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THE EXISTENCE OF POSITIVE SOLUTIONS FOR THE SINGULAR TWO-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we consider the following boundary value problem:

$$\begin{cases} ((-u'(t))^n)' = nt^{n-1}f(u(t)) & \text{for } 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases}$$

where $n > 1$. Using the fixed point theory on a cone and approximation technique, we obtain the existence of positive solutions in which f may be singular at $u = 0$ or f may be sign-changing.

1. Introduction

In this paper, we consider the following problem:

$$\begin{cases} ((-u'(t))^n)' = nt^{n-1}f(u(t)) & \text{for } 0 < t < 1, \\ u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $n > 1$ and f is not identically zero.

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Such a problem arises in the study of radially symmetric solutions to the following Dirichlet problem for the Monge–Ampère equations in $\mathbb{R}^n$:

$$
\begin{cases} \det(D^2u) = \lambda f(-u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}
$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball in $\mathbb{R}^n$ and $D^2u = (\partial^2 u/\partial x_i \partial x_j)$ is the Hessian of u (see [8]).

The Monge–Ampère equation has attracted a growing attention in recent years because of its important role in several areas of applied mathematics. In [11], Lions considered the existence of a unique eigenvalue λ_1 to the boundary value problem (1.2) with $f(u) = u^n$ and showed that λ_1 acts like a bifurcation point for the boundary value problem (1.2). Kutev [9] obtained the existence of a unique nontrivial convex radially symmetric solution to the boundary value problem (1.2) with $f(u) = u^p$, for all $0 < p \neq n$, reducing (1.2) to (1.1). Hu and Wang [8] established sufficient conditions for the existence and multiplicity of positive solutions to problem (1.1), where the function f is continuous on $[0, +\infty)$. In [3], Dai discussed unilateral global bifurcation results for the problem with $f(u) = u^n + g(u)$. In [17]–[18], Wang considered the existence, multiplicity and nonexistence of nontrivial radial convex solutions to systems of Monge–Ampère equations with superlinearity or sublinearity assumptions for an appropriately chosen parameter. In [16], using the Leggett–Williams fixed point theorem, Wang and An investigated the existence of at least three nontrivial radial convex solutions to systems of Monge–Ampère equations. We refer to [4], [7], [12], [20] and references therein for further discussions regarding solutions to the Monge–Ampère equations with continuous nonlinearities. For the case that $f(x)$ is singular at $x = 0$, there are some interesting results also. In [10], using the existing regularity theory and a subsolution-supersolution method, Lazer and McKennar discussed the existence and uniqueness of positive solutions to singular BVP (1.2). Using the sub-super solution technique, Mohammed [13]–[14] established the existence and uniqueness of negative convex solution also to BVP (1.2).

The goal of this paper is to consider the existence of positive solutions under the conditions that $n > 1$ and $f(x)$ is singular at $x = 0$ and sign-changing. Firstly, in order to overcome difficulties caused by singularity of f we pose new conditions which are different from those in [8], [17]–[18], and establish the multiplicity of positive solutions to BVP (1.1) different from that in [10], [13]–[14] under the condition that $f(x)$ is suplinear at $x = +\infty$. Secondly, when f is singular and sign-changing, we establish the existence of at least one positive solution to BVP (1.1) which is different from that in [6], [8], [13]–[14], [17]–[18] where f is supposed to be positive on $(0, +\infty)$.