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EXISTENCE THEORY FOR QUASILINEAR ELLIPTIC EQUATIONS VIA A REGULARIZATION APPROACH

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Abstract. In this paper, we further develop a regularization approach initiated in our earlier work for the study of solution structure of quasilinear elliptic equations containing several special cases of mathematical models.

1. Introduction

We consider the following quasilinear elliptic equation:

$$
\begin{cases} \displaystyle\sum_{i,j=1}^{N} D_j(a_{ij}(x,u)D_i u) \\ \qquad \displaystyle -\frac{1}{2}\sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_i u D_j u + f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain,

$$
D_i = \frac{\partial}{\partial x_i}, \qquad D_s a_{ij}(x,s) = \frac{\partial}{\partial s}\, a_{ij}(x,s), \qquad a_{ij} = a_{ji}.
$$

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The weak form of equation (1.1) means to look for $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ satisfying

$$\int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u)D_iuD_j\varphi + \frac{1}{2}\,D_s a_{ij}(x,u)D_iuD_ju\varphi\bigg)\,dx = \int_\Omega f(x,u)\varphi\,dx, \tag{1.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. Formally the problem has a variational structure given by the functional

$$I(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega F(x,u)\,dx,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

A well-known example is the case of $a_{ij}(x,u) = (1+u^2)\delta_{ij}$ with the related evolution equation called the Modified Nonlinear Schrödinger Equation:

$$-i\,\frac{\partial\Phi}{\partial t} = \Delta\Phi + \frac{1}{2}\,\Phi\Delta|\Phi|^2 + |\Phi|^{q-2}\Phi. \tag{1.3}$$

Quasilinear Schrödinger equations of the form (1.1) with different growth conditions in u for $a_{ij}(x,u)$ appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinearity (e.g. [4], [14], [6], [7], [20], [24], [25], [37] for models in an ultrashort high-intensity laser pulse, in nanotubes fullerenes, in super fluid films, and in laser–plasma interactions).

In the last two decades there has been considerable interest in investigating both the stationary case and the evolutionary case ([2], [3], [8], [9], [12]–[18], [21], [23], [26]–[38]). The local and global existence for Cauchy problems of the evolutionary equations and stability issues for standing waves has been studied in many papers (e.g. [12], [13], [16], [17], [22], [23], [26], [34] and references therein). The quasilinear equation is a correction to the classical semilinear Schrödinger equation in some cases and the quasilinear term plays a stabilizing force for solitary wave solutions. In recent papers [12], [13], [16] it is confirmed that the quasilinear modification indeed stabilizes the solution structure in the sense that the quasilinear growth term raises the stability threshold for the nonlinearity. More precisely, in order to have the standing wave solutions stable, in the semilinear case the threshold for the nonlinearity $f(x,u) = |u|^{q-2}u$ is $2 < q < 2+4/N$ (e.g. [10], [11]) while the threshold for the MNLS is $2 < q < 4+4/N$ due to the presence of the correction ([13], [16]).

The stationary case and standing wave solutions have been intensively studied in recent years (e.g. [2], [15], [18], [19], [27]–[33], [35], [38] and references therein). Compared with the semilinear counterpart difficulties for quasilinear equations lie in the fact that the variational functional is not smooth in the natural variational space H_0^1 and does not have compactness property in any spaces smaller than H_0^1 ([3], [9]). Making it more challenging is the new critical exponents due to the quasilinear growth (e.g. [30]). There have been developed several approaches, for example, minimization with constraints and Nehari manifold (e.g. [2], [29], [35]) both of which do not use much of smoothness of the variational functional but are not suitable for dealing with multiple existence of bound states. A change of variable idea was first used in [15], [28] for the MNLS, which effectively transforms the quasilinear problem to a non-standard semilinear problem for which many techniques for semilinear equations can be adopted. However this approach relies heavily on the special form of the quasilinear term a_{ij} being a scalar matrix and does not work for more general equations of the form (1.1). Finally, in the last several years, the authors of the current paper have proposed and successfully implemented a regularization approach ([27], [31]–[33]) for the systematical investigation of the solution structure of more general quasilinear equations of the form (1.1). Our existence theory provides evidences showing that the quasilinear model is a legitimate one and is quite stable in solution structures as our results allow global perturbations of the special model problem MNLS. Our program of studies is quite effective for several most concerned issues such as multiple existence of bound state solutions, multiple existence in the case of finite potentials, new critical exponent problems, etc. In this paper we continue the regularization program developed in [27], [32], [33]. We will consider more general cases such as the quasilinear term of exponential growth for which our earlier results do not apply yet.

Our existence results for equations (1.1) cover special cases like the following equation with more general h (in this case $a_{ij}(x,s) = (1+h^2(s))\delta_{ij}$):

$$\begin{cases} \Delta u + h(u)\Delta H(u) + f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $H(s) = \int_0^s h(t)\,dt$. The results for equations (1.1) will be modelled on conditions for equations like (1.4), and the quasilinear term $a_{ij}(x,s)$ will be considered as a perturbation of $(1+h^2(s))$ (though it is a global perturbation).

In this paper, we consider the existence of weak solutions to (1.1). We consider two classes of problems depending upon the growth rates of the quasilinear terms. We make the following assumptions on h, f and a_{ij}:

- (h_1) $h \in C(\mathbb{R},\mathbb{R})$, $sh(s) > 0$ for $s \neq 0$, h is increasing, $|h(s)| \leq c|s|^\beta$, $sh(s)/H(s) \leq c$ for $s \in \mathbb{R}$ and some $c, \beta > 0$.
- (f_1) $f \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$. There exists $r \in (2, 2N/(N-2))$ such that
$$|f(x,s)| \leq c(1 + H^{r-1}(s)|h(s)|), \quad \text{for } (x,s) \in \overline{\Omega}\times\mathbb{R}.$$
- (f_2) $\lim\limits_{s\to 0} f(x,s)/s = 0$ uniformly in $x \in \overline{\Omega}$.

