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ON JAMES AND JORDAN–VON NEUMANN TYPE CONSTANTS AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. The weakly convergent sequence coefficient $\mathrm{WCS}(X)$ is estimated by the James type constant $J_{X,t}(\tau)$, Jordan–von Neumann type constant $C_t(X)$ and the Domínguez Benavides coefficient $R(1,X)$, which enable us to obtain some sufficient conditions for normal structure. The results obtained in this paper are more general than other previously known sufficient conditions for normal structure.

1. Introduction

Let X be a nontrivial Banach space, we will use B_X and S_X to denote the unit ball and unit sphere of X, respectively. Recall that a Banach space X is

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called uniformly nonsquare if there exists $\delta > 0$ such that for all $x, y \in S_X$

$$\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\delta).$$

It is well known that if X is uniformly nonsquare then X is reflexive. A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x-y\| : x, y \in H\}.$$

A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. Weak normal structure and normal structure play an important role in metric fixed point theory for nonexpansive mappings, since it was proved by Kirk that every reflexive Banach space with normal structure has the fixed point property (see [10]). Many geometrical properties of Banach spaces implying weak normal structure or normal structure have been studied (see [2]–[4], [6]–[9], [11]–[16], [18]–[20]).

Throughout this paper, we assume that X does not have the Schur property. The weakly convergent sequence coefficient $\mathrm{WCS}(X)$, introduced by Bynum, was reformulated by Sims and Smyth in [13] in the following equivalent form:

$$\mathrm{WCS}(X) = \inf\Big\{\lim_{n,m;n\neq m} \|x_n - x_m\|,\ x_n \xrightarrow{\mathrm{w}} 0,\ \|x_n\| = 1\Big\},$$

where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ exists. It is clear that $1 \leq \mathrm{WCS}(X) \leq 2$ and it is known that $\mathrm{WCS}(X) > 1$ implies that X has weak normal structure.

The coefficient $R(1,X)$ was defined by Domínguez Benavides in [5] as

$$R(1,X) = \sup\Big\{\liminf_{n\to\infty}\{\|x_n + x\|\Big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \leq 1.$$

Obviously, $1 \leq R(1,X) \leq 2$.

The aim of this paper is to estimate the lower bounds for $\mathrm{WCS}(X)$ in terms of James type constant $J_{X,t}(\tau)$, Jordan–von Neumann type constant $C_t(X)$ and Domínguez Benavides coefficient $R(1,X)$. By means of these bounds we identify several geometrical properties implying normal structure. We show that properties obtained in this paper are more general than other previously known sufficient conditions for normal structure.