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CRITICAL BREZIS–NIRENBERG PROBLEM FOR NONLOCAL SYSTEMS

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Abstract. We deal with the existence of solutions to a critical elliptic system involving the fractional Laplacian operator. We consider the primitive of the nonlinearity interacting with the spectrum of the operator. The one side resonant case is also considered. Variational methods are used to obtain the existence, and our result improves earlier results of the authors.

1. Introduction

Let $s \in (0,1)$, $N > 2s$ and let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain. In this paper, we study the existence of solutions to the following fractional system:

$$
\begin{cases}
(-\Delta)^s u = au + bv + \dfrac{2p}{p+q}\,|u|^{p-2}u|v|^q + 2\xi_1|u|^{p+q-2}u & \text{in } \Omega,\\
(-\Delta)^s v = bu + cv + \dfrac{2q}{p+q}\,|u|^p|v|^{q-2}v + 2\xi_2|v|^{p+q-2}v & \text{in } \Omega,\\
u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{1.1}
$$

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where $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x\in\mathbb{R}^N,$$

where $C(N,s)$ is a suitable positive normalization constant, $\xi_1,\xi_2>0$ and $p,q>1$ are constants such that $p+q=2_s^* := 2N/(N-2s)$ denotes the fractional critical Sobolev exponent. By a solution (u,v) to (1.1) we shall always mean a weak solution. Under suitable assumptions, one can also obtain a solution in the viscosity and in the strong sense, as described in [17].

It is convenient to rewrite system (1.1) in the vector and matrix forms such as

$$\begin{cases} (-\overrightarrow{\Delta})^s U = AU + \nabla F(U) & \text{in } \Omega,\\ U=0 & \text{on } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.2}$$

where

$$U^t = \begin{pmatrix} u\\ v\end{pmatrix} \in M_{2\times 1}(\mathbb{R}), \qquad (-\overrightarrow{\Delta})^s U^t = \begin{pmatrix} (-\Delta)^s u\\ (-\Delta)^s v\end{pmatrix},$$

$$A = \begin{pmatrix} a & b\\ b & c\end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

$$F(U) = \frac{2}{p+q}\big(|u|^p|v|^q + \xi_1|u|^{p+q} + \xi_2|v|^{p+q}\big),$$

and ∇ is the gradient operator.

We shall denote by $0<\lambda_{1,s}<\lambda_{2,s}\le\lambda_{3,s}\le\ldots$ the sequence of eigenvalues of the operator $(-\Delta)^s$ with homogeneous Dirichlet boundary datum (that is, $((-\Delta)^s, X(\Omega))$, where $X(\Omega) := \{u\in H^s(\mathbb{R}^N) : u=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\}$), and by μ_1 and μ_2 the eigenvalues of the symmetric matrix A given above. Without loss of generality, we may assume $\mu_1\le\mu_2$.

When $\mu_2<\lambda_{1,s}$, system (1.1) is related to the seminal paper [2], where the authors showed that the critical growth semi-linear problem

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

admits a solution provided that $\lambda\in(0,\lambda_1)$ and $N\ge 4$, λ_1 being the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition and $2^* = 2N/(N-2)$. Furthermore, in dimension $N=3$, the same existence result holds provided that $\mu<\lambda<\lambda_1$, for a suitable $\mu>0$. After that, considerable attention has been paid to (1.3) throughout the years. Later on, in 1984, Cerami,