(f_3) $\lim_{s\to\infty} f(x,s)/s = +\infty$ uniformly in $x \in \overline{\Omega}$. There exists $p > 2(\beta+1)$ such that
$$\frac{1}{p}\, sf(x,s) - F(x,s) \geq -c \quad \text{for } (x,s) \in \overline{\Omega} \times \mathbb{R}.$$

(a_1) $a_{ij}, D_s a_{ij} \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. There exist $c_1, c_0 > 0$ such that
$$c_1(1+h^2(s))|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \leq c_2(1+h^2(s))|\xi|^2$$
for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}$ and $\xi = (\xi_i) \in \mathbb{R}^N$.

(a_2) There exists $\delta > 0$ such that for $(x,s) \in \overline{\Omega} \times \mathbb{R}$, $\xi \in \mathbb{R}^N$
$$\begin{aligned}\delta \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \leq \sum_{i,j=1}^{N} \bigg(a_{ij}(x,s) + \frac{1}{2}\, sD_s a_{ij}(x,s)\bigg)\xi_i\xi_j& \\ \leq \bigg(\frac{p}{2} - \delta\bigg) \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j&.\end{aligned}$$

Here is our main theorem.

Theorem 1.1. *Assume* (h_1), (f_1)–(f_3), (a_1), (a_2) *hold. Then problem* (1.1) *has a nontrivial weak solution.*

We first give two typical examples of direct applications of Theorem 1.1.

Example 1.2. $h(s) = |s|^{\beta-1}s, \beta > 0$; $H(s) = |s|^{\beta+1}/(\beta+1)$, $sh(s)/H(s) = \beta+1$. Assume (f_1)–(f_3), (a_1), (a_2) with $h(s) = |s|^{\beta-1}s$, $\beta > 0$, $p > 2(\beta+1)$. Then problem (1.1) has a nontrivial solution.

Example 1.3. $h(s) = s/\sqrt{1+s^2}$; $H(s) = \sqrt{1+s^2} - 1$, $sh(s)/H(s) = 1 + 1/\sqrt{1+s^2} \leq 2$. Assume ($f_1$)–($f_3$), ($a_1$), ($a_2$) with $h(s) = s/\sqrt{1+s^2}$. Then problem (1.1) has a nontrivial solution. Note that in this case we can use any positive number $\beta > 0$ in (h_1) and (f_3), therefore $p > 2$ suffices.

In Theorem 1.1, the function h is assumed to be controlled by polynomials. We state another result which allows the exponential growth of h.

We make the following alternative assumptions:

(h_2) $h \in C^1(\mathbb{R},\mathbb{R})$, $h(0) = 0$, h is increasing. There exists $c > 0$ such that $H(s)h'(s)/h^2(s) \leq c$.

(f_3') $\lim_{s\to\infty} f(x,s)/s = +\infty$. There exists $p > 2$ such that
$$\frac{1}{p}\, f(x,s)\,\frac{H(s)}{h(s)} - F(x,s) \geq -c. \tag{1.5}$$

The quasilinear term is assumed to be of the form
$$a_{ij}(x,s) = b_{ij}(x,h(s)), \quad (x,s) \in \overline{\Omega} \times \mathbb{R},\ i,j = 1,\dots,N.$$

We assume

(b_1) There exist c_1, c_2 such that

$$c_1(1+s^2)|\xi|^2 \le \sum_{i,j=1}^N b_{ij}(x,s)\xi_i\xi_j \le c_2(1+s^2)|\xi|^2,$$

for $(x,s)\in\overline{\Omega}\times\mathbb{R}$, $\xi\in\mathbb{R}^N$.

(b_2) $0 \le \sum_{i,j=1}^N sD_s b_{ij}(x,s)\xi_i\xi_j \le 2\sum_{i,j=1}^N b_{ij}(x,s)\xi_i\xi_j$, for $(x,s)\in\overline{\Omega}\times\mathbb{R}$, $\xi\in\mathbb{R}^N$.

(b_3) $\sum_{i,j=1}^N \Big(b_{ij}(x,s) - \dfrac{1}{2}sD_s b_{ij}(x,s)\Big)\xi_i\xi_j = o(s^2)$ as $s\to\infty$ uniformly in $(x,\xi)\in\overline{\Omega}\times S$ where $S=\{\xi \mid \xi\in\mathbb{R}^N,\ |\xi|=1\}$.

THEOREM 1.4. *Assume* (h_2), (f_1), (f_2), (f'_3) *hold. With* $a_{ij}(x,s)=b_{ij}(x,h(s))$, *assume* ($b_1$)–($b_3$) *hold. Then problem* (1.1) *has a nontrivial weak solution.*

EXAMPLE 1.5. Let $h(s)=2se^{s^2}$, $H(s)=e^{s^2}-1$, $b_{ij}=(1+s^2)\delta_{ij}$, $f(x,s)=H^{p-1}(s)h(s)$ and $F(x,s)=H^p(s)/p$. Then this is an example to which Theorem 1.4 applies. In particular, the following equation has a nontrivial solution:

$$\begin{cases} \Delta u + h(u)\Delta H(u) + H^{p-1}(u)h(u) = 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

for $2<p<2N/(N-2)$.

