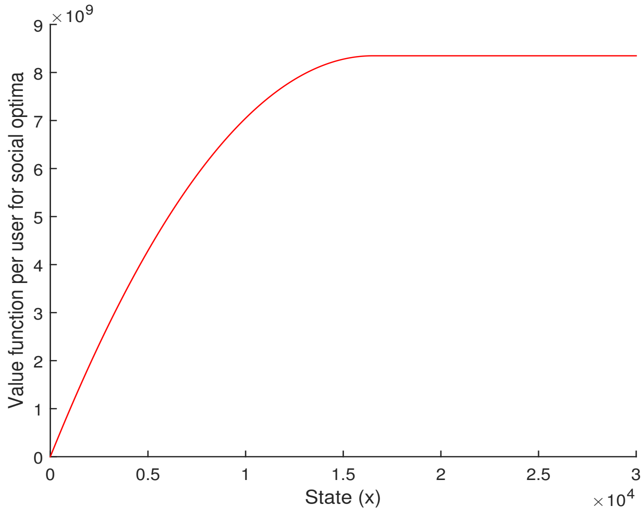

FIGURE 1. The value function per user for the social optimum problem for arbitrary number of players.

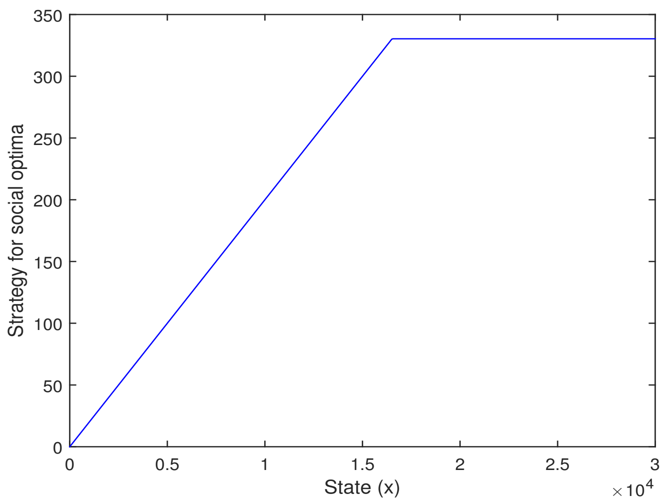

FIGURE 2. The strategy for social optimum problem for arbitrary number of players.

PROOF. (a) The Bellman Equation (3.1) for the value function is

$$\overline{V}(x) = \sup_{\mathbf{s}\in[0,cx]^n} \sum_{i=1}^{n} \left[(A - u^{\mathbf{s}})s_i - \left(f s_i + \frac{s_i^2}{2} \right) \right] + \beta \cdot \overline{V}((1+\xi)x - u^{\mathbf{s}}), \tag{3.8}$$

while the sufficient condition for the optimal profile, given by formula (3.3), is

$$S(x) \in \operatorname*{Argmax}_{\mathbf{s}\in[0,cx]^n} \sum_{i=1}^{n} \left[(A-u^{\mathbf{s}})s_i - \left(fs_i + \frac{s_i^2}{2}\right)\right] + \beta \cdot \overline{V}((1+\xi)x - u^{\mathbf{s}}). \tag{3.9}$$

Since, by Lemma 3.1, the solution is symmetric, so, for simplicity, let us assume symmetry a priori, i.e. take $S_i \equiv S$, and maximize

$$\sum_{t=0}^{\infty} \left((A - S(X(t))) \cdot S(X(t)) - \left(fS(X(t)) + \frac{S^2(X(t))}{2}\right)\right)\beta^t$$

over the set of feedback controls. In this case, the Bellman Equation (3.8) is reduced to the form

$$\overline{V}(x) = \sup_{s\in[0,(1+\xi)x]} n\left((A-s)\cdot s - \left(fs + \frac{s^2}{2}\right)\right) + \beta\overline{V}((1+\xi)x - s). \tag{3.10}$$

First, let us assume that the value function is $\overline{V}(x) = k + gx + hx^2/2$. We look for a solution of the Bellman Equation (3.10) in this class of functions. Afterwards, we find s maximizing the right hand side of the Bellman Equation over the set of available decisions.

We check the first order condition for the internal s from (3.7) and get the value of s as follows:

$$s = \frac{-(1+\xi)(A-f)n + g + hx(1+\xi)}{(h - 3n(1+\xi))}. \tag{3.11}$$

Finally, we substitute the optimal s to the Bellman Equation (3.10), which allows us to calculate the constants for which this equation is fulfilled. In this way, we obtain two sets of values of unknowns as follows:

$$\widehat{k} = 0, \quad \widehat{g} = n(1+\xi)(A-f), \quad \widehat{h} = -3n\xi(1+\xi); \tag{3.12}$$

$$\widetilde{k} = \frac{(A-f)^2 n}{6(1-\beta)}, \quad g, h = 0; \tag{3.13}$$

$$h = 0, \quad \text{arbitrary } g \neq 0, \quad k(g) = \frac{(n(A-f) - \beta g)^2}{6n(1-\beta)}. \tag{3.14}$$

Nevertheless, since $h \leq 0$ for all such sets of constants, s defined by (3.11), if $s \in [0, cx]$, is the global maximizer and it is unique. We consider the following cases.

Case 1. The values of unknowns k, g and h are as in (3.12).

The candidate for the social optimum in this case is equal to ξx, which, obviously, is less than cx and the maximized function is strictly concave. So, it defines the unique maximizer for Case 1. However, the function

$$\overline{V}_1(x) = \frac{\widehat{h}x^2}{2} + \widehat{g}x \tag{3.15}$$

does not fulfil the terminal condition (3.2), since $\lim_{t\to+\infty} \overline{V}_1(X^0(t))\beta^t = -\infty$ for X^0 being the trajectory corresponding to the profile $S \equiv 0$.

Note that replacing this terminal condition by the weakest existing one from Wiszniewska-Matyszkiel [29]:

(i) $\limsup_{t\to+\infty} \overline{V}_1(X(t))\beta^t \leq 0$ for every admissible trajectory X, and

(ii) $\limsup_{t\to+\infty} \overline{V}_1(X(t))\beta^t < 0$ implies that $\sum_{i=1}^{n} \Pi_i(S) = -\infty$ for every S for which X is the corresponding trajectory,

does not solve the problem, since the payoff for $S \equiv 0$ is 0.

Case 2. The values of unknowns k and h are as in (3.13):

$$\overline{V}_2(x) = \widetilde{k}. \tag{3.16}$$

Hence, the terminal condition is obviously fulfilled, since $\overline{V}_2$ is constant. The Bellman Equation (3.1) has the form

$$\overline{V}_2(x) = \sup_{s\in[0,cx]^n} \sum_{i=1}^{n} P_i(s) + \beta\widetilde{k}.$$

Therefore, the optimal strategy of each player is independent of x and equal to $\widehat{s} = (A - f)/3$.

Note that for x close to 0, $\widehat{s} > (1 + \xi)x$, so $\widehat{s}$ cannot be the social optimum for those x. So, $\overline{V}_2(x) = \widetilde{k}$ cannot be the value function for our problem, since (3.9) is also a necessary condition.

Case 3. The values of unknowns k, g and h are as in (3.14). In this case, $\lim_{t\to+\infty} (gX(t) + k(g))\beta^t \neq 0$ for the trajectory X^0 which violated the terminal condition in Case 1.

