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THE BOLZANO PROPERTY AND THE CUBE-LIKE COMPLEXES

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Abstract. Introducing the *Bolzano property*, we present a topological version of the Poincaré–Miranda theorem. One simple, and one algorithmic proof that n-cube-like complexes have this property are given. Moreover, we investigate under what conditions the inverse limit preserves the Bolzano property. Finally, we give a characterization of the Bolzano property for locally connected spaces.

1. Introduction

Bolzano proved that if a continuous function f in a closed interval $[a,b]$ changes sign at the endpoints, i.e. $f(a)\cdot f(b)\leq 0$, then this function equals zero at least at one point of the interval. Nearly a hundred years later Poincaré stated without a proof the following claim [10], [11]:

Let $f_1,\dots,f_n$ be n continuous functions of n variables $x_1,\dots,x_n$; the variable x_i varies between the limits a_i and $-a_i$. Suppose that for every $x_i=a_i$ the function f_i is constantly positive and that for every $x_i=-a_i$ the function f_i is constantly negative; I say there will exist a collection of values of x_i at which all f_i vanish

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The above result provides a solution to the following system of nonlinear equations:

$$\begin{cases} f_1(x_1,\dots,x_n) = 0, \\ \dots\dots\dots\dots\dots\dots \\ f_n(x_1,\dots,x_n) = 0, \end{cases}$$

where $f_1,\dots,f_n$ are continuous and satisfy Poincaré's boundary conditions.

In 1940, Carlo Miranda [9] rediscovered the Poincaré theorem and showed that it is equivalent to the Brouwer fixed point theorem. More information about the history, proofs, and consequences of the above mentioned theorems, the reader may find in Kulpa's paper [7].

In the papers [5], [8], the Poincaré–Miranda theorem was extended from n-cubes to *n-cube-like polyhedrons*. This gives a generalization of some classic results due to Bolzano, Poincaré, Bohl, and Brouwer [1], [2].

Kulpa described the Bolzano–Poincaré–Miranda property for topological spaces in the following way [6, p. 91]:

A family $\{(A_i,B_i) : i = 1,\dots,n\}$ of pairs of non-empty disjoint closed subsets of a topological space X is said to be an n-dimensional boundary system whenever for each continuous map $f\colon X \to R^n$, $f = (f_1,\dots,f_n)$, satisfying for each $i \le n$ the Bolzano condition

$$f_i(A_i) \subset (-\infty,0], \qquad f_i(B_i) \subset [0,\infty),$$

there exists a point $c \in X$ such that $f(c) = 0$.

While Kulpa's definition of the Bolzano property is external, here we present an internal one. Additionally, we compare these two definitions. Next, we show that the Bolzano property holds for n-cube-like-polyhedrons. We provide two different proofs for the latter statement: an existential one, and an algorithmic one. In the first one, we apply very simple arguments inspired by Kulpa's proof of the Poincaré theorem [7]. In the second one, we use concepts introduced in the papers [5], [8]. However, instead of copying and gluing all boundary of the n-cube-like-complex (as in [8]) or making a product (as in [5]), we copy and glue only one face.

In the next part of this paper, we investigate the conditions under which the inverse limit preserves the Bolzano property. In that way we obtain a generalization of Kulpa's theorem [6, p. 90].

Finally, in Theorem 6.1, we provide a characterization of the Bolzano property for the locally connected spaces.

2. The Bolzano property and the Poincaré–Miranda theorem

Definition 2.1. A topological space X is said to have the *n-dimensional Bolzano property* if there exists a family $\{(A_i,B_i) : i = 1,\dots,n\}$ of pairs of