We outline the idea for the regularization approach initiated and developed in our earlier works [27], [32], [33]. Due to the lack of a suitable working space we introduce perturbed functionals which are smooth functionals in a suitable smaller subspace. For $\mu\in(0,1]$ define functionals I_μ on the Sobolev space $W_0^{1,q}(\Omega)$, $q>N$, by

$$\begin{aligned} I_\mu(u) &= \frac{\mu}{2}\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} + I(u) \\ &= \frac{\mu}{2}\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega F(x,u)\,dx. \end{aligned} \tag{1.7}$$

Then it is easy to see that I_μ is a C^1-functional. For $\varphi \in W_0^{1,q}(\Omega)$

$$\begin{aligned}\langle DI_\mu(u),\varphi\rangle =&\mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q-1}\int_\Omega |Du|^{q-2}DuD\varphi\,dx\\&+\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\varphi\,dx\\&+\frac12\int_\Omega \sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\varphi\,dx-\int_\Omega f(x,u)\varphi\,dx.\end{aligned}$$

Then the idea is first to obtain existence of critical points of I_μ for $\mu>0$ by using critical point theory for smooth functionals and then to establish suitable estimates on these critical points so that we can take the limit $\mu\to 0$ to get weak solutions for the original problem. The key ingredient is the convergence results from the perturbed ones to the original one. This step depends on the conditions on h and is somewhat different for the two theorems above.

Remark 1.6. Since we mainly focus on the regularization approach, we consider the existence of nontrivial solutions and do not pursue for multiplicity of solutions. We would like to point out that for the existence of multiple solutions the essential ingredients should be already contained in [27], [32]. With some minor modifications one can easily obtain multiplicity results. We leave details to the interested readers. Also for technical reasons we work here with problems in bounded domains. But our results can be easily extended to the case of the entire space with suitable conditions on the potential functions.

The paper is organized as follows. In Section 2 we prove the convergence theorems for the two situations of the main results, which are the key ingredients in proving the existence results. Section 3 contains proofs of the main theorems.

2. Convergence theorems

In order to carry out our regularization approach we need some convergence estimates. In this section we prove the necessary convergence theorems which will be used later for the existence results. As the proofs are somewhat different for the two situations of the two main theorems, we give the proofs in two subsections below.

2.1. The case of polynomial growth.

Theorem 2.1. *Assume $\mu_n\to 0$, $\{u_n\}\subset W_0^{1,q}(\Omega)$, $I_{\mu_n}(u_n)\le c$, $DI_{\mu_n}(u_n)=0$. Then $\|u_n\|_{L^\infty(\Omega)}\le c$ independently of μ_n. Up to a subsequence*

$$\mu_n\bigg(\int_\Omega |Du_n|^q\,dx\bigg)^{2/q}\to 0,$$

$u_n \to u$ in $H_0^1(\Omega)$ and $I_{\mu_n}(u_n) \to I(u)$ as $n \to \infty$, where $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution to problem (1.1).

Lemma 2.2. *It holds that*

$$\mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} + \int_\Omega (1+h^2(u))|Du|^2\,dx \le c(1+|I_\mu(u)| + \|DI_\mu(u)\|\cdot\|u\|),$$

where the constant c is independent of μ.

Proof. By conditions (f_3), (a_1) and (a_2), we have

$$\begin{aligned} I_\mu(u) - \frac{1}{p}\langle DI_\mu(u), u\rangle &= \bigg(\frac12 - \frac1p\bigg)\mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} \\ &\quad + \int_\Omega \sum_{i,j=1}^N \bigg[\bigg(\frac12 - \frac1p\bigg)a_{ij}(x,u) - \frac{1}{2p}\,uD_s a_{ij}(x,u)\bigg]D_i u D_j u\,dx \\ &\quad + \int_\Omega \bigg(\frac1p\,uf(x,u) - F(x,u)\bigg)\,dx \\ &\ge \bigg(\frac12 - \frac1p\bigg)\mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} + \frac{\delta}{p}\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_i u D_j u\,dx - c \\ &\ge c\mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} + c\int_\Omega (1+h^2(u))|Du|^2\,dx - c. \qquad \square \end{aligned}$$

Lemma 2.3. *Assume $u \in W_0^{1,q}(\Omega)$, $DI_\mu(u) = 0$. Then $u \in L^\infty(\Omega)$, and $\|u\|_{L^\infty(\Omega)} \le c$, the upper bound c depends on $I_\mu(u)$ only.*

Proof. By Lemma 2.2,

$$\int_\Omega |DH(u)|^2\,dx = \int_\Omega h^2(u)|Du|^2\,dx \le c. \tag{2.1}$$

By the Sobolev imbedding theorem, we have $\int_\Omega |H(u)|^{2N/(N-2)}\,dx \le c$, and u satisfies

$$\begin{aligned} \mu\bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q-1} &\int_\Omega |Du|^{q-2}DuD\varphi\,dx \\ + \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_i u D_j\varphi\,dx &+ \frac12\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u)D_i u D_j u\varphi\,dx \\ &= \int_\Omega f(x,u)\varphi\,dx, \end{aligned} \tag{2.2}$$

for all $\varphi \in W_0^{1,q}(\Omega)$. For $k > 1$, set $\varphi = uH^{2k-2}(u)$. Since $q > N$, $W_0^{1,q}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ for some $\alpha > 0$. Using this $\varphi \in W_0^{1,q}(\Omega)$ as a test function in (2.2), and noticing that $sh(s) \geq 0$ for $s \in \mathbb{R}$, we obtain

$$\int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u) + \frac{1}{2}\,uD_s a_{ij}(x,u)\bigg) D_i u D_j u H^{2k-2}(u)\,dx \leq \int_\Omega f(x,u)uH^{2k-2}(u)\,dx. \tag{2.3}$$

By Sobolev and Hölder inequalities, we estimate the terms as follows:

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u) &+ \frac{1}{2}\,uD_s a_{ij}(x,u)\bigg) D_i u D_j u H^{2k-2}(u)\,dx \\ &\geq c\int_\Omega h^2(u)|Du|^2 H^{2k-2}(u)\,dx \\ &\geq \frac{c}{k^2}\int_\Omega |DH^k(u)|^2\,dx \geq \frac{c}{k^2}\bigg(\int_\Omega H^{2k\cdot N/(N-2)}(u)\,dx\bigg)^{(N-2)/N},\end{aligned}$$

and

$$\begin{aligned}\int_\Omega f(x,u)\,uH^{2k-2}(u)\,dx &\leq c\int_\Omega (1+H^{r-1}(u)|h(u)|)|u|H^{2k-2}(u)\,dx \\ &\leq c + c\int_\Omega H^r(u)H^{2k-2}(u)\,dx \\ &\leq c + c\bigg(\int_\Omega H^{2N/(N-2)}(u)\,dx\bigg)^{(r-2)(N-2)/(2N)} \cdot \bigg(\int_\Omega H^{2kd}(u)\,dx\bigg)^{1/d} \\ &\leq c + c\bigg(\int_\Omega H^{2kd}(u)\,dx\bigg)^{1/d},\end{aligned}$$

where $1/d + (r-2)(N-2)/(2N) = 1$. Hence we have

$$\bigg(\int_\Omega H^{2k\cdot N/(N-2)}(u)\,dx\bigg)^{(N-2)/(2kN)} \leq (ck)^{1/k}\bigg(1 + \int_\Omega H^{2kd}(u)\,dx\bigg)^{1/(2kd)}.$$

Notice that $r < 2N/(N-2)$, $d < N/(N-2)$ and $\chi = N/(d(N-2)) > 1$. Choose k_0 such that $2k_0 d = 2N/(N-2)$, $k_0 > 1$. By iterations

$$\begin{aligned}\|H(u)\|_{L^{2k_0 d\chi^j}(\Omega)} &\leq \prod_{i=0}^{j-1}(ck_0\chi^i)^{1/(k_0\chi^i)}(1 + \|H(u)\|_{L^{2k_0 d}(\Omega)}) \\ &\leq c(1 + \|H(u)\|_{L^{2N/(N-2)}(\Omega)}).\end{aligned}$$

Letting $j \to \infty$, we have $\|H(u)\|_{L^\infty(\Omega)} \leq c$ and $\|u\|_{L^\infty(\Omega)} \leq c$, where c is independent of $H(u)$ and depends on $I_\mu(u)$ only. □

Now we give the proof of Theorem 2.1.

Proof. By Lemmas 2.2 and 2.4,

$$\mu_n\bigg(\int_\Omega |Du_n|^q\,dx\bigg)^{2/q} + \int_\Omega |Du_n|^2\,dx \leq c, \quad \|u_n\|_{L^\infty(\Omega)} \leq c_1. \tag{2.4}$$

Assume $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \to u$ for almost every $x \in \Omega$. Note that u_n satisfies the equation

$$\begin{aligned}(2.5)\quad \mu_n\bigg(\int_\Omega |Du_n|^q\,dx\bigg)^{2/q-1}&\int_\Omega |Du_n|^{q-2}Du_nD\varphi\,dx \\ +\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u_n)D_iu_nD_j\varphi\,dx &+ \frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u_n)D_iu_nD_ju_n\varphi\,dx \\ &= \int_\Omega f(x,u_n)\varphi\,dx,\end{aligned}$$

for all $\varphi \in W_0^{1,q}(\Omega)$. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Take $\varphi = \psi e^{-Mu_n}$ in (2.5) as a test function with $M > 0$ to be chosen. We have

$$\begin{aligned}(2.6)\quad \mu_n\bigg(\int_\Omega |Du_n|^q\,dx\bigg)^{2/q-1}&\int_\Omega |Du_n|^{q-2}Du_n(-M\psi Du_n + D\psi)e^{-Mu_n}\,dx \\ +\int_\Omega\sum_{i,j=1}^N \bigg(-Ma_{ij}(x,u_n) &+ \frac{1}{2}\,D_sa_{ij}(x,u_n)\bigg)\psi D_iu_nD_ju_ne^{-Mu_n}\,dx \\ +\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u_n)D_iu_nD_j\psi e^{-Mu_n}\,dx &= \int_\Omega f(x,u_n)\psi e^{-Mu_n}\,dx.\end{aligned}$$

By $\mu_n(\int_\Omega |Du_n|^q dx)^{2/q} \leq c$ we may estimate the first term of (2.6) which tends to zero as $n \to \infty$. For the second term of (2.6) we use Fatou's lemma. In order to use Fatou's lemma, we choose M large enough such that

$$\sum_{i,j=1}^N \bigg(Ma_{ij}(x,s) - \frac{1}{2}\,D_sa_{ij}(x,s)\bigg)\xi_i\xi_j \geq 0, \quad \text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^N,\ |s| \leq c_1.$$

Taking the limit in (2.6), by (2.4) and Fatou's lemma, we have

$$\begin{aligned}(2.7)\quad &\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j(\psi e^{-Mu})\,dx \\ &\qquad + \frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\psi e^{-Mu}\,dx \geq \int_\Omega f(x,u)\psi e^{-Mu}\,dx,\end{aligned}$$

for all $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Given $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$, choose $\{\psi_n\} \subset C_0^\infty(\Omega)$ such that $\psi_n \to \varphi e^{Mu}$ in $H_0^1(\Omega)$, $\psi_n(x) \to \varphi(x)e^{Mu(x)}$ for almost every $x \in \overline{\Omega}$ and $\|\psi_n\|_{L^\infty(\Omega)} \leq c$. Taking ψ_n as a test function in (2.7), we have

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^N a_{ij}(x,u)D_iuD_j\varphi\,dx \\ &\qquad + \frac{1}{2}\int_\Omega\sum_{i,j=1}^N D_sa_{ij}(x,u)D_iuD_ju\varphi\,dx - \int_\Omega f(x,u)\varphi\,dx \geq 0,\end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$. Similarly we have the opposite inequality. By a further approximation we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j \varphi \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u) D_i u D_j u \varphi \, dx - \int_\Omega f(x,u)\varphi \, dx = 0,$$

for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. In particular,

$$\int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u) + \frac{1}{2} D_s a_{ij}(x,u) u \bigg) D_i u D_j u \, dx = \int_\Omega f(x,u) u \, dx.$$