Case 4. Consider a combination of Cases 1 and 2. Let us try the only continuous combination of $\overline{V}_1$ and $\overline{V}_2$ which makes sense, i.e. with $\overline{V}(0) = 0$. The candidate for value function then is

$$\overline{V}(x) = \begin{cases} \overline{V}_1(x) & \text{for } x \in [0, \widetilde{x}], \\ \overline{V}_2(x) & \text{for } x > \widetilde{x}, \end{cases}$$

for $\overline{V}_1(x)$ and $\overline{V}_2(x)$ from equations (3.15)–(3.16). So, $\overline{V}(x) = \overline{V}^{\mathrm{SO}}(x)$. First, note that $\overline{V}^{\mathrm{SO}}$ is not only continuous, but also differentiable. The corresponding candidate for the optimal profile is $\overline{S}^{\mathrm{SO}}(x)$. After derivation of the candidates for value function and optimal profile, we have to prove that the Bellman Equation is really fulfilled by the piecewise defined functions.

To make notation more transparent, given a state x and a decision s (by symmetry, the aggregate extraction will be also equal to s), let us denote the next stage state by $x_{\mathrm{next}}(x, s)$, i.e. $x_{\mathrm{next}}(x, s) = ((1 + \xi)x - s)$.

The set of s for which $x_{\text{next}}(x,s) \leq \widetilde{x}$, is denoted by S_{I} and it is written as $S_{\text{I}} = [s_{\text{Bd}}, (1+\xi)x]$, while the set of the remaining s, $S_{\text{II}} = [0, s_{\text{Bd}})$, where s_{Bd} denotes s for which $x_{\text{next}}(x,s) = \widetilde{x}$, i.e. $\widetilde{x} = (1+\xi)x - s_{\text{Bd}}$, whenever it is non-negative, otherwise we take $s_{\text{Bd}} = 0$ (this holds for some $x < \widetilde{x}$; then S_{II} is empty).

If for some x, $s_{\text{Bd}} = 0$, which may hold only for $x \leq \widetilde{x}$, then for this x, the Bellman Equation (3.10) reduces to

$$\overline{V}_1(x) = \sup_{s \in [0,(1+\xi)x]} \sum_{i=1}^{n} [P(s,s)] + \beta \overline{V}_1(x_{\text{next}}(x,s)),$$

which we have already solved during calculation of coefficients in Case 1. So, let us consider $s_{\text{Bd}} > 0$. In this case, both S_{I} and S_{II} are non-empty. This situation can be decomposed into two cases.

(I) For $x \leq \widetilde{x}$, the Bellman Equation (3.10) can be rewritten as

$$\overline{V}_1(x) = \max\Big\{ \sup_{s \in S_{\text{I}}} nP(s,s) + \beta \overline{V}_1(x_{\text{next}}(x,s)), \\ \sup_{s \in S_{\text{II}}} nP(s,s) + \beta \overline{V}_2(x_{\text{next}}(x,s)) \Big\}.$$

Note that $\sup_{s \in S_{\text{I}}} nP(s,s) + \beta \overline{V}_1(x_{\text{next}}(x,s))$ is attained at ξx, which obviously belongs to S_{I}, while $\sup_{s \in S_{\text{II}}} nP(s,s) + \beta \overline{V}_2(x_{\text{next}}(x,s))$ is attained at s_{Bd}, which does not belong to S_{II}, but to S_{I}.

Since s_{Bd} does not optimize $nP(s,s) + \beta V_2(x_{\text{next}}(x,s))$ on S_{I}, the Bellman Equation (3.10) is fulfilled.

(II) If $x > \widetilde{x}$, then the Bellman Equation (3.10) can be rewritten as

$$\overline{V}_2(x) = \max\Big\{ \sup_{s \in S_{\text{I}}} nP(s,s) + \beta \overline{V}_1(x_{\text{next}}(x,s)), \\ \sup_{s \in S_{\text{II}}} nP(s,s) + \beta \overline{V}_2(x_{\text{next}}(x,s)) \Big\}.$$

First, let us consider optimization over S_{I}. The first order condition of maximization, $\partial(nP(s,s) + \beta \overline{V}_1(x_{\text{next}}(x,s))/\partial s = 0$, is attained at ξx, which is obviously not in S_{I}. So, the supremum over S_{I} is attained at $s_{\text{Bd}} \in \text{Closure}(S_{\text{II}})$.

Since $nP(s,s) + \beta \overline{V}_2(x_{\text{next}}(x,s))$ is strictly concave and $(A-f)/3$ is its only global maximum, $P((A-f)/3, (A-f)/3)$ is greater than at $P(s_{\text{Bd}}, s_{\text{Bd}})$. Since V is continuous, we have that $(A-f)/3$ is the global maximum over $[0,(1+\xi)x]$. Therefore, in Case 4, the Bellman Equation is fulfilled.

The terminal condition given by equation (3.2) is obvious, since $\overline{V}^{\text{SO}}$ is bounded.

(b) Immediate.

(c) For the continuum of players case, we have the following inclusion defining the social optimum:

$$\overline{S} \in \operatorname*{Argmax}_{S\in\Omega} \int_0^1 \sum_{t=0}^{\infty} \beta^t P\big(S_i(X(t)), u^{S(X(t))}\big)\, d\lambda(i).$$

First, note that along the optimal profile, P is positive, since, otherwise, at t for which $P\big(\overline{S}_i(X(t)), u^{\overline{S}(X(t))}\big)$ is negative, we can replace $\overline{S}_i(X(t))$ by 0 and increase the aggregate payoff.

Since $\overline{S}$ is a profile, $\overline{S}(t)$ is measurable, so, $\beta^t P\big(\overline{S}_i(X(t)), u^{\overline{S}(X(t))}\big)$ is integrable. Since P is bounded on the set on which it is positive, along the optimal profile, the series is absolutely convergent, so

$$\int_0^1 \sum_{t=0}^{\infty} \beta^t P\big(\overline{S}_i(X(t))), u^{\overline{S}(X(t))}\big)\, d\lambda(i) = \sum_{t=0}^{\infty} \beta^t \int_0^1 P\big(\overline{S}_i(X(t)), u^{\overline{S}(X(t))}\big)\, d\lambda(i).$$

Since $P(s_i, u)$ is concave in s_i, by the Jensen inequality,

$$\sum_{t=0}^{\infty} \beta^t \int_0^1 P\big(\overline{S}_i(X(t)), u^{\overline{S}(X(t))}\big)\, d\lambda(i) \le \sum_{t=0}^{\infty} \beta^t P\bigg(\int_0^1 \overline{S}_i(X(t))\, d\lambda(i), u^{\overline{S}(X(t))}\bigg).$$

The expression in the right hand side is equal to

$$\sup_{S\in\Omega} \sum_{t=0}^{\infty} P\big(u^{S(X(t))}, u^{S(X(t))}\big)\beta^t = \sup_{S\in\mathbb{S}} \sum_{t=0}^{\infty} P\big(S(X(t)), S(X(t))\big)\beta^t,$$

which reduces the problem for a continuum of players to the social optimum for n players with $n = 1$.

(d) Since the function $\overline{V}^{\mathrm{SO}}(x)$ is the value function, $\overline{S}^{\mathrm{SO}}(x)$ is the social optimum for any finite $n \ge 1$. The result for the continuum of players case is immediate, by reduction of the social optimization problem in this case to the social optimization problem for $n = 1$.

