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MULTIPLICITY OF POSITIVE SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS IN $\mathbb{R}^3$

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Abstract. We are concerned with the multiplicity of positive solutions for the following Kirchhoff type problem:

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u = Q(x)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a parameter, $a, b > 0$ are constants, $p \in (2,6)$, and $Q \in C(\mathbb{R}^3)$ is a nonnegative function. We show how the profile of Q affects the number of positive solutions when ε is sufficiently small.

1. Introduction

In this paper, we study the existence and multiplicity of positive solutions to the following nonlinear problem of Kirchhoff type:

$$\text{(P)} \qquad \begin{cases} -\left(\varepsilon^2 a + \varepsilon b \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)\Delta u + u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $\varepsilon > 0$ is a parameter, $a, b > 0$ are constants, $p \in (2,6)$, and Q is a nonnegative continuous function satisfying

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(Q) $\lim_{|x|\to\infty} Q(x) = Q_\infty > 0$ and there exist some points $x^1, \ldots, x^k$ in $\mathbb{R}^3$ such that $Q(x^i)$ are the strict maxima and satisfy

$$Q(x^i) = Q_m := \max_{x\in\mathbb{R}^3} Q(x) > Q_\infty \quad \text{for all } i = 1, \ldots, k.$$

Problem (P) is a particular case of the following Dirichlet problem of Kirchhoff type:

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain. Such problems are often referred to be nonlocal because of the presence of the Kirchhoff term $(\int_\Omega |\nabla u|^2)\Delta u$ which implies that (1.1) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties that make the study of such problems particularly interesting. Various results on the existence of positive solutions, multiple solutions, sign-changing solutions, ground states have been obtained, see for example [4], [6], [7], [11], [13], [16]–[18], [20], [25], [27] and the references therein.

Recently, there has been increasing interest in studying the following perturbed Kirchhoff type equation (see [12, 10, 9, 23] and the references therein):

$$\begin{cases} -\Big(\varepsilon^2 a + \varepsilon b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.2}$$

where $V \in C(\mathbb{R}^3,\mathbb{R})$, $f \in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$, $a,b > 0$ are constants, and ε is a positive parameter. First, it is important to consider the following autonomous problem with $\varepsilon = 1$ and $V(x) \equiv \nu \in \mathbb{R}$ in (1.2):

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\Big)\Delta u + \nu u = f(u) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

He and Zou in [12] established the existence of ground state solution to (1.2) under the condition that $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies the Ambrosetti–Rabinowitz condition ((AR) in short)

$$\text{there exists} \quad \mu > 4 \quad \text{such that} \quad 0 < \mu\int_0^u f(s)\,ds \le f(u)u,$$

$f(u) = o(u^3)$ as $u \to 0$, $f(u)/|u|^q \to 0$ as $|u| \to \infty$ for some $3 < q < 5$ and $f(u)/u^3$ is increasing for $u > 0$. Ye in [26] extended the above result to the case without the (AR) condition under the conditions that f is superlinear, subcritical and $f(u)/u$ is increasing for $u > 0$. Wang et al. [23], He et al. [10] proved the existence of ground state solution to (1.3) with critical nonlinearity, i.e. $f(u) \sim \lambda|u|^{p-2}u + |u|^4u$ for some $4 < p < 6$, $\lambda > 0$. Latter, He and Li [9] filled the gap when $f(u) = \lambda|u|^{p-2}u + |u|^4u$ where $2 < p \le 4$. Besides, for