By Lebesgue's dominated convergence theorem,

$$\int_\Omega f(x,u_n) u_n \, dx \to \int_\Omega f(x,u) u \, dx.$$

Hence we have

$$\mu_n \bigg(\int_\Omega |Du_n|^q \, dx \bigg)^{2/q} + \int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u_n) + \frac{1}{2} u_n D_s a_{ij}(x,u_n) \bigg) D_i u_n D_j u_n \, dx \to \int_\Omega \sum_{i,j=1}^N \bigg(a_{ij}(x,u) + \frac{1}{2} D_s a_{ij}(x,u) u \bigg) D_i u D_j u \, dx.$$

By (a_2), $\mu_n \big(\int_\Omega |Du_n|^q \, dx \big)^{2/q} \to 0$, and $u_n \to u$ in $H_0^1(\Omega)$. Since $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$, we have

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x,u_n) D_i u_n D_j u_n \, dx \to \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u) D_i u D_j u \, dx$$

and $I_{\mu_n}(u_n) \to I(u)$. □

2.2. The case of exponential growth. The proof of Theorem 1.4 is based on a somewhat different perturbation, and we need to modify the proof of the convergence theorem. The differences are mainly about the energy bound and L^∞ bound. Instead of the perturbed functional I_μ (see (1.4)), we define a new functional J_μ by

$$\begin{aligned} J_\mu(u) &= \frac{1}{2} \mu \bigg(\int_\Omega (1 + |h(u)|^q) |Du|^q \, dx \bigg)^{2/q} + I(u) \\ &= \frac{1}{2} \mu \bigg(\int_\Omega (1 + |h(u)|^q) |Du|^q \, dx \bigg)^{2/q} \\ &\quad + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u)) D_i u D_j u \, dx - \int_\Omega F(x,u) \, dx. \end{aligned} \tag{2.8}$$

J_μ is a C^1-functional on $W_0^{1,q}(\Omega)$ with $q > N$. For $\varphi \in W_0^{1,q}(\Omega)$

$$\begin{aligned}
\text{(2.9)}\qquad \langle DJ_\mu(u), \varphi\rangle = \mu\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q-1} \\
\cdot \int_\Omega \big((1+|h(u)|^q)|Du|^{q-2}DuD\varphi + |h(u)|^{q-2}h(u)h'(u)|Du|^q\varphi\big)\,dx \\
+ \int_\Omega \sum_{i,j=1}^N (b_{ij}(x,h(u))D_iuD_j\varphi \\
+ \frac{1}{2}\,D_sb_{ij}(x,h(u))h'(u)D_iuD_ju\varphi\,dx - \int_\Omega f(x,u)\varphi\,dx.
\end{aligned}$$

Lemma 2.4. *It holds*

$$\mu\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q} + \int_\Omega (1+|h(u)|^2)|Du|^2\,dx \\
\le c(1+|J_\mu(u)| + \|DJ_\mu(u)\|\cdot\|u\|).$$

Proof. For $u \in W_0^{1,q}(\Omega)$, set $\varphi = H(u)/h(u)$. Then

$$|\varphi| = \bigg|\frac{1}{h(u)}\int_0^u h(s)\,ds\bigg| \le |u|,$$
$$D\varphi = Du\bigg(1 - \frac{H(u)h'(u)}{h^2(u)}\bigg), \qquad |D\varphi| \le c|Du|,$$

hence $\varphi \in W_0^{1,q}(\Omega)$. Taking $\varphi = H(u)/h(u)$ in (2.9) as a test function, we have

$$\begin{aligned}
\text{(2.10)}\quad \bigg\langle DJ_\mu(u), \frac{H(u)}{h(u)}\bigg\rangle = \mu\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q} \\
- \mu\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q-1}\int_\Omega |Du|^q\,\frac{H(u)h'(u)}{h^2(u)}\,dx \\
+ \int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u))D_iuD_ju\,dx \\
- \int_\Omega \sum_{i,j=1}^N \bigg(b_{ij}(x,h(u)) - \frac{1}{2}\,h(u)D_sb_{ij}(x,h(u))\bigg)D_iuD_ju\,\frac{H(u)h'(u)}{h^2(u)}\,dx \\
- \int_\Omega f(x,u)\,\frac{H(u)}{h(u)}\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
c(|J_\mu(u)| + \|DJ_\mu(u)\|\,\|u\|) \ge J_\mu(u) - \frac{1}{p}\bigg\langle DJ_\mu(u), \frac{H(u)}{h(u)}\bigg\rangle \\
= \bigg(\frac{1}{2} - \frac{1}{p}\bigg)\mu\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q} \\
+ \frac{\mu}{p}\bigg(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\bigg)^{2/q-1}\int_\Omega |Du|^q\,\frac{H(u)h'(u)}{h^2(u)}\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u))D_iuD_ju\,dx \\
&+\frac{1}{p}\int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u))D_iuD_ju\,\frac{H(u)h'(u)}{h^2(u)}\,dx \\
&-\frac{1}{2p}\int_\Omega \sum_{i,j=1}^N h(u)D_sb_{ij}(x,h(u))D_iuD_ju\,\frac{H(u)h'(u)}{h^2(u)}\,dx \\
&+\int_\Omega\left(\frac{1}{p}\,f(x,u)\,\frac{H(u)}{h(u)}-F(x,u)\right)dx \\
\geq{}&\left(\frac{1}{2}-\frac{1}{p}\right)\mu\left(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\right)^{2/q} \\
&+\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u))D_iuD_ju\,dx-c \\
\geq{}& c\mu\left(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\right)^{2/q}+c\int_\Omega (1+h^2(u))|Du|^2\,dx-c. \qquad \square
\end{aligned}
$$

