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ALMOST PERIODIC SOLUTIONS OF EVOLUTION EQUATIONS

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Abstract. We state existence theorems for almost periodic solutions of evolution problems, namely, quasi-autonomous problems and more generally, time dependent evolution equations. We apply these theorems firstly, to a boundary value quasilinear hyperbolic equation of first order, and secondly, to a boundary value quasi-parabolic equation.

1. Introduction

In this paper we draw general conditions of existence for almost periodic solutions of evolution problems in a real Banach space X.

Firstly, we prove (in Section 3) the existence of almost periodic solutions of the following quasi-autonomous evolution problem:

$$\mathrm{QP}(f) \qquad \frac{du}{dt}(t) \in Au(t) + f(t), \quad t \in \mathbb{R},$$

where $f\colon \mathbb{R} \to X$ is an almost periodic function and $A\colon X \to X$ a multivalued nonlinear densely defined operator such that $A+\omega I$ is, for some $\omega > 0$, dissipative with compact resolvent.

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Secondly, we extend (in Section 4) this existence result for almost periodic solutions to abstract evolution equations (EV) governed by suitable families $(A(t))_{t\in\mathbb{R}}$ of nonlinear multivalued operators from X to X

$$\text{(EV)} \qquad \frac{du}{dt}(t) \in A(t)u, \quad t \in \mathbb{R}.$$

Finally, these theoretical results are applied to two examples of boundary value problems (Sections 5 and 6).

A continuous function $f\colon \mathbb{R} \to X$ is said to be almost periodic if for each $\varepsilon > 0$, the set of almost ε periods is relatively dense in $\mathbb{R}$. An almost ε period of f is a real number p satisfying

$$\text{(1.1)} \qquad \sup_{t\in\mathbb{R}} \|f(t+p) - f(t)\| \leq \varepsilon.$$

A subset $D \subset \mathbb{R}$ is said to be relatively dense if there exists an $l > 0$ such that any interval $[a, a+l]$ has a nonempty intersection with D.

An equivalent definition is that a continuous function f is almost periodic if and only if the set of translated functions $T_s f\colon t \mapsto f(s+t)$ is precompact in $C_{\mathrm{b}}(\mathbb{R}, X)$, the set of bounded functions from $\mathbb{R}$ to X endowed with the supremum norm.

Evidently, a periodic function is an almost periodic function. Almost periodic functions arise naturally in vibrating phenomena (as superposition of harmonics with frequencies that are not all multiple of the same fundamental frequency).

The existence theorems given in this paper complement for instance the existence of almost periodic trajectories given in the autonomous case, in [12] or in [1] and [2] (where X is additionally assumed to be a Hilbert space), associated with autonomous equations $\dot{u}(t) = Au(t)$, $t \in \mathbb{R}^+$, generating suitable semigroups of contractions. More generally, in this paper, we shall consider time dependent problems involving evolution operators $S(s,t)$.

It was underlined in [13] that in the quasi-autonomous case $\mathrm{QP}(f)$, the existence of almost periodic solutions is an open problem. We provide an answer to this question when $A+\omega I$ is, for some $\omega > 0$, dissipative with compact resolvent.

Similar problems were studied by many researchers (see for example [4], [19], [17], [14] and [3]). The closest one is [4]. But there the conditions are formulated in terms of Yosida's approximation, so it seems to be difficult to apply the result of [4] to problems, which we consider in the present paper. In the present paper, conditions are imposed on the operator itself.

We generalize also, in particular, the existence result for almost periodic solutions obtained in [20] in the non-autonomous case, to the special case where the state space X is a Hilbert space and the operator $(t,u) \mapsto A(t,u)$ is uniformly continuous in u and $(-\omega)$-dissipative in t, with $\omega > 0$.