Therefore, the social optimum both for n players and a continuum of players is the profile defined by $\overline{S}_i^{\mathrm{SO}}(x)$. The optimal profile is unique, since maximization of the Bellman Equation is also a necessary condition for a control to be optimal whenever payoffs are well defined (Stokey, Lucas and Prescott [26, Theorem 4.4]). □

COROLLARY 3.3. *The value function for the social optimum problem is continuous, differentiable and strictly increasing, while the social optimum leads to sustainability of the resource.*

REMARK 3.4. Note again that the result is independent of the number of players. This means that our game models properly the solution in which increasing the number of players represents considering a more decomposed decision structure, not introducing additional fishermen.

Remark 3.5. (a) The value function and social optimum do not change if we change the dynamics for $x \geq \widetilde{x}$ and consider the state trajectory

$$X(t+1) = f(X(t), u^{S(X(t))}) \quad \text{with } X(0) = x_0, \tag{3.17}$$

where f is any function with $f(x,u) = (1+\xi)x - u$ for $x < \widetilde{x}$ and such that the interval $[\widetilde{x}, +\infty)$ is invariant under equation (3.17) given $S = \overline{S}^{\mathrm{SO}}$.

(b) For arbitrary $\beta \in (0,1)$, the social optimum remains unchanged at $[\widetilde{x}, +\infty)$, while the value function on this interval is $(A-f)^2 n/(6(1-\beta))$.

Proof. In both cases, for $x > \widetilde{x}$ we have the unconstrained global maximum at $\widehat{s}$ (as the discounted sum of global maxima at each time instant), which is feasible in those cases.

The value function in (b) is, therefore, equal to $P(\widehat{s}, \widehat{s})/(1-\beta)$. $\square$

3.1. A counterexample — a general conclusion for solving dynamic optimization and Nash equilibrium problem in dynamic games. While proving Theorem 3.2, we obtained the following result which can be used as a counterexample for a common simplification used in calculation of optimal controls and Nash equilibria in infinite time horizon problems.

Remark 3.6. The Bellman Equation (3.1) has a unique quadratic solution $\overline{V}_1$ defined by equation (3.15) and for this $\overline{V}_1$, there is a unique S such that

$$S(x) \in \operatorname*{Argmax}_{\mathbf{s} \in [0,cx]^n} \sum_{i=1}^{n} P_i(\mathbf{s}) + \beta \overline{V}_1((1+\xi)x - u^{\mathbf{s}}).$$

Nevertheless, the function $\overline{V}_1$ is *not* the value function for the social optimum problem (Definition 2.1) while this unique S is *not* the social optimum.

Proof. The value function $\overline{V}^{\mathrm{SO}}$ is not equal to $\overline{V}_1$ and the unique social optimum $\overline{S}^{\mathrm{SO}}$ is not equal to S. $\square$

A very important side-effect of this paper, stated in Remark 3.6, is that we have found a simple *counterexample*, showing that skipping checking the terminal condition while looking for the optimal control in the feedback form or a feedback Nash equilibrium, which often appears in literature (e.g. most of the papers in the "Fish Wars" thread — for a detailed discussion see Górniewicz, Wiszniewska-Matyszkiewicz [14]; terminal condition in the infinite time horizon is sometimes also omitted in textbooks), may lead to finding wrong results.

What we have done, is *not* showing that checking the Bellman Equation only, may result in finding a value function which does not fulfil the terminal condition (and, therefore, sufficient condition), *but* that without checking the terminal condition, we, as the *first and most obvious candidate* for the value function, obtain a function which is *not the value function.*

Note also that, if we treated $\overline{V}_1$ as the value function, then the resulting "candidate for optimal control" would lead to the corresponding trajectory which is always constant. Therefore, checking the terminal condition along the "candidate for optimal trajectory" only, practised in some works, is not sufficient and this specific result is also a counterexample to correctness of such an approach.

We want to emphasize that it is not the consequence of introducing state-dependent constraints, since this specific problem can be rewritten so that dependence of constraints on state disappears.

4. Nash equilibria for the continuum of players case

Now, we solve the problem of *Nash equilibrium.* We start from the game with a continuum of players.

THEOREM 4.1. *Consider the game with a continuum of players* ($\mathbb{I} = [0,1]$ *with the Lebesgue measure* λ).

(a) *The profile defined by*

$$\overline{S}_i^{\mathrm{NE}}(x) = \begin{cases} (1+\xi)x & \text{for } x \le \widehat{x}_1, \\ \dfrac{A-f}{2} & \text{otherwise}, \end{cases}$$

for $\widehat{x}_1 = (A-f)/(2(1+\xi))$, *is the only feedback Nash equilibrium* (*up to measure equivalence*).

(b) *The function defined by*

$$\overline{V}_i^{\mathrm{NE}}(x) = \begin{cases} P_{\mathrm{depl}}(x) & \text{for } x \le \widehat{x}_1, \\ \displaystyle\sum_{k=1}^{N} \frac{(A-f)^2 \beta^{k-1}}{8} \\ \qquad +\beta^N P_{\mathrm{depl}}\left((1+\xi)^N x - \frac{1}{2}(A-f)\sum_{k=1}^{N}(1+\xi)^{k-1}\right) \\ & \text{for } x \in (\widehat{x}_N, \widehat{x}_{N+1}), \\ \dfrac{(A-f)^2}{8} \cdot \dfrac{1}{(1-\beta)} & \text{otherwise}, \end{cases}$$

for $P_{\mathrm{depl}}(x) := P((1+\xi)x, (1+\xi)x)$ (*the payoff resulting from immediate depletion of the resource*), *i.e.*

$$P_{\mathrm{depl}}(x) = \left[A - f - \frac{3}{2}x(1+\xi)\right](1+\xi)x \quad \text{and} \quad \widehat{x}_N = \frac{A-f}{2}\sum_{k=1}^{N}\frac{1}{(1+\xi)^{k-1}}$$

for $N \ge 1$, *is the value function for optimization problem for the continuum of players game.*

(c) *For* $x \in (\widehat{x}_N, \widehat{x}_{N+1}]$ *with* $\widehat{x}_0 = 0$, *the resource will be depleted in* $N+1$ *stages, while for* $x \ge \widehat{x}_\infty = \lim_{N\to\infty} \widehat{x}_N$, *the resource will never be depleted.*

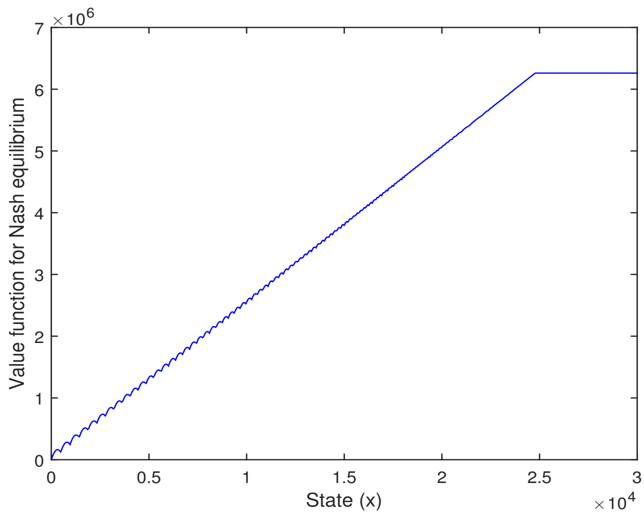

Figure 3. The value function for each player for the Nash equilibrium for the continuum of players game.