Lemma 2.5 (L^∞-bound). *Assume u is a critical point of J_μ. Then $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \leq c$, where c depends on $J_\mu(u)$ only.*

Proof. Assume $DJ_\mu(u)=0$, $J_\mu(u)\leq c$. By Lemma 2.4,

$$
\mu\left(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\right)^{2/q}+\int_\Omega (1+h^2(u))|Du|^2\,dx \leq c. \tag{2.11}
$$

By Sobolev's imbedding theorem,

$$
\begin{aligned}
\left(\int_\Omega |H(u)|^{2N/(N-2)}\,dx\right)^{(N-2)/N} \\
\leq c\int_\Omega |DH(u)|^2\,dx = c\int_\Omega h^2(u)|Du|^2\,dx \leq c.
\end{aligned} \tag{2.12}
$$

We note that u satisfies the equation

$$
\begin{aligned}
\mu&\left(\int_\Omega (1+|h(u)|^q)|Du|^q\,dx\right)^{2/q-1} \\
&\cdot\int_\Omega \big((1+|h(u)|^q)|Du|^{q-2}DuD\varphi+|h(u)|^{q-2}h(u)h'(u)|Du|^q\varphi\big)\,dx \\
&+\int_\Omega \sum_{i,j=1}^N b_{ij}(x,h(u))D_iuD_j\varphi\,dx \\
&+\frac{1}{2}\int_\Omega D_sb_{ij}(x,h(u))h'(u)D_iuD_ju\varphi\,dx=\int_\Omega f(x,u)\varphi\,dx,
\end{aligned} \tag{2.13}
$$

for $\varphi \in W_0^{1,q}(\Omega)$. For $k > 0$, define

$$\varphi = \begin{cases} uH^{2k-2}(u) & \text{for } |u| \le M, \\ c_+ \dfrac{H^{2k-1}(u)}{h(u)} & \text{for } u > M, \\ c_- \dfrac{H^{2k-1}(u)}{h(u)} & \text{for } u < -M, \end{cases}$$

where $c_+ = Mh(M)/H(M)$, $c_- = -Mh(-M)/H(-M)$, and $M > 0$ is to be chosen. Since $q > N$, $W_0^{1,q}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, $\varphi \in W_0^{1,q}(\Omega)$. Taking φ as a test function in (2.13). We estimate the terms on the left and right hand sides of (2.13) as follows:

$$\begin{aligned} &\text{LHS of (2.13)} \\ &\ge \int_{|u|\le M} \sum_{i,j=1}^N b_{ij}(x,h(u)) D_i u D_j u H^{2k-2}(u)\,dx \\ &\quad + c_+ \int_{u>M} \sum_{i,j=1}^N b_{ij}(x,h(u)) D_i u D_j u H^{2k-2}(u)\,dx \\ &\quad - c_+ \int_{u>M} \sum_{i,j=1}^N \Big(b_{ij}(x,h(u)) - \frac{1}{2} h(u) D_s b_{ij}(x,h(u)) \Big) \\ &\qquad\qquad \cdot \frac{H(u)h'(u)}{h^2(u)} D_i u D_j u H^{2k-2}(u)\,dx \\ &\quad + c_- \int_{u<-M} \sum_{i,j=1}^N b_{ij}(x,h(u)) D_i u D_j u H^{2k-2}(u)\,dx \\ &\quad - c_- \int_{u<-M} \sum_{i,j=1}^N \Big(b_{ij}(x,h(u)) - \frac{1}{2} h(u) D_s b_{ij}(x,h(u)) \Big) \\ &\qquad\qquad \cdot \frac{H(u)h'(u)}{h^2(u)} D_i u D_j u H^{2k-2}(u)\,dx. \end{aligned}$$

Recall that $H(s)h'(s)/h^2(s) \le c$ for $s \in \mathbb{R}$ from (h_2). Due to (b_3) and the homogeneity of the quadratic expression, by choosing $M > 0$ large enough, we have for $|t| \ge M$

$$\begin{aligned} c \sum_{i,j=1}^N \Big(b_{ij}(x,h(t)) - \frac{1}{2} h(t) D_s b_{ij}(x,h(t)) \Big) \xi_i \xi_j \\ \le \frac{1}{2} \sum_{i,j=1}^N b_{ij}(x,h(t)) \xi_i \xi_j, \end{aligned} \tag{2.14}$$

for $\xi \in \mathbb{R}^N$. It follows from (2.14) that

$$
\begin{aligned}
&\text{LHS of (2.13)} \\
&\geq c \int_\Omega \sum_{i,j=1}^N b_{ij}(x, h(u)) D_i u D_j u H^{2k-2}(u)\,dx \\
&\geq c \int_\Omega h^2(u) |Du|^2 H^{2k-2}(u)\,dx = \frac{c}{k^2} \int_\Omega |DH^k(u)|^2\,dx \\
&\geq \frac{c}{k^2} \bigg(\int_\Omega H^{2kN/(N-2)}(u)\,dx \bigg)^{(N-2)/N}.
\end{aligned}
$$

