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AN INDEFINITE CONCAVE-CONVEX EQUATION UNDER A NEUMANN BOUNDARY CONDITION II

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Abstract. We proceed with the investigation of the problem

$$(\mathrm{P}_\lambda) \quad -\Delta u = \lambda b(x)|u|^{q-2}u + a(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$, $1 < q < 2 < p$, $\lambda \in \mathbb{R}$, and $a, b \in C^\alpha(\overline{\Omega})$ with $0 < \alpha < 1$. Dealing now with the case $b \geq 0$, $b \not\equiv 0$, we show the existence (and several properties) of an unbounded subcontinuum of nontrivial nonnegative solutions of (P_λ). Our approach is based on *a priori* bounds, a regularisation procedure, and Whyburn's topological method.

1. Introduction and statements of main results

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. This paper is devoted to the study of nontrivial nonnegative solutions for the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda b(x)u^{q-1} + a(x)u^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ is the usual Laplacian in $\mathbb{R}^N$;

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- $\lambda \in \mathbb{R}$;
- $1 < q < 2 < p < \infty$;
- $a, b \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $a, b \not\equiv 0$, and $b \geq 0$;
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

By a *nonnegative* (*classical*) *solution* of (P_λ) we mean a nonnegative function $u \in C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$ which satisfies (P_λ) in the classical sense. When $\lambda \geq 0$, the strong maximum principle and the boundary point lemma apply to (P_λ), and as a consequence a nontrivial nonnegative solution of (P_λ) is positive on $\overline{\Omega}$. In the sequel we call it a *positive solution* of (P_λ).

In this article, we proceed with the investigation of (P_λ) made in [13]. We are now concerned with the case where $b \geq 0$ and we investigate the existence of an unbounded subcontinuum $\mathcal{C}_0 = \{(\lambda, u)\}$ of nontrivial nonnegative solutions of (P_λ), bifurcating from the trivial line $\{(\lambda, 0)\}$. Note that since $q < 2$ the nonlinearity in (P_λ) is not differentiable at $u = 0$, so that we cannot apply the standard local bifurcation theory from [5] directly. When $a \equiv 0$, $\Gamma_0 = \{(0, c) : c$ is a positive constant$\}$ is a continuum of positive solutions of (P_λ) bifurcating at $(0,0)$, and there is no positive solution for any $\lambda \neq 0$. Throughout this paper we shall then assume $a \not\equiv 0$, and we shall observe that the existence and behavior of $\mathcal{C}_0$ depend on the sign of a.

To state our main results we introduce the following sets:

$$\Omega_{\pm}^{a} = \{x \in \Omega : a(x) \gtrless 0\}, \qquad \Omega_{+}^{b} = \{x \in \Omega : b(x) > 0\}.$$

We remark that $\Omega_{\pm}^{a}, \Omega_{+}^{b}$ are all open subsets of Ω. We shall use the following conditions on these sets:

(H_1) $\Omega_{\pm}^{a}$ are both smooth subdomains of Ω, with either

$$\overline{\Omega_{+}^{a}} \subset \Omega \quad \text{and} \quad \Omega = \overline{\Omega_{+}^{a}} \cup \Omega_{-}^{a}, \quad \text{or} \tag{1.1}$$

$$\overline{\Omega_{-}^{a}} \subset \Omega \quad \text{and} \quad \Omega = \overline{\Omega_{-}^{a}} \cup \Omega_{+}^{a}. \tag{1.2}$$

(H_2) Under (H_1) there exist a function α^+ which is continuous, positive, and bounded away from zero in a tubular neighbourhood of $\partial\Omega_{+}^{a}$ in Ω_{+}^{a} and $\gamma > 0$ such that

$$a^{+}(x) = \alpha^{+}(x)\,\mathrm{dist}(x, \partial\Omega_{+}^{a})^{\gamma},$$

where $\mathrm{dist}(x, A)$ denotes the distance function to a set A, and moreover,

$$2 < p < \min\left\{\frac{2N}{N-2}, \frac{2N+\gamma}{N-1}\right\} \quad \text{if } N > 2.$$

Assumptions (H_1) and (H_2) are used to obtain *a priori* bounds on positive solutions of $(\mathrm{Q}_{\lambda,\varepsilon})$ below, cf. Amann and López-Gómez [2].

Remark 1.1. In (H_1) we may allow $\Omega_{+}^{a} = \emptyset$ (resp. $\Omega_{-}^{a} = \emptyset$). In this case it is understood that $\Omega = \Omega_{-}^{a}$ (resp. $\Omega = \Omega_{+}^{a}$).