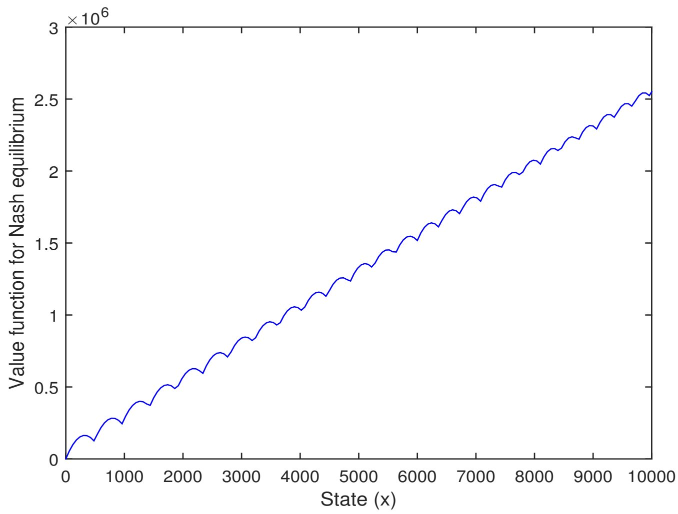

Figure 4. The value function for each player for the Nash equilibrium for the continuum of players game — zoomed view.

We illustrate the results in Figures 3–6 for the same constants as before, i.e. $A = 1000$, $f = 9$, $\beta = 1/(1+\xi)$ and $\xi = 0.02$, and we draw the value function with accuracy resulting from taking maximal $N = 1000$. Changing the constants within ranges assumed in the formulation does not change the character of the graph.

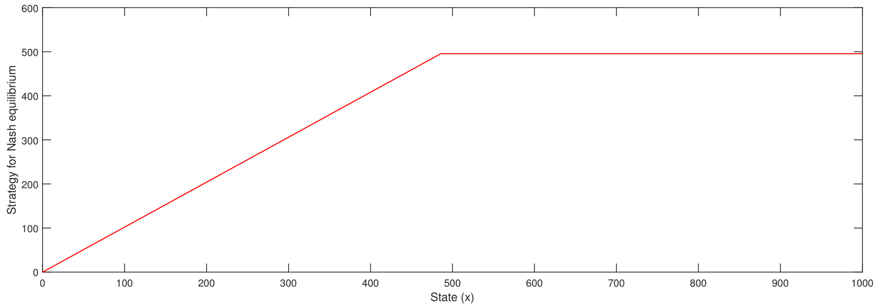

Figure 5. The strategy for the Nash equilibrium for the continuum of players game.

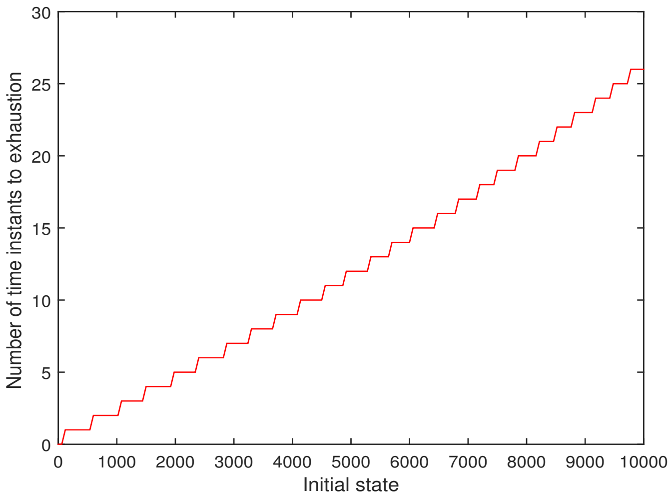

Figure 6. The number of time moments to resource exhaustion at the Nash equilibrium for the continuum of players game.

We do not present any figures comparing those results to the results for the social optima, since drawing both value functions (for the Nash equilibria and social optima) on one graph does not make sense because of values 10^3 times larger for the social optimum. Similarly, drawing both the Nash equilibrium and the social optimum strategies on one graph also does not make sense because of values 25 times larger for the Nash equilibrium.

Proof. (a) We use Decomposition Theorem A.1 applicable to dynamic games with a continuum of players. Reduced to our simple game (with identical players, regular current payoff functions, compact decision sets and continuous, independent of x and finite payoffs) it states that a profile is a Nash equilibrium if and only if it is a sequence of Nash equilibria in one stage games. So, we look for Nash equilibria in one stage games. For those games, in each stage game with state x, a Nash equilibrium is a profile of decisions $\bar{\mathbf{s}}$ such that, for almost every i,

$$\bar{s}_i \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} P(s_i, u^{\bar{\mathbf{s}}}).$$

Consider any $u^{\bar{\mathbf{s}}}$. Note that the influence of any single player on $u^{\bar{\mathbf{s}}}$ is negligible. First, note that given $u^{\bar{\mathbf{s}}}$, every player faces the same decision making problem with unique solution. Therefore, all the profiles are symmetric. So, $u^{\bar{\mathbf{s}}} = s_i$.

Consider

$$\overline{S}^{\mathrm{NE}}(x) = \begin{cases} (1+\xi)x & \text{for } x \le \widehat{x}_1, \\ \dfrac{A-f}{2} & \text{otherwise.} \end{cases}$$

Assume that for some x, a static profile at x yields some other aggregate $u \neq S(x)$.

Case 1. If $x \le \widehat{x}_1$ and $u < (1+\xi)x$, then the best response of every player i in this static game is $(1+\xi)x$, so $u^{\mathbf{s}} > u$, which is a contradiction.

Case 2. If $x > \widehat{x}_1$ and $u < (A-f)/2$, then the best response of every player i in this static game is $s_i > (A-f)/2$. So $u^{\mathbf{s}} > u$, which is a contradiction.

Case 3. If $x > \widehat{x}_1$ and $u > (A-f)/2$, then the best response of every player i in this static game is $s_i < (A-f)/2$. So $u^{\mathbf{s}} < u$, which is a contradiction.

Case 4. Finally, consider $u = \overline{S}^{\mathrm{NE}}(x)$.

(i) Consider $X(t) = x \le \widehat{x}_1$. At this stage, $\operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} P(s_i, u)$ for player i is $(1+\xi)x$ and in the next stage $X(t+1) = 0$.

(ii) Consider $X(t) = x > \widehat{x}_1$. At this stage, $\operatorname{Argmax}_{s_i \in [0,(1+\xi)x]} P(s_i, u)$ for player i is $(A-f)/2$.

So, for every state x, $\overline{S}^{\mathrm{NE}}(x)$ is a static Nash equilibrium at x. Therefore, by Theorem A.1, S is a Nash equilibrium.

(b) and (c) are proved together. We consider the following cases.

(i) Consider $x \le \widehat{x}_1$. Then the optimal decision is $(1+\xi)x$. So, the resource is immediately depleted. In this case, $\overline{V}_i^{\mathrm{NE}}(x) = P_{\mathrm{depl}}(x)$ and it equals to

$$V_i^{\mathrm{NE}}(x) = \left(A - f - \frac{3}{2}(1+\xi)x\right)(1+\xi)x.$$

(ii) Consider $\widehat{x}_1 < x < \widehat{x}_\infty$. Then the optimal choice for every player is $(A-f)/2$. We define $\widehat{x}_2$ such that $\widehat{x}_1 = (1+\xi)\widehat{x}_2 - (A-f)/2$, then for $x \in (\widehat{x}_1, \widehat{x}_2]$, the state in the next stage is in $(0, \widehat{x}_1]$.

Recursively, we have $\widehat{x}_N = (1+\xi)\widehat{x}_{N+1} - (A-f)/2$, then for $x \in (\widehat{x}_N, \widehat{x}_{N+1}]$, the state in the next stage is in $(\widehat{x}_{N-1}, \widehat{x}_N]$ and the resource will be depleted in N stages. Consequently, we have the recurrence relation for $N \ge 1$,

$$\widehat{x}_{N+1} = \frac{1}{1+\xi}\left(\widehat{x}_N + \frac{A-f}{2}\right),$$

which yields

$$\widehat{x}_N = \frac{A-f}{2}\sum_{k=1}^{N} \frac{1}{(1+\xi)}^{k-1}$$

and for x in the interval $(\widehat{x}_{N+1}, \widehat{x}_{N+2}]$, the resource will be depleted in $N+2$ stages. The limit $\widehat{x}_\infty$ of the sequence $\widehat{x}_N$ is

$$\widehat{x}_\infty = \lim_{N\to\infty} \widehat{x}_N = \frac{A-f}{2\xi}.$$

Since for $x \geq \widehat{x}_\infty$, $(1+\xi)x - (A-f)/2 > x$, the resource will never be depleted.

We return to the recurrence equation for $\overline{V}_i^{\mathrm{NE}}$ for $\widehat{x}_1 < x < \widehat{x}_\infty$. This recurrence equation is

$$\overline{V}_i^{\mathrm{NE}}(x) = P\biggl(\frac{A-f}{2}, \frac{A-f}{2}\biggr) + \beta \cdot \overline{V}_i^{\mathrm{NE}}\biggl((1+\xi)x - \frac{A-f}{2}\biggr).$$

For $X(t) \in (\widehat{x}_1, \widehat{x}_2]$, in the next stage, $X(t+1) \in [0, \widehat{x}_1]$, so $X(t+2) = 0$. Therefore, the value function in $(\widehat{x}_1, \widehat{x}_2]$ is

$$\overline{V}_i^{\mathrm{NE}}(x) = P\biggl(\frac{A-f}{2}, \frac{A-f}{2}\biggr) + \beta P_{\mathrm{depl}}\biggl((1+\xi)x - \frac{A-f}{2}\biggr).$$

By induction, proceeding in the same manner, we have

$$\overline{V}_i^{\mathrm{NE}}(x) = \sum_{k=1}^{N} \frac{(A-f)^2 \beta^{k-1}}{8} + \beta^N P_{\mathrm{depl}}\biggl((1+\xi)^N x - \frac{A-f}{2}\sum_{k=1}^{N}(1+\xi)^{k-1}\biggr)$$

for $x \in [\widehat{x}_N, \widehat{x}_{N+1}]$.

(iii) Finally, we consider $x \geq \widehat{x}_\infty$. Then

$$\overline{V}_i^{\mathrm{NE}}(x) = \sum_{t=0}^{\infty} \beta^t P\biggl(\frac{A-f}{2}, \frac{A-f}{2}\biggr) = \frac{(A-f)^2}{8} \cdot \frac{1+\xi}{\xi}. \qquad \square$$

Remark 4.2. The form of the value function obtained in Theorem 4.1 is very unusual for LQ games and its strange shape is caused only by constraints and possibility of extinction of the exploited species.

Remark 4.3. The value function and the Nash equilbrium profile do not change if we change the dynamics for $x \geq \widehat{x}_\infty$ and consider the state trajectory

$$X(t+1) = f(X(t), u^{S(X(t))}) \quad \text{with } X(0) = x_0, \tag{4.1}$$

where f is any function with $f(x,u) = (1+\xi)x - u$ for $x < \widehat{x}_\infty$ and such that the interval $[\widehat{x}_\infty, +\infty)$ is invariant under equation (4.1) given $S = \overline{S}^{\mathrm{NE}}$.

Proof. For $x > \widetilde{x}$, for each player, given the strategies of the other players $\overline{S}_{\sim i}^{\mathrm{NE}}$, $\overline{S}^{\mathrm{NE}}$ is the unconstrained global maximum (as the discounted sum of global maxima at each time instant), which is feasible.

Corollary 4.4. *The value function for the Nash equilibrium in the continuum of players case is not differentiable, but it is continuous. For some values of parameters it is also not monotone.*

Proof. Immediate for $x < \widehat{x}_\infty$. For $\widehat{x}_\infty$,

$$\begin{aligned}\lim_{x\to x_\infty^-} \overline{V}_i^{\mathrm{NE}}(x) &= \lim_{N\to\infty} \beta^N P_{\mathrm{depl}}\bigg((1+\xi)^N x - \frac{A-f}{2}\sum_{k=1}^{N}(1+\xi)^{k-1}\bigg) \\ &\qquad + \sum_{k=1}^{\infty}\beta^k P\bigg(\frac{A-f}{2},\frac{A-f}{2}\bigg) \\ &= 0 + \frac{(A-f)^2}{8(1-\beta)} = \overline{V}_i^{\mathrm{NE}}(x_\infty) = \lim_{x\to x_\infty^+} \overline{V}_i^{\mathrm{NE}}(x). \qquad \square\end{aligned}$$

We can also emphasize the following relations.

Corollary 4.5. *For every $x > 0$ and almost every $i \in \mathbb{I}$:*
(a) $\overline{V}^{\mathrm{SO}}(x)/n > \overline{V}_i^{\mathrm{NE}}(x)$,
(b) $\overline{S}_i^{\mathrm{SO}}(x) < \overline{S}_i^{NE}(x)$.

Proof. The proof is immediate by comparing the calculated results. $\square$

Moreover, as we can see in Figures 1–5, this difference is usually of substantial order.

5. Nash equilibria for finitely many players

In this section, we show that the problem of finding a Nash equilibrium for n players is more compound than both the social optimum problem for an arbitrary number of players and finding a Nash equilibrium for a continuum of players.

Moreover, it cannot be calculated by any of the methods used before: neither the undetermined coefficient method with quadratic value function (which is regarded as inherent in LQ games) nor with the decomposition method specific to large games. We also indicate a potential trap while applying the first method.

Therefore, now we switch to the problem of Nash equilibria for n players.

Theorem 5.1. *Consider $\beta = 1/(1+\xi)$ and $\mathbb{I} = \{1,\dots,n\}$ for $n \geq 2$. At symmetric Nash equilibria, the symmetric Nash equilibrium strategy is not piecewise linear with less than three intervals of constant coefficients with the value function quadratic with less than three intervals of constant coefficients.*

Proof. To prove this, we shall use necessity of the Bellman Equation. For unbounded payoffs proved by, e.g. Stokey, Lucas and Prescott [26]: Theorem 4.2 showing that the value function has to fulfil the Bellman Equation and Theorem 4.4 stating that the optimal solution should be a maximizer of the right hand side of the Bellman Equation.

The Bellman Equation for the value function for optimization of player i given strategies of the others $S_{\sim i}(x)$ is

$$\overline{V}_i(x) = \sup_{s_i \in [0,cx]} P_i[s_i, S_{\sim i}(x)] + \beta \cdot \overline{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{1}{n}\sum_{j \neq i} S_j(x)\right), \tag{5.1}$$

while the Nash equilibrium strategy of player i, the optimal strategy given the strategies of the others, has to fulfil

$$S_i(x) \in \operatorname*{Argmax}_{s_i \in [0,cx]} P_i[s_i, S_{\sim i}(x)] + \beta \cdot \overline{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{1}{n}\sum_{j \neq i} S_j(x)\right). \tag{5.2}$$

Since we look for piecewise linear equilibria, first, we assume that strategies of the others are of the form $S_{\sim i}(x) = (ax+b, \ldots, ax+b)$.

Note that if we have linear strategies of all the players, then the payoff of player i is quadratic in their decision, so is the right hand side of the Bellman Equation. Therefore, the value function of player i at a Nash equilibrium is of the form $k + gx + hx^2/2$. So, we assume the equilibrium strategy of player i is linear in the state variable, while the value function is quadratic.