On the other hand, we have the estimate on the left hand side of (2.13) as follows:

$$
\begin{aligned}
&\text{RHS of (2.13)} \\
&\leq c \int_{|u|\leq M} (1 + H^{r-1}(u)|h(u)|)|u| H^{2k-2}(u)\,dx \\
&\quad + c \int_{|u|\geq M} (1 + H^{r-1}(u)|h(u)|)\, \frac{H^{2k-1}(u)}{|h(u)|}\,dx \\
&\leq c\bigg(1 + \int_\Omega H^{2k+r-2}(u)\,dx\bigg) \\
&\leq c\bigg(1 + \bigg(\int_\Omega H^{2N/(N-2)}\,dx\bigg)^{(r-2)(N-2)/(2N)} \bigg(\int_\Omega H^{2kd}(u)\,dx\bigg)^{1/d}\bigg) \\
&\leq c\bigg(1 + \int_\Omega H^{2kd}(u)\,dx\bigg)^{1/d},
\end{aligned}
$$

where $(r-2)(N-2)/(2N) + 1/d = 1$. Since $r < 2N/(N-2)$, we have $d < N/(N-2)$. It follows from the above two inequalities that

$$
\bigg(\int_\Omega H^{2k\cdot N/(N-2)}(u)\,dx\bigg)^{(N-2)/(2kN)} \leq (ck)^{1/k}\bigg(1 + \int_\Omega H^{2kd}(u)\,dx\bigg)^{1/(2kd)}.
$$

Choose k_0 with $2k_0 d = 2N/(N-2)$, $k_0 > 1$. By standard iteration, we have $\|H(u)\|_{L^\infty(\Omega)} \leq c\|H(u)\|_{L^{2N/(N-2)}(\Omega)} \leq c$, and $\|u\|_{L^\infty(\Omega)} \leq c$, where c depends on $\|H(u)\|_{L^{2N/(N-2)}(\Omega)}$, hence on $J_\mu(u)$ only. □

With the aids of Lemmas 2.4 and 2.5 we may follow the proof of Theorem 2.1 to prove the following convergence result.

Theorem 2.6. *Assume* (h_2), (f_1), (f_2), (f_3') *hold. With* $a_{ij}(x,s) = b_{ij}(x, h^2(s))$ *assume* (b_1), (b_2) *and* (b_3) *hold. Suppose* $\mu_n \to 0$, $\{u_n\} \subset W_0^{1,q}(\Omega)$, $DJ_{\mu_n}(u_n) = 0$, $J_{\mu_n}(u_n) \leq c$. *Then* $\|u_n\|_{L^\infty(\Omega)} \leq c$ *independently of* μ_n. *Up to a subsequence* $\mu_n\big(\int_\Omega |Du_n|^q\,dx\big)^{2/q} \to 0$, $u_n \to u$ *in* $H_0^1(\Omega)$ *and* $J_{\mu_n}(u_n) \to I(u)$ *as* $n \to \infty$ *where* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is a weak solution to* (1.1).

3. Existence theory

3.1. The proof of Theorem 1.1.

Lemma 3.1. *I_μ satisfies the PS condition.*

Proof. Let $\{u_n\} \subset W_0^{1,q}(\Omega)$ be a PS sequence of I_μ, that is, $I_\mu(u_n) \to c$, $\|DI_\mu(u_n)\| \to 0$. By Lemma 2.2,

$$\mu\bigg(\int_\Omega |Du_n|^q\,dx\bigg)^{2/q} + \int_\Omega (1+h^2(u_n))|Du_n|^2\,dx \le c,$$

and $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$. If $\int_\Omega |Du_n|^q\,dx \to 0$, we are done. Otherwise assume $\int_\Omega |Du_n|^q\,dx \to c_2 > 0$. Up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$, $u_n \to u$ in $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$.

$$\begin{aligned}
o(1) &= \langle DI_\mu(u_n) - DI_\mu(u_m), u_n - u_m\rangle \\
&= \mu c_2^{2/q-1}\int_\Omega \big(|Du_n|^{q-2}Du_n - |Du_m|^{q-2}Du_m, Du_n - Du_m\big)\,dx \\
&\quad + \int_\Omega \sum_{i,j=1}^N a_{ij}(x,u)D_i(u_n-u_m)D_j(u_n-u_m)\,dx \\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N D_s a_{ij}(x,u)(D_iu_nD_ju_n - D_iu_mD_ju_m)(u_n-u_m)\,dx \\
&\quad - \int_\Omega (f(x,u_n) - f(x,u_m))(u_n-u_m)\,dx + o(1) \\
&\ge c\int_\Omega |Du_n - Du_m|^q\,dx + c\int_\Omega |Du_n - Du_m|^2\,dx + o(1).
\end{aligned}$$

Hence $u_n \to u$ in $W_0^{1,q}(\Omega)$. □

Lemma 3.2. *I_μ has a nontrivial critical point u with $I_\mu(u) \ge \alpha$ independently of μ.*

Proof. We apply the Mountain Pass Lemma [1]. By (f_1), (f_2) and (a_1),

$$\begin{aligned}
I_\mu(u) &\ge c\int_\Omega (1+h^2(u))|Du|^2\,dx - c\int_\Omega (\varepsilon u^2 + H^r(u))\,dx \\
&\ge c\int_\Omega |DH(u)|^2\,dx - c\int_\Omega H^r(u)\,dx \\
&\ge c_0\bigg(\int_\Omega H^r(u)\,dx\bigg)^{2/r} - c\int_\Omega H^r(u)\,dx.
\end{aligned}$$

Set $D_\rho = \{u \in W_0^{1,q}(\Omega) \mid (\int_\Omega H^r(u)\,dx)^{1/r} \le \rho\}$. Then for $u \in S_\rho = \partial D_\rho$ we have

$$I_\mu(u) \ge c_0\rho^2 - c\rho^r \ge \frac{1}{2}\,c_0\rho^2 := \alpha, \quad \text{if } \rho \text{ small.}$$

On the other hand, by (a_1), (h_2) and (f_3),

$$I_1(u) \le \bigg(\int_\Omega |Du|^q\,dx\bigg)^{2/q} + c\int_\Omega (1+|u|^{2\beta})|Du|^2\,dx - c\int_\Omega |u|^p\,dx + c,$$

hence $I_1(tu) \to -\infty$ as $t \to \infty$. Define

$$c_\mu = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\mu(\gamma(t)),$$

where $\Gamma = \{\gamma \mid \gamma \in C([0,1],\ W_0^{1,q}(\Omega)),\ \gamma(0)=0,\ I_1(\gamma(1))<0\}$. By the Mountain Pass Lemma, for $\mu\in(0,1]$, c_μ is a critical value of I_μ and $c_1 \ge c_\mu \ge \alpha$. $\square$

