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INFINITELY MANY POSITIVE SOLUTIONS OF FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. We are concerned with the qualitative analysis of solutions of a class of fractional boundary value problems with Dirichlet boundary conditions. By combining a direct variational approach with the theory of the fractional derivative spaces, we establish the existence of infinitely many distinct positive solutions whose E^{α}-norms and L^{∞}-norms tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively).

1. Introduction and statement of main result

In this paper, we consider the fractional boundary value problem of the following form:

$$\text{(P)}\quad \begin{cases} \dfrac{d}{dt}\bigg(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\dfrac{1}{2}\,{}_0D_T^{-\beta}(u'(t))\bigg)+\nabla F(t,u(t))=0 & \text{a.a. } t\in[0,T],\\ u(0)=u(T)=0, \end{cases}$$

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where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann–Liouville fractional integrals of order $0 \le \beta < 1$, respectively, $F\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t,x)$ is the gradient of F at x.

Fractional differential equations are very efficient tools for the mathematical description of numerous phenomena in various fields of science and engineering, such as, viscoelasticity, electrochemistry, electromagnetism, economics, optimal control, porous media, etc., see [2], [4], [7], [14], [16]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details and examples, we refer to [8], [23], [13], [21], [10] and the references therein.

During the past years, there are many papers dealing with the existence of multiple solutions of fractional boundary value problems, for example [1], [3], [5], [6], [9], [15], [18]–[20], [22]. Chen and Tang [1] studied the existence and multiplicity of solutions for system (P) when the nonlinearlity F is superquadradic, asymptotically quadratic, and subquadratic. Jiao and Zhou [11] obtained the existence of solutions for (P) by the mountain pass theorem under the Ambrosetti–Rabinowitz condition. Nyamoradi obtained in [18] the existence of infinitely many non-negative solutions of problem (P). In [3], by using the variational method, the existence of at least two nontrivial solutions for (P) is established.

The aim of the present paper is to prove the existence of infinitely many distinct positive solutions for problem (P) under suitable oscillatory assumptions on the potential F at zero or at infinity. Indeed, our main results (see Theorems 1.3 and 1.6 below) give sufficient conditions on the oscillatory terms such that problem (P) has infinitely many positive solutions. As a byproduct, these solutions can be constructed in such a way that their norms in a suitable space E^α tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively). These results correspond to the existence of infinitely many low-energy (respectively, high-energy) solutions, according to the oscillation properties of the nonlinear term.

In the present paper, in order to establish the existence of infinitely many solutions of system (P) we consider two distinct cases, according to the growth of the nonlinear term near the origin, respectively, in a neighborhood of infinity.

1.1. Oscillation near the origin. Now we are in the position to state our first main result which deals with the case when the nonlinearity F exhibits an oscillation at the origin. For this case, we make the following assumptions.

H(F)$_1$ $F\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $F(t,0) = 0$ for almost all $t \in [0,T]$ and it satisfies the following founded facts:

(1) For all $x \in \mathbb{R}^N$, $t \mapsto F(t,x)$ is measurable.

(2) For almost all $t \in [0,T]$, $x \mapsto F(t,x)$ is continuously differentiable.