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HÉNON TYPE EQUATIONS WITH ONE-SIDED EXPONENTIAL GROWTH

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Abstract. We deal with the following class of problems:

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u_+) + f(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1 is the unit ball in $\mathbb{R}^2$, g is a C^1-function in $[0,+\infty)$ which is assumed to be in the subcritical or critical growth range of Trudinger–Moser type and $f \in L^\mu(B_1)$ for some $\mu > 2$. Under suitable hypotheses on the constant λ, we prove existence of at least two solutions to this problem using variational methods. In case of f radially symmetric, the two solutions are radially symmetric as well.

1. Introduction

In this paper we study the solvability of problems of the type

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u_+) + f(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1}$$

where $\lambda, \alpha \geq 0$ and $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. Here we assume that g has the maximum growth which allows us to treat problem (1.1) variationally in

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suitable Sobolev spaces, due to the well-known Trudinger–Moser inequality (see [18], [28]), which, in two dimensions, is given by

$$\sup_{\substack{u\in H_0^1(B_1)\\ \|\nabla u\|_2=1}} \int_{B_1} e^{\beta u^2}\,dx \begin{cases} < +\infty & \text{if } \beta \le 4\pi, \\ = +\infty & \text{if } \beta > 4\pi. \end{cases} \tag{1.2}$$

Working with a Hénon type problem in $H^1_{0,\mathrm{rad}}(B_1) \subset H_0^1(B_1)$, we observe that the weight $|x|^\alpha$ changes this fact. Indeed, one has

$$\sup_{\substack{u\in H_{0,\mathrm{rad}}^1(B_1)\\ \|\nabla u\|_2=1}} \int_{B_1} |x|^\alpha e^{\beta u^2}\,dx \begin{cases} < +\infty & \text{if } \beta \le 2\pi(2+\alpha), \\ = +\infty & \text{if } \beta > 2\pi(2+\alpha), \end{cases} \tag{1.3}$$

see [3] and [8]. Motivated by (1.2)–(1.3), we say that g has subcritical growth at $+\infty$ if

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta, \tag{1.4}$$

and g has critical growth at $+\infty$ if there exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta > \beta_0; \qquad \lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = +\infty \quad \text{for all } \beta < \beta_0. \tag{1.5}$$

1.1. Hypotheses. Before stating our main results, we shall introduce the following assumptions on the non-linearity g:

(g_0) $g \in C(\mathbb{R}, \mathbb{R}^+)$, $g(s) = 0$ for all $s \le 0$.

(g_1) There exist s_0 and $M > 0$ such that

$$0 < G(s) = \int_0^s g(t)\,dt \le Mg(s) \quad \text{for all } s > s_0.$$

(g_2) $|g(s)| = o(|s|)$ when $|s| \to 0$.

Following the well-established notation in the present literature, we denote by $\lambda_1 < \lambda_2 \le \ldots \le \lambda_j \le \ldots$ the sequence of eigenvalues of $(-\Delta, H_0^1(B_1))$, and by ϕ_j a j^{th} eigenfunction of $(-\Delta, H_0^1(B_1))$.

We observe that, using assumption (g_0), one can see that ψ is a non-positive solution to (1.1) if and only if it is a non-positive solution to the linear problem

$$\begin{cases} -\Delta\psi = \lambda\psi + f(x) & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1. \end{cases} \tag{1.6}$$

In order to get such solutions to (1.6), let us assume that

(f_1) $f(x) = h(x) + t\phi_1(x)$, where $h \in L^\mu(B_1)$, $\mu > 2$ and $\int_{B_1} h\phi_1\,dx = 0$.

For that matter, the parameter t plays a crucial role. We shall use this hypothesis in the first theorem of this paper.