Proof of Theorem 1.1. By Lemma 3.2, I_μ has a nontrivial critical point u_μ with $0<\alpha\le I_\mu(u_\mu)\le c_1$ for $\mu\in(0,1]$. By Theorem 2.1, for a sequence $\{\mu_n\}$, $\mu_n\to 0$, $u_n\to u$ in $H_0^1(\Omega)$, $I_{\mu_n}(u_n)\to I(u)$ where $u\in H_0^1(\Omega)\cap L^\infty(\Omega)$ is a nontrivial weak solution to (1.1) with $I(u)\ge\alpha>0$. $\square$

3.2. The proof of Theorem 1.4. The proof of Theorem 1.4 is similar to that of Theorem 1.1. We sketch it here and point out necessary modifications.

First of all, J_μ satisfies the PS condition. Indeed, by Lemma 2.4, a PS sequence $\{u_n\}$ of J_μ is bounded in $W_0^{1,q}(\Omega)$. Assume $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ and $u_n\to u$ in $C^\alpha(\overline{\Omega})$ for some $\alpha>0$. Then the proof is similar to that of Lemma 3.1.

Next, as in the proof of Theorem 1.1, we show that J_{μ_n} has a nontrivial critical point u_{μ_n} with $0<\alpha\le J_{\mu_n}(u_n)$. Then by Theorem 2.6, as $\mu_n\to 0$, u_n converges to a solution u to (1.1). We prove the existence of nontrivial solution of J_μ by the Mountain Pass Lemma. Set $B=\big\{u \mid u\in W_0^{1,q}(\Omega), \int_\Omega H^r(u)\,dx \le \rho^r\big\}$. By $(f_1),(f_2)$, $F(x,u)\le \varepsilon u^2 + cH^r(u)$, we have for $u\in\partial B$, if ρ is small enough,

$$\begin{aligned} J_\mu(u) &\ge c\int_\Omega (1+h^2(u))|Du|^2\,dx - \int_\Omega (\varepsilon u^2 + cH^r(u))\,dx \\ &\ge c_1\bigg(\int_\Omega H^r(u)\,dx\bigg)^{2/r} - c_2\int_\Omega H^r(u)\,dx \\ &= c_1\rho^2 - c_2\rho^r \ge \frac{1}{2}\rho^2 := \alpha. \end{aligned} \tag{3.1}$$

Define the variable change $G=G(s)$, $s\ge 0$, by

$$\frac{dG}{ds} = \sqrt{1+h^2(s)}, \qquad G(0)=0.$$

G and its inverse are C^1-functions. Fix $s_0>0$, for $s\ge s_0$, $h(s)\ge h(s_0)>0$ and $\sqrt{1+h^2(s)}\le ch(s)$. We have

$$G(s)-G(s_0) = \int_{s_0}^s \sqrt{1+h^2(t)}\,dt \le c\int_{s_0}^s h(t)\,dt = c(H(s)-H(s_0)),$$

hence $G(s)\le c(H(s)+1)$ and by (1.5)

$$G^p(s) \le c(H^p(s)+1) \le c(F(x,s)+1), \quad \text{for } (x,s)\in\overline{\Omega}\times\mathbb{R}^+.$$

Choose $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$. Define φ_t, $t \geq 0$, by $t\psi = G(\varphi_t)$, $\varphi_t \in C_0^1(\overline{\Omega})$.

$$\begin{aligned}
J_\mu(\varphi_t) &= \frac{1}{2}\,\mu\bigg(\int_\Omega (1 + |h(\varphi_t)|^q)|D\varphi_t|^q\,dx\bigg)^{2/q} \\
&\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^N b_{ij}(x, h(\varphi_t))D_i\varphi_t D_j\varphi_t\,dx - \int_\Omega F(x,\varphi_t)\,dx \\
&\leq G\bigg(\int_\Omega (1 + h^2(\varphi_t))^{q/2}|D\varphi_t|^q\,dx\bigg)^{2/q} \\
&\quad + c\int_\Omega (1 + h^2(\varphi_t))|D\varphi_t|^2\,dx - c\int_\Omega G^p(\varphi_t)\,dx + c \\
&\leq c\bigg(\int_\Omega |tD\psi|^q\,dx\bigg)^{2/q} + c\int_\Omega (1 + h^2(\varphi_t))|D\varphi_t|^2\,dx - c\int_\Omega G^p(\varphi_t)\,dx + c \\
&= ct^2\bigg(\int_\Omega |D\psi|^q\,dx\bigg)^{2/q} + c\int_\Omega t^2|D\psi|^2\,dx - c\int_\Omega t^p\psi^p\,dx + c.
\end{aligned}$$

$J_\mu(\varphi_t) \to -\infty$, as $t \to \infty$. Now choose $T > 0$ such that

$$J_1(\varphi_T) < 0, \qquad \int_\Omega H^r(\varphi_T)\,dx > \rho^r.$$

For $\mu \in (0,1]$, define

$$c_\mu = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J_\mu(\gamma(t)),$$

where $\Gamma = \{\gamma \mid \gamma \in C([0,1], W_0^{1,q}(\Omega)),\ \gamma(0) = 0,\ \gamma(1) = \varphi_T\}$. Then by the Mountain Pass Lemma, c_μ is a critical value of J_μ and

$$0 < \alpha \leq c_\mu \leq \beta := \sup_{t\in[0,1]} J_1(\varphi_t).$$

Finally, the convergence result Theorem 2.6 gives the existence of a solution to the original equation, finishing the proof of Theorem 1.4.

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