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MULTIPLE NODAL SOLUTIONS FOR SEMILINEAR ROBIN PROBLEMS WITH INDEFINITE LINEAR PART AND CONCAVE TERMS

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Abstract. We consider a semilinear Robin problem driven by Laplacian plus an indefinite and unbounded potential. The reaction function contains a concave term and a perturbation of arbitrary growth. Using a variant of the symmetric mountain pass theorem, we show the existence of smooth nodal solutions which converge to zero in $C^1(\overline{\Omega})$. If the coefficient of the concave term is sign changing, then again we produce a sequence of smooth solutions converging to zero in $C^1(\overline{\Omega})$, but we cannot claim that they are nodal.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem:

$$
\text{(1.1)} \qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = \vartheta(z)|u(z)|^{q-2}u(z) + f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega, \\ & \text{for } 1 < q < 2. \end{cases}
$$

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In this problem the potential function $\xi \in L^s(\Omega)$, with $s > N$, and it is sign changing, so the linear part of the problem is indefinite. In the reaction part of the problem (the right hand side), there is a "concave"' (that is, strictly sublinear) term, which is the term $\vartheta(z)|u(z)|^{q-2}u(z)$ $(1 < q < 2)$ with the weight $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) > 0$ for almost all $z \in \Omega$. In the last part of the paper we allow ϑ to be sign changing. There is also a perturbation term f which is assumed to be a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). A special feature of our work is that we do not impose any growth condition on $f(z,\cdot)$. We only impose conditions near zero and we assume that $f(z,\cdot)$ is odd and the whole reaction function minus the potential term (that is, the function $x \mapsto \vartheta(z)|x|^{q-2}x + f(z,x) - \xi(z)x$) exhibits a kind of oscillatory behaviour near zero. Suitable truncations make the behaviour of $f(z,\cdot)$ near $\pm\infty$ irrelevant. Using a variant of the symmetric mountain pass theorem, which is due to Heinz [3] and Kajikiya [5], we produce a whole sequence of distinct smooth (that is, they belong in $C^1(\overline{\Omega})$) nodal (that is, sign changing) solutions, which converge to zero in $C^1(\overline{\Omega})$. Finally we see what happens when the weight function ϑ is sign changing. In this case, again we produce a sequence of smooth solutions in $C^1(\overline{\Omega})$ converging to zero, but we no longer claim that these solutions are nodal.

Infiniteness of the set of solutions for indefinite semilinear Dirichlet equations was established by Yu, Yongqing [16], Zhang, Liu [17], Qin, Tang, Zhang [13], Zhang, Tang, Zhang [18]. In all these works, the reaction term is superlinear but with subcritical polynomial growth in the x variable. However, nodality of solutions is not shown. We also mention the related works of Wang [15], Qian [11] and Qian, Li [12]. In [15] the problem is Dirichlet with zero potential (that is, $\xi \equiv 0$) and the reaction function f_0 is continuous on $\Omega \times \mathbb{R}$ and no growth condition is imposed on $f_0(z,\cdot)$. In [15], the author produces infinitely many distinct solutions but also does not show that they are nodal. Infinitely many nodal solutions were produced by Qian [11] for Neumann problems with a coercive differential operator of the form

$$u \to -\Delta u + au, \quad \text{for all } u \in H^1(\Omega) \text{ with } 0 < a < +\infty.$$

In [11], the reaction function $f_0(z,x)$ is assumed to be continuous on $\Omega \times \mathbb{R}$ and superlinear in $x \in \mathbb{R}$, but with subcritical polynomial growth. For Robin problems with zero potential term (that is, $\xi \equiv 0$), there is the work of Qian, Li [12], where the reaction function $f_0(z,x)$ is continuous on $\overline{\Omega} \times \mathbb{R}$ and superlinear in $x \in \mathbb{R}$, but again with subcritical polynomial growth. Qian, Li [12] produce a whole sequence of distinct solutions but they again do not show that these solutions are nodal.

So, the above survey of the relevant literature reveals that only Wang [15] deals with a problem where the reaction function $f_0(z,x)$ is of arbitrary growth