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MICHAEL'S SELECTION THEOREM FOR A MAPPING DEFINABLE IN AN O-MINIMAL STRUCTURE DEFINED ON A SET OF DIMENSION 1

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Abstract. Let R be a real closed field and let some o-minimal structure extending R be given. Let $F\colon X \rightrightarrows R^m$ be a definable multivalued lower semicontinuous mapping with nonempty definably connected values defined on a definable subset X of R^n of dimension 1 (X can be identified with a finite graph immersed in R^n). Then F admits a definable continuous selection.

1. Introduction

Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [3] or [1].)

Let $F\colon X \rightrightarrows R^m$ be a multivalued mapping defined on a subset X of R^n; i.e. a mapping which assigns to each point $x \in X$ a nonempty subset $F(x)$ of R^m. F can be identified with its graph; i.e. a subset of $R^n \times R^m$. If this subset is definable we will call F *definable*. F is called *lower semicontinuous* if for each $x \in X$ and each $u \in F(x)$ and any neighbourhood U of u, there exists

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a neighbourhood V of x such that $U \cap F(y) \neq \emptyset$, for each $y \in V$. A mapping $\varphi\colon A \to R^m$, where $A \subset X$, is called a *selection* of F on A if $\varphi(x) \in F(x)$, for each $x \in A$.

The aim of the present article is the following version of Michael's Selection Theorem.

Theorem 1.1. (Main Theorem) *Let $F\colon X \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping with nonempty definably connected* (1) *values defined on a definable subset X of R^n of dimension 1 (X can be identified with a finite graph in R^n). Let $\varphi\colon A \to R^m$ be any continuous definable selection of F on a definable closed subset A of X. Then there exists a continuous definable selection $f\colon X \to R^m$ of F on X such that $f|A = \varphi$.*

Let us notice that our Main Theorem is independent of classical Michael's Selection Theorem (cf. [4, Theorem 1.2]). To see this, consider as an example the following semialgebraic multivalued mapping $F\colon R \rightrightarrows R^2$ defined by the formula

$$F(x) := \begin{cases} \{(y,z) \in R^2 : y^2 - zx^2 = 0\}, & \text{when } x \neq 0,\\ \{(y,z) \in R^2 : y = 0,\ z \geq 0\}, & \text{when } x = 0. \end{cases}$$

(The graph of F is the famous *Whitney umbrella.*) By our theorem, for any semialgebraic closed subset $A \subset R$ and any semialgebraic continuous selection $\varphi\colon A \to R^2$ of F on A there exists a semialgebraic continuous selection of F on R extending φ. However, the family $\{F(x) : x \in R\}$ is obviously not equi-LC^0 in the sense of Michael [4] and if we consider the following (non-semialgebraic) continuous selection $\varphi\colon A \to R^2$ on $A = \{1/n : n = 1, 2, \dots\} \cup \{0\}$ defined by:

$$\varphi(x) := \begin{cases} \left(\dfrac{1}{n}, 1\right) & \text{when } x = \dfrac{1}{n},\ n \text{ is even},\\ \left(-\dfrac{1}{n}, 1\right) & \text{when } x = \dfrac{1}{n},\ n \text{ is odd},\\ (0,1) & \text{when } x = 0, \end{cases}$$

then it is easy to see that there is no extension of φ to a continuous selection of F on a neighbourhood of 0.

As an application of Main Theorem we can see that in the counterexample from [2] the dimension 2 of the domain is the smallest possible.

2. Proof of Main Theorem

The proof is based on the following three fundamental tools of the o-minimal geometry: Curve Selection Lemma (see [3, Chapter 6, (1.5)] or [1, Theorem 3.2]),

(1) In fact any definably connected subset is definably arcwise connected; i.e. arcwise connected by definable arcs. Besides, if R is the field of real numbers $\mathbb{R}$, then definable connectedness coincides with usual connectedness.