Topological Methods in Nonlinear Analysis
Volume 50, No. 1, 2017, 125–150
DOI: 10.12775/TMNA.2017.020

WEAK FORMS OF SHADOWING IN TOPOLOGICAL DYNAMICS

Danila Cherkashin — Sergey Kryzhevich

Abstract. We consider continuous maps of compact metric spaces. It is proved that every pseudotrajectory with sufficiently small errors contains a subsequence of positive density that is point-wise close to a subsequence of an exact trajectory with the same indices. Also, we study homeomorphisms such that any pseudotrajectory can be shadowed by a finite number of exact orbits. In terms of numerical methods this property (we call it multishadowing) implies possibility to calculate minimal points of the dynamical system. We prove that for the non-wandering case multishadowing is equivalent to density of minimal points. Moreover, it is equivalent to existence of a family of ε-networks ($\varepsilon > 0$) whose iterations are also ε-networks. Relations between multishadowing and some ergodic and topological properties of dynamical systems are discussed.

2010 *Mathematics Subject Classification.* Primary: 37B05; Secondary: 37B20, 37A05.

Key words and phrases. Topological dynamics; minimal points; invariant measure; shadowing; chain recurrence; ε-networks; syndetic sets.

Danila Cherkashin was supported by the Saint-Petersburg Mathematical Society (Rokhlin Prize), by the Banach Center of Polish Academy of Sciences and by Saint-Petersburg University, grant 6.37.208.2016.

Sergey Kryzhevich was supported by the Russian Foundation for Basic Research, grants 14-01-00202 and 15-01-03797-a, by the Fulbright Visiting Scholar Program and by Saint-Petersburg State University, grant 6.60.1355.2016.

Both coauthors acknowledge Saint-Petersburg State University for research grant 6.38.223.2014.

1. Introduction

Shadowing is a very important property of dynamical systems, closely related to problems of structural stability and modelling. For review on general shadowing theory we refer to [27], [35]–[37]. Though the most evident application of shadowing is related to numerical methods, first results involving the concept of pseudotrajectories were obtained by Anosov [2], Bowen [11] and Conley [13] as a tool to study qualitative properties of dynamical systems.

In a nutshell, shadowing is existence of an exact trajectory point-wise near a given pseudotrajectory, i.e. a trajectory with errors. This property is closely related to structural stability. Indeed, it is well known that structural stability implies shadowing [44], [48]. Such shadowing is Lipschitz [38]. Sakai [46] demonstrated that the C^1-interior of the set of all diffeomorphisms with shadowing coincides with the set of all structurally stable diffeomorphisms. Osipov, Pilyugin and Tikhomirov [34], [38] showed that the so-called Lipschitz periodic shadowing property is equivalent to Ω-stability, see also [36]. Moreover, the corresponding set of dynamical systems coincides with the interior of the set of systems with periodic shadowing property and with the set of systems with orbital limit shadowing property. Pilyugin and Tikhomirov [42] proved that Lipschitz shadowing is equivalent to structural stability.

Shadowing is not C^1-generic. Bonatti, Diaz and Turcat [10] provided a C^1-open set of diffeomorphisms of the 3-torus where none of diffeomorphisms satisfies the shadowing property. Yuan and Yorke [51] established a similar result for C^r-diffeomorphisms ($r > 1$). Surprisingly, shadowing is generic in the C^0-topology of homeomorphisms of a smooth manifold. This was proved by Pliyugin and Plamenevskaya [39]. Similar results were obtained for continuous mappings of manifolds [25], [30] and for continuous maps of Cantor set [5]. This fact inspired studying shadowing by means of topological dynamics. This approach gave many important results mostly obtained in the last two decades. Mai and Ye [28] demonstrated that odometers have shadowing. This is the only example of such type infinite minimal systems. Of course, there are many non-minimal infinite systems with shadowing, e.g. the Bernoulli shift. On the other hand, Moothathu [31] proved that minimal points are dense for every non-wandering system with shadowing. Moothathu and Oprocha [32] demonstrated that non-wandering systems with shadowing have a dense set of regularly recurrent points. Dastjerdi and Hosseini [14], [15] studied "almost identical" mappings. They proved that if a chain transitive dynamical system has an equicontinuity point then it is a distal, equicontinuous and minimal homeomorphism (see also [18] and [20]). Thus any transitive system with shadowing is either sensitive or equicontinuous.