We obtain s as a function of a, b, g and h,

$$\begin{aligned} s = \big(&-(1+\xi)(A - ax - b - f)n^2 + (((h-a)x - b)\xi + ((-a+1)h - a)x \\ &- b - hb + g)n + h(ax+b)\big)/((-1-\xi)n^2 + (-2\,\xi - 2)n + h). \end{aligned}$$

We substitute the symmetry assumption, $s = ax + b$, and we get

$$a = \frac{h\,(1+\xi)}{(-2\,n-1)\,\xi - 2\,n - 1 + h} \quad \text{and} \quad b = \frac{-\,(1+\xi)\,(A-f)\,n + g}{(-2\,\xi - 2)\,n + h - 1 - \xi}$$

as functions of h and g.

We write the Bellman Equation (5.1) with substituted a, b and s we calculate the coefficients h, g and k by equating the coefficients at x^2, x and the constants. We obtain three possible values of h: positive h^+, negative h^- and 0. They are equal to

$$h^+ = \frac{1}{2}\left(-3\,\xi + 4\,n - 1 + 4\sqrt{(1+\xi)\left(\frac{9}{16}\,\xi + \left(n - \frac{1}{4}\right)^2\right)}\right)(1+\xi),$$

$$h^- = -\frac{1}{2}\left(3\,\xi - 4\,n + 1 + 4\sqrt{(1+\xi)\left(\frac{9}{16}\,\xi + \left(n - \frac{1}{4}\right)^2\right)}\right)(1+\xi).$$

The corresponding g is

$$g = \frac{(1+\xi)\,((-1-2\,n^2)\xi - 1 - 2\,n^2 + h)\,(A-f)}{((1-4\,n)\xi + h - 4\,n + 1)},$$

whenever h is nonzero, and it is arbitrary otherwise. The resultant real constant k is unique given h and g. For h^- and the resultant g^-, k is equal to 0. For h^+, the function on the right hand side of Bellman Equation (5.1) is strictly convex, so the point of zero derivative is not a maximizer. Therefore, we consider only the cases of negative h^- and $h = 0$. After substitution, we obtain a and b for each of those sets of constants.

First, we exclude the case $h = 0$ and $g \neq 0$. In this case, if we solve the Bellman Equation, the resultant s_i is constant. If we substitute this constant s_i into the payoff function, we obtain a constant candidate for the value function, i.e. with $g = 0$. At this moment, let us formulate necessary conditions for the value function of each of the players at any symmetric Nash equilibrium resulting from the analysis of the problem without solving the Nash equilibrium problem explicitly.

Lemma 5.2. *At a symmetric Nash equilibrium the value function $\overline{V}_i$ of player i fulfils the following conditions*:

(a) $\overline{V}_i(0) = 0$.
(b) $\overline{V}_i(x) \geq 0$ *for every* x.
(c) $\overline{V}_i(x) < \overline{V}_i^{SO}(x)/n$ *for every* x.

Proof. (a) Immediate.

(b) Since 0 strategy is always available to player i.

(c) In the calculation of the social optimum, the value function is the solution of the problem

$$\overline{V}^{\mathrm{SO}}(x) = \max_{S \in \Omega} \sum_{i=1}^{n} \Pi_i(S, x),$$

while for every Nash equilibrium,

$$\overline{V}_i(x) = \max_{S_i \in \mathbb{S}} \Pi_i(S, x) \quad \text{for } i = 1, \ldots, n \text{ and } \Omega = \mathbb{S}^n.$$

Since

$$\frac{\overline{V}^{\mathrm{SO}}(x)}{n} = \frac{1}{n} \max_{S \in \Omega} \sum_{j=1}^{n} \Pi_j(S, x) \geq \frac{1}{n} \sum_{j=1}^{n} \Pi_j(S, x)$$

for every S with strict inequality whenever $S \neq \overline{S}^{\mathrm{SO}}$, so, also for a symmetric Nash equilibrium. In this case,

$$\frac{1}{n} \sum_{j=1}^{n} \Pi_j(S, x) = \frac{1}{n} \sum_{j=1}^{n} \overline{V}_j(x).$$

By the symmetry assumption, all $\Pi_j(S, x)$ are equal, so

$$\overline{V}_i(x) = \frac{1}{n} \sum_{j=1}^{n} \overline{V}_j(x) < \frac{\overline{V}^{\mathrm{SO}}(x)}{n}. \qquad \square$$

Note that neither of the functions $k + gx + hx^2/2$ that we have obtained, fulfils both (a) and (b), so we have to combine them. The only such combination consisting of two pieces which fulfils (a) and (b) is

$$\overline{V_i}^{\text{cand1}} = \begin{cases} g^- x + \dfrac{1}{2}\, h^- x^2 & \text{for } x \le \overline{x}, \\ \widetilde{k} & \text{otherwise.} \end{cases}$$

where $\widetilde{k} = (A - f - 3\widehat{s}/2)\,\widehat{s}/(1-\beta)$ is the positive real constant which corresponds to the solution when $h = 0$ and $g = 0$, $\widehat{s} = (A-f)n/(2n+1)$ (which is equal to the static Cournot–Nash equilibrium — the Nash equilibrium in a one stage game) and for some $\overline{x} > 0$ (note that otherwise, the Bellman Equation does not hold). The resultant candidate for the Nash equilibrium strategy is

$$\overline{S_i}^{\text{cand1}} = \begin{cases} ax + b & \text{for } x \le \overline{x}, \\ \widehat{s} & \text{otherwise.} \end{cases}$$

At this moment, we realize that we have not checked that $ax + b \le (1+\xi)x$, which does not hold for x close to 0, since $b > 0$. We denote the point at which $ax + b = (1+\xi)x$ by $\widetilde{x}$. It belongs to $(0, (A-f)n/(2n+1)\xi)$. For $x \le \widetilde{x}$, the calculated $ax + b$ is greater than $(1+\xi)x$. So, if $\widetilde{x} < \overline{x}$, then the candidate for the symmetric Nash equilibrium strategy has at least three parts:

$$\overline{S}_i^{\text{cand2}} = \begin{cases} (1+\xi)x & \text{for } x \le \widetilde{x}, \\ ax + b & \text{for } \widetilde{x} < x \le \overline{x}, \\ \widehat{s} & \text{otherwise;} \end{cases}$$

and the corresponding candidate for the value function is

$$\overline{V}_i^{\text{cand2}} = \begin{cases} \left(A - f - \dfrac{3}{2}\,(1+\xi)x\right)(1+\xi)x & \text{for } x \le \widetilde{x}, \\ g^- x + \dfrac{1}{2}\, h^- x^2 & \text{for } \widetilde{x} < x \le \overline{x}, \\ \widetilde{k} & \text{otherwise.} \end{cases}$$

(Nevertheless, even in this case, for $x = \widetilde{x} + \epsilon$ for small $\epsilon > 0$, the Bellman Equation does not hold, since $(1+\xi)x - (ax+b) < \widetilde{x}$.)

