Topological **M**ethods in **N**onlinear **A**nalysis
Volume 50, No. 1, 2017, 217–229
DOI: 10.12775/TMNA.2017.027

GENERALIZED RECURRENCE IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

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Abstract. We aim to introduce the generalized recurrence into the theory of impulsive semidynamical systems. Similarly to Auslander's construction in [J. Auslander, *Generalized recurrence in dynamical systems*, Contrib. Differential Equations **3** (1964), 65–74], we present two different characterizations, respectively, by Lyapunov functions and higher prolongations. In fact, we show that if the phase space is a locally compact separable metric space, then the generalized recurrent set is the same as the quasi prolongational recurrent set. Also, we see that many new phenomena appear for the impulse effects in the semidynamical system.

1. Introduction

Since at least the time of Poisson, mathematicians have pondered the notion of recurrence for differential equations. Solutions that exhibit recurrent behavior provide insight into the behavior of general solutions. In the theory of dynamical systems, the different notions of recurrence all express the idea of a point returning to itself, in some sense, for arbitrarily large time. Using continuous real valued functions on the phase space, Auslander [1] introduced the concept of generalized recurrence in dynamical systems. In the literature, the generalized recurrence is also called to be the Auslander recurrence or prolongational recurrence. The generalized recurrent set contains periodic points, recurrent

2010 *Mathematics Subject Classification.* 37B20, 37C10, 54H20.

Key words and phrases. Impulsive semidynamical system; generalized recurrence; Lyapunov function; higher prolongation.

(or Poisson stable) points and non-wandering points. It is known that the generalized recurrence is a very important concept in the theory of stabilities, for example, Nitecki [15] showed the role of generalized recurrence in a completely unstable flow, and Peixoto [16] perturbed a vector field with a non-periodic prolongational recurrent point to get a periodic orbit. Recently, in [9] we have used prolongational recurrence to generalize Birkhoff center and its depth.

An impulsive semidynamical system is a discontinuous semidynamical system, which is a natural generalization of a classical dynamical system. Systems with impulses may present many interesting and unexpected phenomena such as 'beating', 'merging' and 'noncontinuation of solutions'. Since an impulsive system admits abrupt perturbations, its dynamical behavior is much richer than that of the corresponding system. Kaul [11] began to investigate limit sets and the periodicity of impulsive orbits. Later, using a discrete dynamical system associated to the given impulsive semidynamical system, he studied recursive properties in [12]. Ciesielski applied his section theory to obtain the continuity of impulsive time functions and stabilities in [4], [5]. The second author of this paper presented some results on the structure of limit sets in [7], [8]. Now, the theory of impulsive systems is an important and flourishing area of investigation.

The aim of this paper is to introduce the notion of generalized recurrence for impulsive dynamical systems. Since there exist impulse effects in the impulsive systems, analogous results to those established by Auslander in [1] in the impulsive case are not true, our examples also show that many new phenomena occur. In this paper, we will define two different prolongational recurrent sets, and show that if the phase space is a locally compact separable metric space, the generalized recurrent set is the same as the quasi prolongational recurrent set (for definition, see Section 4).

This paper is organized as follows. In Section 2, we recall the definition of an impulsive dynamical system, and fix some notations that will be used in the sequel. In Section 3, following Auslander, we use Lyapunov functions of an impulsive system to define the generalized recurrence. Finally, in Section 4, we introduce the high prolongations and present their fundamental properties, which lead to a variant characterization of generalized recurrence.

2. Impulsive dynamics

Throughout the paper, $X = (X, d)$ denotes a metric space with metric d. For a subset $A \subseteq X$, $\overline{A}$ denotes the closure of A. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ be the open ball with center x and radius $r > 0$. Let $\mathbb{R}$ be the real line, and $\mathbb{R}^+$ be the subset of $\mathbb{R}$ consisting of nonnegative real numbers.

A *semidynamical system* (or *semiflow*) on X is a triple $(X, \pi, \mathbb{R}^+)$, where π is a continuous mapping from $X \times \mathbb{R}^+$ onto X satisfying the following axioms: