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A NONCOMMUTATIVE VERSION OF FARBER'S TOPOLOGICAL COMPLEXITY

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Abstract. Topological complexity for spaces was introduced by M. Farber as a minimal number of continuity domains for motion planning algorithms. It turns out that this notion can be extended to the case of not necessarily commutative C^*-algebras. Topological complexity for spaces is closely related to the Lusternik–Schnirelmann category, for which we do not know any noncommutative extension, so there is no hope to generalize the known estimation methods, but we are able to evaluate the topological complexity for some very simple examples of noncommutative C^*-algebras.

1. Introduction

Gelfand duality between compact Hausdorff spaces and unital commutative C^*-algebras allows to translate some topological constructions and invariants into the noncommutative setting. The most successful example is K-theory, which became a very useful tool in C^*-algebra theory. Homotopies between $*$-homomorphisms of C^*-algebras also play an important role, but there is no nice general homotopy theory for C^*-algebras due to the fact that the loop functor has no left adjoint [11], Appendix A. Nevertheless, there are some homotopy invariants that allow noncommutative versions.

The aim of our work is to show that M. Farber's topological complexity [4] is one of those. In Section 2 we recall the original commutative definition of

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topological complexity, and in Section 3 we use Gelfand duality to reverse arrows in this definition, and show that the resulting noncommutative definition generalizes the commutative one. In the remaining two sections we calculate topological complexity for some simple examples of C^*-algebras. In particular, we show that introducing noncommutative coefficients may decrease topological complexity. Although in most our examples topological complexity is either 1 or ∞, we provide a noncommutative example with topological complexity 2.

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2. Farber's topological complexity

The topological approach to the robot motion planning problem was initiated by M. Farber in [4]. Let us recall his basic construction. Let X be the configuration space of a mechanical system. A continuous path $\gamma\colon [0,1] \to X$ represents a motion of the system, with $\gamma(0)$ and $\gamma(1)$ being the initial and the final state of the system. If X is path-connected then the system can be moved to an arbitrary state from a given state. Let PX denote the space of paths in X with the compact-open topology, and let

$$\pi\colon PX \to X \times X \tag{2.1}$$

be the map given by $\pi(\gamma) = (\gamma(0), \gamma(1))$. A continuous *motion planning algorithm* is a continuous section

$$s\colon X \times X \to PX$$

of π. Typically, there may be no continuous motion planning algorithm, so one may take a covering of $X \times X$ by sets $V_1, \ldots, V_n$ (domains of continuity) and require existence of continuous sections

$$s_i\colon V_i \to PX|_{V_i}$$

of maps $\pi_i\colon PX|_{V_i} \to V_i$, $i = 1, \ldots, n$. Here $PX|_{V_i}$ denotes the restriction of π onto V_i, i.e. the subset of paths $\gamma\colon [0,1] \to X$ such that $(\gamma(0), \gamma(1)) \in V_i$. In this case, the collection of the sections s_i, $i = 1, \ldots, n$, is called a (discontinuous) motion planning algorithm. There are several versions of the definition, which use various kinds of coverings, e.g. coverings by open or closed sets, or by Euclidean neighbourhood retracts, etc., but most of them agree on simplicial polyhedra (cf. [5], Theorem 13.1). The topological complexity $TC(X)$ of X is the minimal number n of domains of continuity, i.e. the minimal number n, for which there exists a covering $V_1, \ldots, V_n$ and continuous sections s_i as above. This number measures the complexity of the problem of navigation in X.