If $\widetilde{x} \ge \overline{x}$, then we have

$$\overline{S}_i^{\text{cand3}} = \begin{cases} (1+\xi)x & \text{for } x \le \overline{x}, \\ \widehat{s} & \text{otherwise} \end{cases}$$

and the resultant candidate for the value function consisting of at most two pieces is

$$\overline{V}_i^{\text{cand3}} = \begin{cases} \left(A - f - \dfrac{3}{2}\,(1+\xi)x\right)(1+\xi)x & \text{for } x \le \overline{x}, \\ \widetilde{k} & \text{otherwise.} \end{cases}$$

The necessary conditions (5.1) and (5.2) in this case are also not fulfilled. To prove this, note that for $x > \widehat{s}(n-1)/n(1+\xi)$, $(1+\xi)x$ is not the best response to $S_{\sim i}(x) \equiv (1+\xi)x$. So, we must have $\overline{x} \leq \widehat{s}(n-1)/(n(1+\xi))$. But then $\overline{x} < \widehat{s}/\xi$ — the critical value below which the constant $S(x) \equiv \widehat{s}$ is not a strategy profile, since it is not feasible. Therefore, the value function cannot equal $\widetilde{k}$ on $[\overline{x}, \infty)$. □

While proving Theorem 5.1, we noticed a potential trap when looking for Nash equilibrium in feedback strategies for infinite time horizon using undetermined coefficient method.

Remark 5.3. Consider the procedure of undetermined coefficient for finding symmetric feedback Nash equilibria, shortly described as follows. "Assume the general form of the value function and fix strategies of the other players. Next, given this general form of the value function, find the best response (dependent on the state variable) to the strategies of the other players. Find coefficients such that all the value functions are equal, equate the best response to the strategies of the others and solve for symmetric Nash equilibrium solution".

If we used this procedure for our game, then it would return the best response equal to ξx for every player for x below some level, then constant, equal to static Cournot–Nash equilibrium (see Figure 7). However, if we consider the first interval, ξx is not the best response to strategies of the other players equal to ξx, since the Bellman Equation after substitution of strategies of the other players is not fulfilled.

Therefore, our game is a *counterexample* to such a way of solving dynamic games in the case of infinite time horizon. However, if we correct the algorithm by starting from "Assume the general form of the value function *and the general form of the equilibrium strategies of the other players*", this problem disappears. Similarly, if we equate the strategies of the others to the best response before calculating the coefficients.

In Figure 7, we present graphically results of using the incorrect procedure for looking for Nash equilibria, described in Remark 5.3.

Since it is not the main subject of the paper, we shall not present here the whole proof of Remark 5.3. The sketch of its proof is analogous to what we did at a beginning of the proof of Theorem 5.1 but without writing strategies of the other players as $ax + b$. Nevertheless, the level of complication is much lower, similar rather to the proof of Theorem 3.2, with two possible values of the quadratic coefficient h.

Obviously, this trap is specific only to dynamic games and it does not cause problems in dynamic optimization problems.

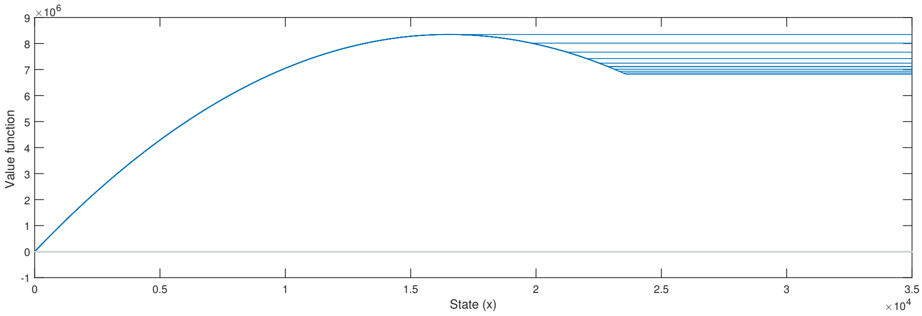

Figure 7. A wrong value function for Nash equilibria for n players resulting from falling into the trap while using the undetermined coefficient method described in Remark 5.3 for various numbers of players.

6. Enforcing social optimality by a tax system

As we have proved in Theorems 3.2 and 4.1, for every x, the social optimum for the golden rule $\beta = 1/(1+\xi)$ guarantees sustainability of fish (the biomass is always nondecreasing), while at every Nash equilibrium for a continuum of players, for every $x < \widehat{x}_\infty$ the biomass is always decreasing and the fishery is depleted in finitely many stages.

We are interested in enforcing the social optimum by a tax system, linear in player's strategy, i.e. $\mathrm{tax}(s) = \tau s$. Such a tax system is called Pigouvian. It is of purely regulatory character. Formally, *introduction of a tax* (or a *tax-subsidy system*) is a modification of the game by changing the payoffs. In our game the current payoff of player i changes to $\mathrm{price}(s) - \mathrm{cost}(s_i) - \mathrm{tax}(s_i, x)$. We are interested in a linear tax, $\mathrm{tax}(s_i, x) = \tau(x) s_i$.

We say that a *tax system enforces* a profile $\overline{S}$ if $\overline{S}$ is a Nash equilibrium in the game modified by introduction of this tax.

We prove that for the continuum of players Nash equilibrium, we cannot enforce social optimality for all states by one τ. We can calculate such a constant rate only for $x \geq (A-f)/(3\xi)$.

If we want to enforce the social optimum for all levels of biomass of fish, we have to consider a variable tax rate $\tau(x)$. This is, however, variability which can be justified, since the tax rate which we obtain is higher for low levels of x — the more the species is endangered, the more is paid for its exploitation.

Proposition 6.1. *Consider the game with a continuum of players.*

(a) *The rate of tax enforcing the socially optimal profile* $\overline{S}^{\mathrm{SO}}$ *(defined in Theorem* 3.2 (d)*) for* $\beta = 1/(1+\xi)$ *is given by*

$$\tau(x) = \begin{cases} A - f - 2\xi x & \text{if } x \leq (A-f)/(3\xi), \\ (A-f)/3 & \text{otherwise.} \end{cases}$$

(b) *The same tax rate enforces* $\overline{S}^{\mathrm{SO}}$ *(defined in Theorem* 3.2 (d)*) for any* $\beta \in (0,1)$ *and it guarantees sustainability of the fishery.*

The linear tax rate enforcing $\overline{S}^{\mathrm{SO}}$ is illustrated by Figures 8 and 9, for the same constants as before: $A = 1000$, $f = 9$, $\xi = 0.02$ and $\beta = 1/(1+\xi)$. Changing the constants within the range assumed in the formulation does not change the character of the graph.

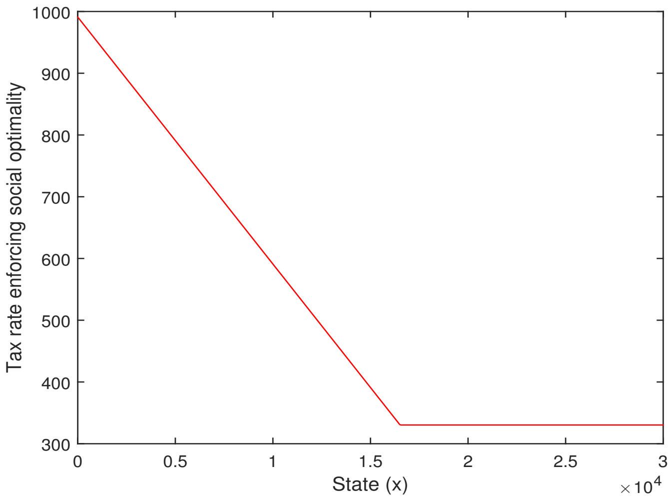

Figure 8. The rate of tax enforcing $\overline{S}^{\mathrm{SO}}$ and guaranteeing sustainability for a continuum of players.

