Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 2, 2017, 817–833
DOI: 10.12775/TMNA.2017.011

LOCALIZATION OF POSITIVE CRITICAL POINTS IN BANACH SPACES AND APPLICATIONS

Radu Precup — Csaba Varga

Abstract. Two critical point theorems of M. Schechter in a ball of a Hilbert space are extended to uniformly convex Banach spaces by exploiting the properties of the duality mapping. Moreover, the critical points are sought in the intersection of a ball with a wedge, in particular with a cone, making possible applications to positive solutions of variational problems. The extension from Hilbert to Banach spaces not only requires a major refining of reasoning, but also a different statement by adding a third possibility to the original two alternatives from Schechter's results. The theory is applied to positive solutions for p-Laplace equations.

1. Introduction

Fixed point theory offers a large number of useful methods for the study of nonlinear equations. Such a method is the Leray–Schauder continuation principle, see [8], [11], consisting in embedding the original equation in a one-parameter family of equations in a such way that the solution of a simpler equation can be continued inside a given set until a solution of the initial equation. This continuation process is guaranteed by the robustness of the simpler equation and

2010 *Mathematics Subject Classification.* 58E05, 47J30, 35J60.

Key words and phrases. Critical point; mountain pass lemma; positive solution; p-Laplace equation.

The first author, Radu Precup, was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0094. The research of Csaba Varga has been supported by a grant of Babeş-Bolyai University, "Granturi pentru Susţinerea Cercetării de Excelenţă", project number GSCE-30252/2015.

by a condition making closed the boundary of the given set. Here is a simple version of the Leray–Schauder continuation principle.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space, $R > 0$ and let X_R be the closed ball of radius R centered in the origin. Assume that $N\colon X_R \to X$ is a compact map such that the following Leray–Schauder boundary condition is satisfied*:

$$u \neq \lambda N(u) \quad \text{for } \|u\| = R, \text{ and } \lambda \in (0,1). \tag{1.1}$$

Then N has at least one fixed point in X_R.

This theorem allows us to obtain the existence and localization of a solution of the equation

$$N(u) = u, \tag{1.2}$$

in the Banach space X, for a completely continuous operator $N\colon X \to X$, via the so-called "a priori bounds" technique. Indeed, if there exists a number $R > 0$ such that all the solutions $u \in X$ of the equations $\lambda N(u) = u$ for $\lambda \in (0,1)$ are a priori bounded by R, i.e. $\|u\| < R$, then the condition (1.1) is trivially satisfied and thus, according to Theorem 1.1, the equation (1.2) has at least one solution satisfying $\|u\| \leq R$. There is a huge literature devoted to the applications of the Leray–Schauder continuation principle to lots of classes of nonlinear problems, see [11], [12]. Variational versions of the Leray–Schauder principle are due to Schechter [15], [16] (for the role of the Leray–Schauder boundary condition in critical point theory, see also [13]). This kind of results allows to establish the existence and localization of solutions to (1.2), of a precise level of energy, when the equation (1.2) has a variational form, i.e. $N(u) = u - E'(u)$ for some C^1 (energy) functional $E\colon X \to \mathbb{R}$, where X is a Hilbert space identified to its dual and with inner product $(\,\cdot\,;\,\cdot\,)$. Clearly, in this case, the fixed points of the operator N coincide with the critical points of the functional E. In order to recall Schechter's results, we introduce some notions and notations.

We say that a C^1 functional $E\colon X_R \to \mathbb{R}$ satisfies the *Schechter–Palais–Smale condition at the level* λ, $(\mathrm{SPS})_\lambda$ for short, in X_R if any sequence of elements $u_k \in X_R \setminus \{0\}$ for which

$$E(u_k) \to \lambda, \qquad E'(u_k) - \frac{(E'(u_k); u_k)}{\|u_k\|^2}\, u_k \to 0, \qquad (E'(u_k); u_k) \to \nu \leq 0$$

as $k \to \infty$, has a convergent subsequence.

We say that the functional $E\colon X_R \to \mathbb{R}$ satisfies the *mountain pass geometry* in X_R if there are elements $u_0, u_1 \in X_R$ and a number $r > 0$ such that $\|u_0\| < r < \|u_1\|$ and

$$\inf\{\, E(u) : \|u\| = r\} > \max\{E(u_0), E(u_1)\}.$$