Proof. (a) We look for a linear tax enforcing social optimum with rate τ. We modify the current payoff function by subtracting $\tau(x)s_i$ and we want the Nash equilibrium in the modified game to be equal to the social optimum in the original game.

Note that the results of applying a linear tax of rate $\tau(x)$ are equivalent to increasing f by $\tau(x)$.

Case 1. If $x > (A-f)/(3\xi)$ then $\overline{S}^{\mathrm{SO}}(x) = (A-f)/3$. In this case,

$$\frac{A-f}{3} = \frac{A-f-\tau(x)}{2}.$$

Solving for τ yields $\tau(x) = (A-f)/3$.

Case 2. If $x \le (A-f)/(3\xi)$ then $\overline{S}^{\mathrm{SO}}(x) = \xi x$. In this case, we want the new Nash equilibrium strategy to be equal to ξx, which requires solving for $\tau(x)$

$$\xi x = \frac{A-f-\tau(x)}{2},$$

which yields $\tau(x) = A - f - 2\xi x$.

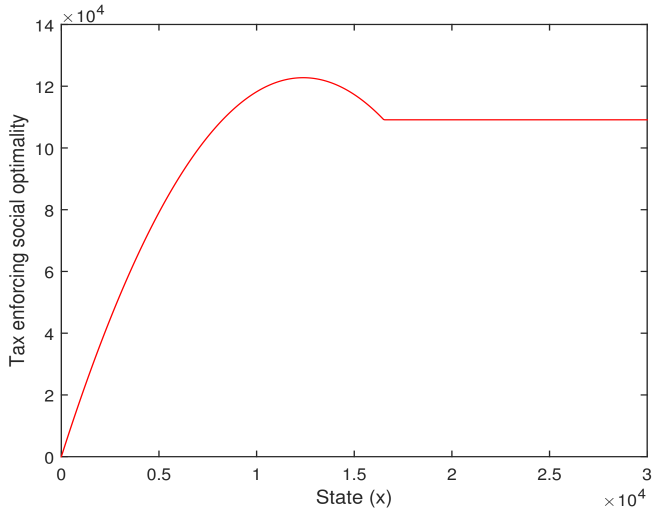

FIGURE 9. The tax enforcing $\overline{S}^{\mathrm{SO}}$ and guaranteeing sustainability for a continuum of players.

(b) Analogously, only without social optimality of $\overline{S}^{\mathrm{SO}}$. Sustainability of $\overline{S}^{\mathrm{SO}}$ has already been mentioned in Corollary 3.3. □

7. Conclusions and further research

In this paper, we consider a discrete time dynamic game of exploitation of a common renewable resource — a fishery — over the infinite time horizon, with a quadratic payoff, a linear state dynamics, with constraints on strategies implied by the problem considered and a possibility of depletion of the fishery. These constraints make the problem of finding a feedback Nash equilibrium difficult.

We calculate the social optima and some Nash equilibria as well as the value functions for them. When the social optimum is considered, we are able to calculate it for arbitrary number of players, either finite, positive integer or continuum. On the other hand, for the Nash equilibrium, we are not able to do this for finite $n \geq 2$ — only negative results can be proven, with stating that the number of different intervals in the analytic formula for the value function and strategy at any symmetric equilibrium is non less than three. However, we are able to calculate Nash equilibria for the continuum of players case. Although the Nash equilibrium for the continuum of players case is quite simple, the value function is very complicated and irregular, which, to the best of the authors' knowledge, never appears in the literature on linear quadratic dynamic games.

It is worth emphasizing again that our research, although initially it was assumed to be solely an in-depth analysis of a specific problem of exploitation of common renewable resources, results also in important theoretical findings:

- showing that presence of even a very simple and obvious constraint on strategies may result in a very complicated form of the value functions and Nash equilibria and
- finding out a very simple counterexample to correctness of a procedure often used while looking for Nash equilibria and/or optimal controls.

There are several potential continuations of this analysis. Besides working on new methods to determine Nash equilibria in games with n players, a belief distorted Nash equilibrium can be computed numerically. It may also be interesting to introduce a more compound spatial distribution of fish. In such a case, current decisions of each of the players have more influence on the future level of biomass in their zone than decisions of any other player. Obtaining Nash equilibria in such a model, however, may turn out to be possible only in finite time horizon problems.

Appendix A. Decomposition theorem for games with a continuum of players

Here, we cite a theorem concerning dynamic games with a continuum of players which we use in this paper — a decomposition theorem from Wiszniewska-Matyszkiel [30].

First, we define the more general environment of dynamic games with a continuum of players in which those theorems were stated in [30]. We use a slightly reduced form because of high complexity of the games considered in [30] and we cite the result restricted to the infinite horizon case only.

We consider dynamic games with a measure space of players $\mathbb{I} = [0,1]$ with the Lebesgue measure λ. Players' decision sets $\mathbb{D}_i$ are measurable subsets of a measurable space $(\mathbb{D}, \mathcal{D})$. The state space is $\mathbb{X}$. Currently available decisions are $D_i(x)$ for $D_i\colon \mathbb{X} \multimap \mathbb{D} \in \mathcal{D}$ with $D_i(x) \subseteq \mathbb{D}_i$. A profile of decisions available at state x is any measurable function $\mathbf{s}\colon \mathbb{I} \to \mathbb{D}_i$ with $s_i \in D_i(x)$. Current payoffs are $P_i\colon \mathbb{D} \times \mathbb{U} \times \mathbb{X} \to \mathbb{R}$, where $\mathbb{U}$ is the set of statistics of profiles of decisions, i.e. $u^{\mathbf{s}} = \int_{\mathbb{I}} g(i, s_i)\, d\lambda(i)$. The trajectory of the state variable corresponding to a profile of strategies S is described by $X^S(t+1) = \psi(X^S(t), u^S(t))$, where $u^S(t) = u^{S(X(t))}$ for $\psi\colon \mathbb{X} \times \mathbb{U} \to \mathbb{X}$, with $X(0) = x_0$ and the payoffs in the game are

$$\Pi_i(S, x_0) = \sum_{t=0}^{\infty} P_i(S_i(X^S(t)), u^S(t), X^S(t))\beta^t.$$

Theorem A.1 (Wiszniewska-Matyszkiel [30, Theorem 3.2]).

(a) *If S is a profile of strategies and for all t, the profiles of decisions $S(X^S(t))$ are static equilibria (i.e. equilibria in one stage games) at state of the system $X^S(t)$, then S is a Nash equilibrium.*

(b) *Let the space of decisions* $\mathbb{D}$ *be such that the set* $\{(d, d) : d \in \mathbb{D}\}$ *is* $\mathcal{D} \otimes \mathcal{D}$*-measurable and* $\mathbb{D}$ *is a measurable image of a measurable space* $(\mathbb{Z}, \mathcal{Z})$ *that is an analytic subspace of a separable compact topological space* $\mathbb{W}$ (*with the* σ*-field of Borel subsets*). *Assume that, for almost every* i *and every* u, x, *the function* $P_i(\,\cdot\,, u, x)$ *is upper semi-continuous, for almost every* i, *the function* P_i *is such that inverse images of measurable sets are* $\mathcal{D} \otimes \mathcal{B}(\mathbb{U}) \otimes \mathcal{X}$*-analytic and the correspondence* D_i *has an* $\mathcal{X} \otimes \mathcal{D}$*-analytic graph and compact values. Every Nash equilibrium* S *such that, for almost every player* i, *the payoff* $\Pi_i(S, x_0)$ *is finite, satisfies the following condition: for all* t, *static profiles* $S(X^S(t))$ *are static equilibria at the state of the system* $X^S(t)$.

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